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Blow-Up of Solutions for Coupled Nonlinear Klein-Gordon Equations with Weak Damping Terms

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Abstract: In this paper, we consider coupled nonlinear Klein-Gordon equations with weak damping terms, in a bounded domain. The blow up of the solution with negative initial energy is established.

Keywords: Klein-Gordon Equation, Blow up

1 Introduction

In this paper we consider the following coupled nonlinear Klein-Gordon equations

$$\begin{cases} u_{tt}-\triangle u+m_1^2u+|u_t|^{p-1}u_t=f_1(u,v), & (x,t)\in\Omega\times(0,T), \\ v_{tt}-\triangle v+m_2^2v+|v_t|^{q-1}v_t=f_2(u,v), & (x,t)\in\Omega\times(0,T), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), & x\in\Omega, \\ u(x,t)=v(x,t)=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n=1,2,3)$, $m_1,\ m_2>0$ and $p,q\geq 1$ are constants. The coupled nonlinear Klein-Gordon equation which models the motion of charged mesons in an electromagnetic field is investigated [1].

For $p=q$, Ye [2] studied the global existence and asymptotic stability of solutions of the problem (1). In [3], Pişkin proved the global existence, decay and blow up of solutions of the problem (1). Also, In the case of $p=q=1$, problem was studied by Korpusov [4], Miranda and Medeiros [5] and Wu [6]. When $m_1=\ m_2=0$, the problem (1) was considered by many authors [7,8,9,10].

In this work, the blow up of the solution with negative initial energy is proved for $p=q=1$, by using the technique of [11].

This paper will be organized as follows. In Section 2, we present some lemmas and the local existence theorem. In Section 3, we show the blow up properties of solutions in the case of $p=q=1$.

2 Preliminaries

In this section, we give some assumptions and lemmas which will be used throughout this work. Hereafter we denote by $\|.\|$ and $\|.\|_p$ the norm of $L^2(\Omega)$ and $L^p(\Omega)$, respectively.

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$f_1(u,v)=(r+1)\left[a|u+v|^{r-1}(u+v)+b|u|^{\frac{r-3}{2}}u|v|^{\frac{r+1}{2}}\right],$$

$$f_2(u,v)=(r+1)\left[a|u+v|^{r-1}(u+v)+b|u|^{\frac{r+1}{2}}|v|^{\frac{r-3}{2}}v\right],$$

where $a,b>0$ are constants and r satisfies

$$\begin{cases} 1<r & \text{if } n\leq 2, \\ 1<r\leq\frac{n}{n-2} & \text{if } n>2. \end{cases} \tag{2}$$

One can easily verify that

$$u\ f_1(u,v)+vf_2(u,v)=(r+1)F(u,v),\ \forall(u,v)\in R^2, \tag{3}$$

where

$$F(u,v)=\left[a|u+v|^{r+1}+2b|uv|^{\frac{r+1}{2}}\right]. \tag{4}$$

We have the following result.

Lemma 1. *[12]. There exist two positive constants c_0 and c_1 such that*

$$c_0\left(|u|^{r+1}+|v|^{r+1}\right)\leq F(u,v)\leq c_1\left(|u|^{r+1}+|v|^{r+1}\right) \tag{5}$$

is satisfied.

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We define the energy function as follows

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2\|u\|^2 + m_2^2\|v\|^2\right) - \int_\Omega F(u,v)\,dx. \quad (6)$$

The next lemma shows that our energy functional (6) is a nonincreasing function along the solution of (1).

Lemma 2. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -\left(\|u_t\|^2 + \|v_t\|^2\right) \leq 0. \quad (7)$$

Proof. Multiplying the first equation of (1) by u_t and the second equation by v_t, integrating over Ω, using integrating by parts and summing up the product results, we get

$$E(t) - E(0) = -\int_0^t \left(\|u_\tau\|^2 + \|v_\tau\|^2\right) d\tau \text{ for } t \geq 0. \quad (8)$$

Next, we state the local existence theorem of the problem (1), which can be obtained in a similar way as done in [7].

Theorem 1. *(Local existence). Suppose that (2) holds, and further $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$. Then problem (1) has a unique local solution*

$$u, v \in C\left([0,T); H_0^1(\Omega)\right),$$

$u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p+1}(\Omega \times [0,T))$ *and* $v_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{q+1}(\Omega \times [0,T))$.

Moreover, at least one of the following statements holds true:

i) $T = \infty$,
ii) $\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2\|u\|^2 + m_2^2\|v\|^2 \longrightarrow \infty$ as $t \longrightarrow T^-$.

3 Blow up of solutions

In this section, we are going to consider the blow up of the solution for the problem (1), when $p = q = 1$.

Lemma 3. ***[11].*** *Suppose that $\psi(t)$ is a twice continuously differentiable function satisfying*

$$\begin{cases} \psi''(t) + \psi'(t) \geq C_0 \psi^{1+\alpha}(t), & t > 0, \\ \psi(0) > 0, \quad \psi'(0) \geq 0, \end{cases}$$

where $C_0 > 0$, $\alpha > 0$ are constants. Then, $\psi(t)$ blows up in finite time.

Theorem 2. *Let the assumptions of Theorem 1 hold. Assume further that $p = q = 1$. If initial data satisfies*

$$E(0) \leq 0, \quad \int_\Omega (u_0 u_1 + v_0 v_1)\,dx \geq 0,$$

then the corresponding solution blows up in finite time. In other words, there exists a positive constant T^* such that $\lim_{t \longrightarrow T^*}\left(\|u\|^2 + \|v\|^2\right) = \infty$.

Proof. To apply Lemma 3, we define

$$\psi(t) = \frac{1}{2}\int_\Omega \left(|u|^2 + |v|^2\right) dx. \quad (9)$$

Therefore

$$\psi'(t) = \int_\Omega (u u_t + v v_t)\,dx, \quad (10)$$

and

$$\psi''(t) = \int_\Omega \left(u_t^2 + v_t^2\right) dx + \int_\Omega (u u_{tt} + v v_{tt})\,dx. \quad (11)$$

Then, eq (1) is used to estimate (11) as follows

$$\psi''(t) = \left(\|u_t\|^2 + \|v_t\|^2\right) - \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - \left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) - \int_\Omega (u u_t + v v_t)\,dx + (r+1)\int_\Omega F(u,v)\,dx. \quad (12)$$

Now, we exploit (6) to substitute for $m_1^2\|u\|^2 + m_2^2\|v\|^2$; we have

$$\begin{aligned}\psi''(t) + \psi'(t) &= 2\left(\|u_t\|^2 + \|v_t\|^2\right) - 2E(t) + (r-1)\int_\Omega F(u,v)\,dx \\ &\geq c_0(r-1)\left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1}\right),\end{aligned} \quad (13)$$

where $c_0\left(|u|^{r+1} + |v|^{r+1}\right) \leq F(u,v)$ is used.

Now, Hölder's inequality is used to estimates $\|u\|_{r+1}^{r+1}$ and $\|v\|_{r+1}^{r+1}$ as follows

$$\int_\Omega |u|^2 dx \leq \left(\int_\Omega |u|^{r+1} dx\right)^{\frac{2}{r+1}} \left(\int_\Omega 1dx\right)^{\frac{r-1}{r+1}}.$$

W_n is called the volume of the domain Ω, then

$$\|u\|_{r+1}^{r+1} \geq \left(\int_\Omega |u|^2 dx\right)^{\frac{r+1}{2}} (W_n)^{-\left(\frac{r-1}{2}\right)}, \quad (14)$$

and similarly

$$\|v\|_{r+1}^{r+1} \geq \left(\int_\Omega |v|^2 dx\right)^{\frac{r+1}{2}} (W_n)^{-\left(\frac{r-1}{2}\right)}. \quad (15)$$

Substituting the estimate (14), (15) into (13), we conclude

$$\psi''(t) + \psi'(t) \geq c_0(r-1)(W_n)^{-\left(\frac{r-1}{2}\right)}\left[\left(\int_\Omega |u|^2 dx\right)^{\frac{r+1}{2}} + \left(\int_\Omega |v|^2 dx\right)^{\frac{r+1}{2}}\right]. \quad (16)$$

In order to estimate the right-hand side in (16), we make use of the following inequality

$$(X+Y)^{\rho} \le 2^{\rho-1}\left(X^{\rho}+Y^{\rho}\right),$$

$X,Y \ge 0,\ 1 \le \rho < \infty$, applying the above inequality we have

$$2^{-\left(\frac{r-1}{2}\right)}\left(\int_{\Omega}|u|^{2}dx+\int_{\Omega}|v|^{2}dx\right)^{\frac{r+1}{2}} \le \left(\int_{\Omega}|u|^{2}dx\right)^{\frac{r+1}{2}}+\left(\int_{\Omega}|v|^{2}dx\right)^{\frac{r+1}{2}}.$$

Consequently, (16) becomes

$$\begin{aligned}\psi''(t)+\psi'(t) &\ge 2^{-\left(\frac{r-1}{2}\right)}c_{0}(r-1)(W_{n})^{-\left(\frac{r-1}{2}\right)}\left(\int_{\Omega}|u|^{2}dx+\int_{\Omega}|v|^{2}dx\right)^{\frac{r+1}{2}}\\ &= 2c_{0}(r-1)(W_{n})^{-\left(\frac{r-1}{2}\right)}\psi^{\frac{r+1}{2}}(t).\end{aligned}$$

It is easy to verify that the requirements of Lemma 3 are satisfied by

$$C_{0}=2c_{0}(r-1)(W_{n})^{-\left(\frac{r-1}{2}\right)}>0 \text{ and } \alpha=\frac{r+1}{2}>0.$$

Therefore $\psi(t)$ blows up in finite.

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