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Copson and Converses Copson Type Inequalities Via Conformable Calculus

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Abstract: In this paper, we prove some inequalities of Copson and converses Copson's type by using conformable calculus and obtain some results of special cases of fractional orders. The main results will be proved by employing Hölder's inequality for α-integrable functions and the integration by parts rule for α-integrable functions.

Keywords: Copson inequality, converses Copson inequality, conformable derivative, conformable integral, Hölder's inequality.

1 Introduction, motivation and preliminaries

In 1925 Hardy [1] proved the continuous inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1}$$

where $f \ge 0$ and integrable over any finite interval $(0,x)$, f^p is integrable and convergent over $(0,\infty)$ and $p>1$. The constant $(p/(p-1))^p$ is the best possible constant. In 1927 Copson [2] proved that if $f(x)>0$, $p>1$ and that $f^p(x)$ is integrable over $(0,\infty)$, then

$$\int_0^\infty \left(\int_x^\infty \frac{f(t)}{t}dt\right)^p dx \le p^p \int_0^\infty f^p(x)dx. \tag{2}$$

In 1976 Copson [3, Theorems 1 , 3], proved that: if f is a nonnegative integrable function on $(0,\infty)$, $p \ge 1$ and $c>1$, then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}\left(\int_0^x \lambda(t)f(t)dt\right)^p dx \le \left(\frac{p}{c-1}\right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x)dx, \tag{3}$$

where $\Lambda(x)=\int_0^x \lambda(t)dt$, and if $p>1$ and $0\le c<1$, then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}\left(\int_x^\infty \lambda(t)f(t)dt\right)^p dx \le \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x)dx. \tag{4}$$

Also Copson in [3] proved that if $0<p\le 1$ and $c<1$, then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}\left(\int_x^\infty \lambda(t)f(t)dt\right)^p dx \ge \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x)dx \tag{5}$$

and if $0<p\le 1$, $c>1$ and $\Lambda(x)\to\infty$, as $x\to\infty$, then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}\left(\int_0^x \lambda(t)f(t)dt\right)^p dx \ge \left(\frac{p}{c-1}\right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x)dx. \tag{6}$$

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In recent years, fractional inequalities using the fractional Riemann-Liouville calculus has been studied by some authors. In particular Sarikaya et al in [4], proved the Hadamard type inequality for Riemann-Liouville fractional integrals

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \leq \frac{f(a)+f(b)}{2},$$

where $f:[a,b]\to\mathbb{R}$, is convex and positive function with $f\in L_1[a,b]$ and $J_{a^+}^{\alpha}f$, $J_{b^-}^{\alpha}f$ are the Riemann-Liouville integrals of order $\alpha>0$ with $0\leq a<b$.

In [5], Dahmani and Tabharit proved Grüss inequality by using Riemann-Liouville fractional integral. In particular, they proved that if f and g are two integrable functions on $[0,\infty)$, then for all $t>0$ and $\alpha>0$ we have

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\right| \leq \frac{t^{\alpha}}{2\Gamma(\alpha+1)}(M-m)(P-p),$$

where

$$m\leq f(t)\leq M,\ \ p\leq g(t)\leq P,\ \ m, M, p, P\in\mathbb{R}, t\in[0,\infty),$$

and $J^{\alpha}$is the Riemann-Liouville integral. For some recent results connected with fractional integral inequalities see [6], [7], [8], [9], [10], and [11].

In [12] and [13] the authors introduced the conformable calculus. The interest of this new approach was born from the notion that makes a dependency just on the basic limit definition of the derivative. In recent years conformable calculus has been used in various fields, such as partial differential equations, numerical analysis and mathematical models (see [14], [15], [16], [17] and [18]). Very recently, some authors have applied this calculus and proved some of inequalities such as Opial's inequality (see [19] and [20]), Hermite-Hadamard's inequality (see [21], [22] and [23]), Chebyshev's inequality (see [24]) and Steffensen's inequality (see [25]).

In this paper, we present new Copson and conversed Copson's type inequalities via conformable calculus using a different approach. The paper is organized as follows: In Section 2, we present some preliminaries about the conformable calculus which will be needed in the proofs of the main results. In Section 3, we prove Copson-type inequalities and some of its generalizations (3) and (4). In Section 4, we prove converses of Copson-type inequalities (5) and (6). We derive the classical inequalities as special cases from our results when $\alpha=1$.

2 Basic concepts and lemmas

In this section, we present some basic definitions concerning conformable calculus. For more details, we refer the reader to [12] and [13].

Definition 1. *Let $f:[0,\infty)\to\mathbb{R}$. The conformable derivative of order α of f is defined by*

$$D_{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},$$

for all $t>0$ and $0<\alpha\leq 1$, and $D_{\alpha}f(0)=\lim_{t\to 0^+}D_{\alpha}f(t)$.

Let $\alpha\in(0,1]$ and f, g be $\alpha-$differentiable at a point t. Then

$$D_{\alpha}(fg)(t)=f(t)D_{\alpha}g(t)+g(t)D_{\alpha}f(t). \tag{7}$$

Further, let $\alpha\in(0,1]$ and f, g be $\alpha-$differentiable at a point t, with $g(t)\neq 0$. Then

$$D_{\alpha}\left(\frac{f}{g}\right)(t)=\frac{g(t)D_{\alpha}f(t)-f(t)D_{\alpha}g(t)}{g^2(t)}. \tag{8}$$

Remark. If f is a differentiable function, then

$$D_{\alpha}f(t)=t^{1-\alpha}\frac{df(t)}{dt}.$$

Definition 2. *Let $f:[0,\infty)\to\mathbb{R}$. The conformable integral of order α of f is defined by*

$$I_{\alpha}f(t)=\int_0^t f(x)d_{\alpha}x=\int_0^t x^{\alpha-1}f(x)dx, \tag{9}$$

for all $t>0$ and $0<\alpha\leq 1$.

Now, we state an integration by parts formula (see [12] and [13]) which will be used later.

Lemma 1. *Assume that* $u, v : [0,\infty) \to \mathbb{R}$ *are two functions such that* u, v *are* $\alpha-$*differentiable and* $0 < \alpha \le 1$. *Then for any* $b > 0$, *we have*

$$\int_0^b u(x) D_\alpha v(x) d_\alpha x = u(x)v(x)|_0^b - \int_0^b v(x) D_\alpha u(x) d_\alpha x. \tag{10}$$

Next, we state the Hölder type inequality that will be needed in the next sections (of course it is the usual Hölder inequality for the functions considered (i.e. $x^{\frac{(\alpha-1)}{p}} f(x)$ and $x^{\frac{(\alpha-1)}{q}} g(x)$).

Lemma 2. *Let* $f, g : [0,\infty) \to \mathbb{R}$ *and* $0 < \alpha \le 1$. *Then for any* $b > 0$, *we have*

$$\int_0^b |f(x)g(x)| d_\alpha x \le \left(\int_0^b |f(x)|^p d_\alpha x \right)^{\frac{1}{p}} \left(\int_0^b |g(x)|^q d_\alpha x \right)^{\frac{1}{q}}, \tag{11}$$

where $1/p + 1/q = 1$ *(provided the integrals exist (and are finite)).*

Throughout the paper, we will assume that the functions are nonnegative locally $\alpha-$integrable and the integrals throughout are assumed to exist (and are finite i.e. convergent).

3 Inequalities of Copson's type

In this section, we begin with the fractional version of the classical Copson type inequality.

Theorem 1. *Let* $x^{\alpha-1} f(x)$ *is continuous on* $[0,\infty)$, $\lim_{t\to\infty} t^{2\alpha-1} f(t) = 0$ *and* $\int_0^\infty \frac{f(s)}{s} d_\alpha s < \infty$. *If* $p > 1$, *then*

$$\int_0^\infty \left(\int_x^\infty \frac{f(s)}{s} d_\alpha s \right)^p d_\alpha x \le \left(\frac{p}{\alpha} \right)^p \int_0^\infty (x^{\alpha-1} f(x))^p d_\alpha x, \tag{12}$$

where

$$F(x) := \int_x^\infty \frac{f(s)}{s} d_\alpha s.$$

Proof. Integrating the term $\int_0^t F^p(x) d_\alpha x$ by the parts formula (10) with

$$u(x) = F^p(x), \text{ and } v(x) = \frac{x^\alpha}{\alpha},$$

we get (with $D_\alpha v(x) = 1$) that

$$\begin{aligned}
\int_0^t F^p(x) d_\alpha x &= \left. \frac{F^p(x) x^\alpha}{\alpha} \right|_0^t - \int_0^t \frac{x^\alpha}{\alpha} D_\alpha F^p(x) d_\alpha x \\
&= \left. \frac{F^p(x) x^\alpha}{\alpha} \right|_0^t - \frac{p}{\alpha} \int_0^t x F^{p-1}(x) F'(x) d_\alpha x.
\end{aligned}$$

Now by using the fact $\int_0^\infty \frac{f(s)}{s} d_\alpha s < \infty$, we get

$$\int_0^t F^p(x) d_\alpha x = \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t x F^{p-1}(x) F'(x) d_\alpha x.$$

Let $t \to \infty$, and noting that

$$\begin{aligned}
\lim_{t\to\infty} t^\alpha F(t) &= \lim_{t\to\infty} \frac{\int_t^\infty s^{\alpha-2} f(s) ds}{t^{-\alpha}} = \lim_{t\to\infty} \frac{-t^{\alpha-2} f(t)}{-\alpha t^{-\alpha-1}} \\
&= \lim_{t\to\infty} \left(\frac{1}{\alpha} \right) t^{2\alpha-1} f(t) = 0,
\end{aligned}$$

then

$$\int_0^{\infty} F^p(x)d_\alpha x = -\frac{p}{\alpha}\int_0^{\infty} xF^{p-1}(x)F'(x)d_\alpha x. \tag{13}$$

From the definition of F, we see that

$$F'(x) = -x^{\alpha-2}f(x).$$

Substituting into (13), we obtain

$$\int_0^{\infty} F^p(x)d_\alpha x = \frac{p}{\alpha}\int_0^{\infty} x^{\alpha-1}f(x)F^{p-1}(x)d_\alpha x. \tag{14}$$

Applying Hölder's inequality (2) on the term on the right hand side of (14) with indices p and $p/(p-1)$, we see that

$$\int_0^{\infty} F^p(x)d_\alpha x \le \frac{p}{\alpha}\left(\int_0^{\infty} F^p(x)d_\alpha x\right)^{\frac{p-1}{p}}\left(\int_0^{\infty}\left(x^{\alpha-1}f(x)\right)^p d_\alpha x\right)^{\frac{1}{p}},$$

then,

$$\left(\int_0^{\infty} F^p(x)d_\alpha x\right)^{\frac{1}{p}} \le \frac{p}{\alpha}\left(\int_0^{\infty}\left(x^{\alpha-1}f(x)\right)^p d_\alpha x\right)^{\frac{1}{p}}.$$

So that

$$\int_0^{\infty} F^p(x)d_\alpha x \le \left(\frac{p}{\alpha}\right)^p\int_0^{\infty}\left(x^{\alpha-1}f(x)\right)^p d_\alpha x,$$

which is the desired inequality (12). The proof is complete.

Remark. In Theorem 1 when $\alpha = 1$, then we obtain the classical Copson inequality (2).

Theorem 2. *If $p > 1$ and $\int_0^{\infty}\frac{\lambda(s)f(s)}{\Lambda(s)}d_\alpha s < \infty$, then*

$$\int_0^{\infty}\lambda(x)\left(\int_x^{\infty}\frac{\lambda(s)f(s)}{\Lambda(s)}d_\alpha s\right)^p d_\alpha x \le p^p\int_0^{\infty}\lambda(x)f^p(x)d_\alpha x, \tag{15}$$

where

$$\Lambda(x) := \int_0^x \lambda(s)d_\alpha s, \text{ and } F(x) := \int_x^{\infty}\frac{\lambda(s)f(s)}{\Lambda(s)}d_\alpha s.$$

Proof. Integrating the term $\int_0^{\infty}\lambda(x)F^p(x)d_\alpha x$, by parts formula (10) with

$$u(x) = F^p(x), \text{ and } v(x) = \Lambda(x),$$

we get that

$$\begin{aligned}\int_0^{\infty}\lambda(x)F^p(x)d_\alpha x &= \Lambda(x)F^p(x)|_0^{\infty} - \int_0^{\infty}\Lambda(x)x^{1-\alpha}pF^{p-1}(x)F'(x)d_\alpha x.\\ &= -\int_0^{\infty}\Lambda(x)x^{1-\alpha}pF^{p-1}(x)F'(x)d_\alpha x,\end{aligned} \tag{16}$$

where

$$\Lambda(0) = 0, \Lambda(\infty) < \infty, F(0) < \infty, \text{ and } F(\infty) = 0.$$

From the definition of F, we see that

$$F'(x) = -x^{\alpha-1}\frac{\lambda(x)f(x)}{\Lambda(x)}.$$

Substituting into (16), we obtain

$$\int_0^{\infty}\lambda(x)F^p(x)d_\alpha x = p\int_0^{\infty}\frac{\lambda(x)f(x)}{\lambda^{\frac{p-1}{p}}(x)}\lambda^{\frac{p-1}{p}}(x)F^{p-1}(x)d_\alpha x. \tag{17}$$

Applying Hölder's inequality (2) on the right hand side of (17) with indices p and $p/(p-1)$, we see that

$$\int_0^\infty \lambda(x)F^p(x)d_\alpha x$$
$$\leq p\left(\int_0^\infty \left(\frac{\lambda(x)f(x)}{\lambda^{\frac{p-1}{p}}(x)}\right)^p d_\alpha x\right)^{\frac{1}{p}}\left(\int_0^\infty \left(\lambda^{\frac{p-1}{p}}(x)F^{p-1}(x)\right)^{\frac{p}{p-1}} d_\alpha x\right)^{\frac{p-1}{p}}$$
$$= p\left(\int_0^\infty \lambda(x)f^p(x)d_\alpha x\right)^{\frac{1}{p}}\left(\int_0^\infty \lambda(x)F^p(x)d_\alpha x\right)^{\frac{p-1}{p}}.$$

Then

$$\left(\int_0^\infty \lambda(x)F^p(x)d_\alpha x\right)^{\frac{1}{p}} \leq p\left(\int_0^\infty \lambda(x)f^p(x)d_\alpha x\right)^{\frac{1}{p}},$$

and so that

$$\int_0^\infty \lambda(x)F^p(x)d_\alpha x \leq p^p\int_0^\infty \lambda(x)f^p(x)d_\alpha x,$$

which is the desired inequality (15). The proof is complete.

Remark. In Theorem 2 when $\alpha = 1$ and $\lambda(x) = 1$, we obtain the classical Copson inequality (2).

Theorem 3. *If* $p,\ c > 1$ *and* $p > c-1$, *then*

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x \leq \left(\frac{p}{c-1}\right)^p\int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)}f^p(x)d_\alpha x, \tag{18}$$

where

$$\Lambda(x) := \int_0^x \lambda(s)d_\alpha s,\ F(x) := \int_0^x \lambda(s)f(s)d_\alpha s,\ F(\infty) < \infty, \text{ and } \Lambda(x) \to \infty \text{ as } x \to \infty.$$

Proof. Integrating the term

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x,$$

by parts formula (10) with

$$u(x) = F^p(x)\Lambda^{-c}(x), \text{ and } v(x) = \Lambda(x),$$

where

$$D_\alpha u(x) = x^{1-\alpha}(pF^{p^{-1}}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x)), \text{ and } D_\alpha v(x) = \lambda(x)$$

we obtain,

$$\int_0^t \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x = \Lambda^{1-c}(x)F^p(x)\Big|_0^\infty$$
$$-\int_0^t x^{1-\alpha}(pF^{p^{-1}}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x))\Lambda(x)d_\alpha x.$$

By using

$$\Lambda(0) = 0,\ \Lambda(\infty) = \infty,\ F(0) = 0,\ F(\infty) < \infty, \text{ and } c > 1,$$

and noting that

$$\lim_{x\to 0^+}\Lambda^{\frac{1-c}{p}}(x)F(x) = \lim_{x\to 0^+}\frac{\int_0^x s^{\alpha-1}\lambda(s)f(s)ds}{(\Lambda(x))^{\frac{c-1}{p}}} = \lim_{x\to 0^+}\frac{x^{\alpha-1}\lambda(x)f(x)}{\frac{c-1}{p}(\Lambda(x))^{\frac{c-1}{p}-1}x^{\alpha-1}\lambda(x)}$$
$$= \lim_{x\to 0^+}\frac{p(\Lambda(x))^{1-\frac{c-1}{p}}f(x)}{c-1} = 0$$

we obtain

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x =$$
$$-\int_0^\infty x^{1-\alpha}(pF^{p-1}(x)F'(x)\Lambda^{1-c}(x) - cF^p(x)\Lambda^{-c}(x)\Lambda'(x))d_\alpha x.$$

Since

$$F'(x) = x^{\alpha-1}\lambda(x)f(x), \text{ and } \Lambda'(x) = x^{\alpha-1}\lambda(x),$$

we have that

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x =$$
$$-p\int_0^\infty \lambda(x)\Lambda^{1-c}(x)f(x)F^{p-1}(x)d_\alpha x + c\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x,$$

and then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x = \tag{19}$$
$$-\left(\frac{p}{1-c}\right)\int_0^\infty \left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{-\frac{p-1}{p}} \lambda(x)\Lambda^{1-c}(x)f(x)\left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{\frac{p-1}{p}} F^{p-1}(x)d_\alpha x.$$

Applying Hölder's inequality (2) on the right hand side of (19) with indices p and $p/(p-1)$, we see that

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x \le \left(\frac{p}{c-1}\right)\left(\int_0^\infty \lambda(x)\Lambda^{p-c}(x)f^p(x)d_\alpha x\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x\right)^{\frac{p-1}{p}}.$$

This implies that

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x\right)^{\frac{1}{p}} \le \left(\frac{p}{c-1}\right)\left(\int_0^\infty \lambda(x)\Lambda^{p-c}(x)f^p(x)d_\alpha x\right)^{\frac{1}{p}},$$

and so that

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)}F^p(x)d_\alpha x \le \left(\frac{p}{c-1}\right)^p \int_0^\infty \lambda(x)\Lambda^{p-c}(x)f^p(x)d_\alpha x,$$

which is the desired inequality (18). The proof is complete.

Remark. From Theorem 3, we see that if $x^{\alpha-1}\lambda(x)f(x)$ and $x^{\alpha-1}\lambda(x)$ are continuous on $[0,\infty)$ replaced by either
(i). $x^{\alpha-1}\lambda(x)f(x)$, $x^{\alpha-1}\lambda(x)$ continuous on $(0,\infty)$ and $\lim_{x\to 0^+}(\Lambda(x))^{1-\frac{c-1}{p}} f(x) = 0$,
or
(ii). $\lim_{x\to 0^+}\Lambda^{1-c}(x)F^p(x) = 0$,
then (18) is again true.

Remark. In Theorem 3 when $\alpha = 1$ and $\lambda(x) = 1$, we have the Hardy - Littelwood inequality

$$\int_0^\infty \frac{1}{x^c}F^p(x)dx \le \left(\frac{p}{c-1}\right)^p \int_0^\infty x^{p-c}f^p(x)dx, \tag{20}$$

which is Theorem 330 in [26] where $F(\infty) < \infty$, with $c > 1$.

Remark. In Theorem 3 when $\alpha = 1$, $\lambda(x) = 1$, and $c = p$, we have the classical Hardy inequality

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{21}$$

where $F(\infty) < \infty$.

Theorem 4. *If* $0 \leq c < 1$ *and* $p > 1$*, then*

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} \left(\int_x^\infty \lambda(s) f(s) d_\alpha s \right)^p d_\alpha x \leq \left(\frac{p}{1-c} \right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x) d_\alpha x, \tag{22}$$

where

$$\Lambda(x) := \int_0^x \lambda(s) d_\alpha s, \text{ and } F(x) := \int_x^\infty \lambda(s) f(s) d_\alpha s.$$

Proof. Integrating the term

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x,$$

by parts formula (10) with

$$u(x) = F^p(x)\Lambda^{-c}(x), \text{ and } v(x) = \Lambda(x),$$

where

$$D_\alpha u(x) = x^{1-\alpha}(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x)), \text{ and } D_\alpha v(x) = \lambda(x),$$

and we obtain,

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = \Lambda^{1-c}(x) F^p(x)\Big|_0^\infty$$
$$- \int_0^\infty x^{1-\alpha}(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x))\Lambda(x) d_\alpha x.$$

By using

$$\Lambda(0) = 0, F(0) < \infty, F(\infty) = 0, \text{ and } \Lambda(\infty) < \infty,$$

then we have

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = - \int_0^\infty x^{1-\alpha}(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x))\Lambda(x) d_\alpha x.$$

Since

$$F'(x) = x^{\alpha-1}\lambda(x) f(x) \text{ and } \Lambda'(x) = x^{\alpha-1}\lambda(x),$$

we obtain

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = p \int_0^\infty \lambda(x)\Lambda^{1-c}(x) f(x) F^{p-1}(x) d_\alpha x$$
$$+ c \int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x,$$

and then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x \tag{23}$$
$$= \left(\frac{p}{1-c} \right) \int_0^\infty \frac{\lambda(x)\Lambda^{1-c}(x) f(x)}{\left(\frac{\lambda(x)}{\Lambda^c(x)} \right)^{\frac{p-1}{p}}} \left(\frac{\lambda(x)}{\Lambda^c(x)} \right)^{\frac{p-1}{p}} F^{p-1}(x) d_\alpha x.$$

Applying Hölder's inequality (2) on the right hand side of (23) with indices p and $p/(p-1)$, we have

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x$$
$$\leq \left(\frac{p}{1-c} \right) \left(\int_0^\infty \lambda(x)\Lambda^{p-c}(x) f^p(x) d_\alpha x \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x \right)^{\frac{p-1}{p}},$$

and so

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right)^{\frac{1}{p}} \leq \left(\frac{p}{1-c}\right)\left(\int_0^\infty \lambda(x)\Lambda^{p-c}(x) f^p(x) d_\alpha x\right)^{\frac{1}{p}}.$$

Hence

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x \leq \left(\frac{p}{1-c}\right)^p \int_0^\infty \lambda(x)\Lambda^{p-c}(x) f^p(x) d_\alpha x,$$

which is the desired inequality (22). The proof is complete.

Remark. In Theorem 4 when $\alpha = 1$ and $\lambda(x) = 1$, we have the Hardy - Littelwood inequality

$$\int_0^\infty \frac{1}{x^c} F^p(x) dx \leq \left(\frac{p}{1-c}\right)^p \left(\int_0^\infty x^{p-c} f^p(x) dx\right). \tag{24}$$

which is Theorem 330 in [26] with $c < 1$.

Remark. In Theorem 4 when $\alpha = 1$, $\lambda(x) = 1$ and $c = p$, we have the classical Hardy inequality

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{1-p}\right)^p \left(\int_0^\infty f^p(x) dx\right). \tag{25}$$

4 Reversed inequalities of Copson type

In this section, we prove the fractional version of the reversed Copson type inequalities (5) and (6) by using conformable calculus.

Theorem 5. *If $p \leq 1$ and $c < 1$, then*

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} \left(\int_x^\infty \lambda(s) f(s) d_\alpha t\right)^p d_\alpha x \geq \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x) d_\alpha x, \tag{26}$$

where

$$\Lambda(x) := \int_0^x \lambda(s) d_\alpha s \text{ and } F(x) := \int_x^\infty \lambda(s) f(s) d_\alpha s.$$

Proof. Integrating by parts formula (10) the term

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x,$$

with

$$u(x) = F^p(x)\Lambda^{-c}(x), \text{ and } v(x) = \Lambda(x),$$

where

$$D_\alpha u(x) = x^{1-\alpha}(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x)), \text{ and } D_\alpha v(x) = \lambda(x)$$

we obtain

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = \Lambda^{1-c}(x) F^p(x)\Big|_0^\infty$$
$$- \int_0^\infty x^{1-\alpha}\left(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x)\right)\Lambda(x) d_\alpha x.$$

By using

$$\Lambda(0) = 0, F(0) < \infty, F(\infty) = 0, \text{ and } \Lambda(\infty) < \infty,$$

and

$$F'(x) = -x^{\alpha-1}\lambda(x) f(x), \text{ and } \Lambda'(x) = x^{\alpha-1}\lambda(x),$$

we get that

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = p \int_0^\infty \lambda(x) \Lambda^{1-c}(x) f(x) F^{p-1}(x) d_\alpha x + c \int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x.$$

Then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = \left(\frac{p}{1-c}\right) \int_0^\infty \lambda(x) \Lambda^{1-c} f(x) F^{p-1}(x) d_\alpha x,$$

which can be rewritten in the form

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right)^p = \left(\frac{p}{1-c}\right)^p \left(\int_0^\infty \left(\frac{(\lambda(x) f(x))^p}{\Lambda^{p(c-1)}(x) F^{p(1-p)}(x)}\right)^{\frac{1}{p}} d_\alpha x\right)^p. \tag{27}$$

Applying Hölder's inequality

$$\int_0^\infty \Phi(x) \Psi(x) d_\alpha x \leq \left(\int_0^\infty \Phi^u(x) d_\alpha x\right)^{\frac{1}{u}} \left(\int_0^\infty \Psi^v(x) d_\alpha x\right)^{\frac{1}{v}}, \tag{28}$$

with indices $u = 1/p$ and $v = 1/(1-p)$, where

$$\Phi(x) = \frac{(\lambda(x) f(x))^p}{\Lambda^{p(c-1)}(x) F^{p(1-p)}(x)}, \text{ and } \Psi(x) = \left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{1-p} F^{p(1-p)}(x),$$

we get that

$$\begin{aligned}
\left(\int_0^\infty \Phi^{\frac{1}{p}}(x) d_\alpha x\right)^p &= \left(\int_0^\infty \left(\frac{(\lambda(x) f(x))^p}{\Lambda^{p(c-1)}(x) F^{p(1-p)}(x)}\right)^{\frac{1}{p}} d_\alpha x\right)^p \\
&\geq \frac{\int_0^\infty \Phi(x) \Psi(x) d_\alpha x}{\left(\int_0^\infty \Psi^{\frac{1}{1-p}}(x) d_\alpha x\right)^{1-p}} \\
&= \frac{\int_0^\infty \frac{(\lambda(x) f(x))^p}{\Lambda^{p(c-1)}(x) F^{p(1-p)}(x)} \left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{1-p} F^{p(1-p)}(x) d_\alpha x}{\left(\int_0^\infty \left(\left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{1-p} F^{p(1-p)}(x)\right)^{\frac{1}{1-p}} d_\alpha x\right)^{1-p}}.
\end{aligned}$$

This implies that

$$\left(\int_0^\infty \left(\frac{(\lambda(x) f(x))^p}{\Lambda^{p(c-1)}(x) F^{p(1-p)}(x)}\right)^{\frac{1}{p}} d_\alpha x\right)^p \geq \left(\int_0^\infty \frac{\lambda(x) f^p(x)}{\Lambda^{c-p}(x)} d_\alpha x\right) \times \left(\int_0^\infty \left(\frac{\lambda(x)}{\Lambda^c(x)}\right) F^p(x) d_\alpha x\right)^{p-1}. \tag{29}$$

Substituting (29) into (27), we get

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right)^p \geq \left(\frac{p}{1-c}\right)^p \left(\int_0^\infty \frac{\lambda(x) f^p(x)}{\Lambda^{c-p}(x)} d_\alpha x\right) \times \left(\int_0^\infty \left(\frac{\lambda(x)}{\Lambda^c(x)}\right) F^p(x) d_\alpha x\right)^{p-1}.$$

Hence

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right) \geq \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{\lambda(x) f^p(x)}{\Lambda^{c-p}(x)} d_\alpha x,$$

which is the desired inequality (31). The proof is complete.

Remark. In Theorem 5 when $\alpha = 1$ and $\lambda(x) = 1$, we have the inequality

$$\int_0^\infty x^{-c} F^p(x)dx \geq \left(\frac{p}{1-c}\right)^p \int_0^\infty x^{p-c} f^p(x)dx, \tag{30}$$

which is Theorem 347 in [26] with $c < 1$.

Theorem 6. *If* $p \leq 1$, $c > 1$ *and* $p < c-1$, *then*

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x \geq \left(\frac{p}{c-1}\right)^p \int_0^\infty \frac{\lambda(x)}{\Lambda^{c-p}(x)} f^p(x) d_\alpha x, \tag{31}$$

where

$$\Lambda(x) := \int_0^x \lambda(s) d_\alpha s,\ F(x) := \int_0^x \lambda(s) f(s) d_\alpha s \text{ and } F(x) \to 0 \text{ as } x \to \infty.$$

Proof. Integrating by parts formula (10) the term

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x,$$

with

$$u(x) = F^p(x)\Lambda^{-c}(x), \text{ and } v(x) = \Lambda(x),$$

where

$$D_\alpha u(x) = x^{1-\alpha}(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x)),$$
and $D_\alpha v(x) = \lambda(x)$,

we have

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = \Lambda^{1-c}(x) F^p(x)\Big|_0^\infty$$
$$-\int_0^\infty x^{1-\alpha}\left(pF^{p-1}(x)F'(x)\Lambda^{-c}(x) - cF^p(x)\Lambda^{-c-1}(x)\Lambda'(x)\right)\Lambda(x) d_\alpha x.$$

By using

$$\Lambda(0) = 0,\ \Lambda(\infty) < \infty,\ F(0) = 0, \text{ and } F(\infty) = 0,$$

and

$$F'(x) = x^{\alpha-1}\lambda(x) f(x) \text{ and } \Lambda'(x) = x^{\alpha-1}\lambda(x),$$

and noting that

$$\lim_{x\to 0^+} \Lambda(x) F^{\frac{p}{1-c}}(x) = \lim_{x\to 0^+} \frac{\int_0^x s^{\alpha-1}\lambda(s)\,ds}{(F(x))^{\frac{p}{c-1}}} = \lim_{x\to 0^+} \frac{x^{\alpha-1}\lambda(x)}{\frac{p}{c-1}(F(x))^{\frac{p}{c-1}-1} x^{\alpha-1}\lambda(x) f(x)}$$
$$= \lim_{x\to 0^+} \frac{(c-1)(F(x))^{1-\frac{p}{c-1}}}{pf(x)} = 0,$$

we get

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x$$
$$= -p\int_0^\infty \lambda(x)\Lambda^{1-c}(x) f(x) F^{p-1}(x) d_\alpha x + c\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x,$$

and then

$$\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x = \left(\frac{-p}{1-c}\right)\int_0^\infty \lambda(x)\Lambda^{1-c}(x) f(x) F^{p-1} d_\alpha x,$$

which can be rewritten in the form

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right)^p = \left(\frac{p}{c-1}\right)^p \left(\int_0^\infty \left(\frac{(\lambda(x)f(x))^p}{\Lambda^{p(c-1)}(x)F^{p(1-p)}(x)}\right)^{\frac{1}{p}} d_\alpha x\right)^p. \tag{32}$$

Applying the Hölder's inequality (28) with indices $1/p$ and $1/(1-p)$ where

$$\Phi(x) = \frac{(\lambda(x)f(x))^p}{\Lambda^{p(c-1)}(x)F^{p(1-p)}(x)}, \text{ and } \Psi(x) = \left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{1-p} F^{p(1-p)}(x),$$

we see that

$$\begin{aligned}\left(\int_0^\infty \Phi^{\frac{1}{p}}(x) d_\alpha x\right)^p &= \left(\int_0^\infty \left(\frac{(\lambda(x)f(x))^p}{\Lambda^{p(c-1)}(x)F^{p(1-p)}(x)}\right)^{\frac{1}{p}} d_\alpha x\right)^p \\ &\geq \frac{\int_0^\infty \Phi(x)\Psi(x) d_\alpha x}{\left(\int_0^\infty \Psi^{\frac{1}{1-p}}(x) d_\alpha x\right)^{1-p}} \\ &= \left(\int_0^\infty \frac{(\lambda(x)f(x))^p}{\Lambda^{p(c-1)}(x)F^{p(1-p)}(x)} \left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{1-p} F^{p(1-p)}(x) d_\alpha x\right) \\ &\quad \times \left(\int_0^\infty \left(\left(\frac{\lambda(x)}{\Lambda^c(x)}\right)^{1-p} F^{p(1-p)}(x)\right)^{\frac{1}{1-p}} d_\alpha x\right)^{p-1}.\end{aligned}$$

Then,

$$\begin{aligned}\left(\int_0^\infty \left(\frac{(\lambda(x)f(x))^p}{\Lambda^{p(c-1)}(x)F^{p(1-p)}(x)}\right)^{\frac{1}{p}} d_\alpha x\right)^p \geq \left(\int_0^\infty \frac{\lambda(x)f^p(x)}{\Lambda^{c-p}(x)} d_\alpha x\right) \times \\ \left(\int_0^\infty \left(\frac{\lambda(x)}{(\Lambda(x))^c}\right) F^p(x) d_\alpha x\right)^{p-1}.\end{aligned} \tag{33}$$

Substituting (33) into (32), we get

$$\begin{aligned}\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right)^p \geq \left(\frac{p}{c-1}\right)^p \left(\int_0^\infty \frac{\lambda(x)f^p(x)}{\Lambda^{c-p}(x)} d_\alpha x\right) \times \\ \left(\int_0^\infty \left(\frac{\lambda(x)}{\Lambda^c(x)}\right) F^p(x) d_\alpha x\right)^{p-1}.\end{aligned}$$

Hence

$$\left(\int_0^\infty \frac{\lambda(x)}{\Lambda^c(x)} F^p(x) d_\alpha x\right) \geq \left(\frac{p}{c-1}\right)^p \int_0^\infty \frac{\lambda(x)f^p(x)}{\Lambda^{c-p}(x)} d_\alpha x,$$

which is the desired inequality (31). The proof is complete.

Remark. From Theorem 6 we see that if $x^{\alpha-1}\lambda(x)f(x)$ and $x^{\alpha-1}\lambda(x)$ are continuous on $[0,\infty)$ replaced by either
(i). $x^{\alpha-1}\lambda(x)f(x)$, $x^{\alpha-1}\lambda(x)$ is continuous on $(0,\infty)$ and $\lim_{x\to 0^+} \frac{(F(x))^{1-\frac{p}{c-1}}}{f(x)} = 0$,
or
(ii). $\lim_{x\to 0^+} \Lambda^{1-c}(x)F^p(x) = 0$,
then (31) is again true.

Remark. In Theorem 6 when $\alpha = 1$ and $\lambda(x) = 1$, we have the inequality

$$\int_0^\infty x^{-c}F^p(x)dx \geq \left(\frac{p}{c-1}\right)^p \int_0^\infty x^{p-c}f^p(x)dx, \tag{34}$$

which is Theorem 347 in [26] where $F(x) \to 0$ as $x \to \infty$ with $c > 1$.

References

[1] G. H. Hardy, Notes on some points in the integral calculus, LX. An inequality between integrals, *Mess. Math.* **54**, 150–156 (1925).
[2] E. T. Copson, Note on series of positive terms, *J. London. Math. Soc.* **2**, 9–12 (1927).
[3] E. T. Copson, Some integral inequalities, *Prof. Roy. Soc. Edinburg. Sect. A* **75**, 157–164 (1975/1976).
[4] M. Z. Sarıkaya, E. Set, H. Yaldiz and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model. Dyn. Syst.* **57** (9-10), 2403–2407 (2013).
[5] Z. Dahmani, L. Tabharit and S. Taf, New generalizations of Grüuss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.* **2**(3), 93–99 (2010).
[6] G. A. Anastassiou. *Advances on fractional inequalities*, Springer, 2011.
[7] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, *J. Ineq. Pure Appl. Math.* **10**(3), Art. 86 (2009).
[8] K. Bogdan and B. Dyda, The best constant in a fractional Hardy inequality, *Math. Nach.* **284**, 629–638 (2011).
[9] Z. Dahmani, New inequalities in fractional integrals, *Int. J. Nonlin. Sci.* **9** (4), 493–497 (2010).
[10] M. Jleli and B. Samet, Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions, *Math. Ineq. Appl.* **18**, 443–451 (2015).
[11] C. Yildiz, M. E. Ozdemir and H. K. Onelan, Fractional integral inequalities for different functions, *New Trend. Math. Sci.* **3**, 110–117 (2015).
[12] T. Abdeljawad, On conformable fractional calculus, *J. Comp. Appl. Math.* **279**, 57–66 (2015).
[13] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comp. Appl. Math.* **264**, 65–70 (2014).
[14] O. Acan and D. Baleanu, A new numerical technique for solving fractional partial differential equations, arXiv preprint arXiv:1704.02575 (2015).
[15] O. Acan, M. Al Qurashi and D. Baleanu, New exact solution of generalized biological population model, *J. Nonlin. Sci. Appl.* **10**, 3916–3929 (2017).
[16] A. Atangana, D. Baleanu and A. Alsaedi, New properties of conformable derivative, *Open Math.* **13**, 889–898 (2015).
[17] Y. Çenesiz, D. Baleanu, A. Kurt and O. Tasbozan, New exact solutions of Burgers' type equations with conformable derivative, *Waves Rand. Compl. Med.* **27**(1), 103–116 (2017).
[18] J. Singh, D. Kumar, M. Al Qurashi and D. Baleanu, Analysis of a new fractional model for damped Bergers' equation, *Open Phys.* **15**(1), 35–41 (2017).
[19] M. Z. Sarikaya and H. Budak, New inequalities of Opial type for conformable fractional integrals, *Turkish J. Math.* **41** (5), 1164–1173 (2017).
[20] M. Z. Sarikaya and H. Budak, Opial type inequalities for conformable fractional integrals, *RGMIA Res. Rep. Collect.* **19**, Article 93 (2016).
[21] Y. M. Chu, M. A. Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, *J. Ineq. Appl.***1**, 93 (2017).
[22] M. A. Khan, T. Ali, S. S. Dragomir and M. Z. Sarikaya, Hermite-Hadamard type inequalities for conformable fractional integrals, *Rev. Real Acad. Cienc. Exact. Fisic. Nat. Ser. A Matem.* 1–16 (2017).
[23] E. Set, A. Gözpınar and A. Ekinci, Hermite- Hadamard type inequalities via conformable fractional integrals, *Acta Math. Univ. Comenian.* **86**(2), 309–320 (2017).
[24] A. Akkurt, M. E. Yildirim and H. Yildirim, On some integral inequalities for conformable fractional integrals, *RGMIA Res. Rep. Collect.* **19**, Article 107 (2016).
[25] M. Z. Sarikaya, H. Yaldiz and H. Budak, Steffensen's integral inequality for conformable fractional integrals, *Int. J. Anal. Appl.* **15**(1), 23–30 (2017).
[26] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Second Edition, Cambridge University Press, (1952).
[27] F. J. Martin-Reyes and E. Sawyer, Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater, *Proc. Amer. Math. Soc.* **106**, 727-733 (1989).