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Odd Burr-G Poisson Family of Distributions

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Abstract: We introduce a new class of univariate continuous distribution called Odd Burr-G Poisson family of distributions (in short OBGP). Four special sub models are considered odd Burr Weibull Poisson, odd Burr Lomax Poisson, odd Burr Gamma Poisson and odd Burr beta Poisson. We gave the mixture representation of the pdf and cdf of OBGP density, we also discuss the shapes of pdf and hrf of POBG family. We gave a comprehensive treatment of mathematical properties, such as, the rth moment, sth incomplete moment, moment generating function and mean deviations. We also discussed the Renyi and Shannon entropies and stochastic ordering. The model parameters are estimated by using maximum likelihood method and the expression for ith order statistics are given. A special model Odd Burr Lomax Poisson is discussed in detail. Simulation is carried out by using monte carlo method, to check the performance of the maximum likelihood estimates. Two real life data applications are carried out to check the efficiency of the proposed family.

Keywords: Burr XII distribution, G-class of distributions, Poisson distribution, quantile function, maximum Likelihood estimation.

1 Introduction

From last few years, there has been practice to combine two or more distributions for the purpose of exploring the more shapes of distribution. An approach was introduced which deals with compounding the discrete distribution truncated at zero, with a continuous univariate lifetime model. The basic idea of introducing the compounded models or families is that a lifetime of a system with N (discrete random variable) components and the positive continuous random variable, say y_i (the lifetime of ith components), can be denoted by the non-negative random variable $U_N = Min(Y_1,Y_2,...,Y_N)$ (the minimum of a fixed number of any continuous random variables) or $V_N = Max(Y_1,Y_2,...,Y_N)$ (the maximum of a of fixed number any continuous random variable), based on whether the components are in series or in parallel structure.
Many compounded classes were proposed by many authors, such as, Alkarni *et al.*(2012) proposed a compound class of Poisson and lifetime distribution, Al-Zahrani (2014) proposed an extended Poisson Lomax distribution, Al-Zahrani (2015) gave the Poisson Lomax distribution, Asgharzadeh *et al.* (2013) proposed Pareto Poisson Lindley distribution with application, Barreto-Souza *et al.* (2009) introduced a generalization of the exponential Poisson distribution, Bereta *et al.* (2011) introduced the Poisson Weibull distribution, Bidram (2013) gave a bivariate compound class of geometric Poisson and lifetime distribution, Cancho *et al.*(2011) proposed the Poisson exponential lifetime distribution, Cordeiro *et al.*(2014) proposed the Poisson generalized linear failure rate model, da-Silva *et al.*(2015) proposed the exponentiated Burr XII Poisson distribution, Gomes *et al.*(2015) proposed the exponentiated-G Poisson model, Gui *et al.* (2014) gave the Lindley-Poisson distribution in lifetime analysis and its properties, Gupta *et al.*(2014) gave exponentiated generalized Poisson distribution with application in survival data analysis, Hashimoto *et al.* (2014) gave Poisson Birnbaum-Saunders model with long term survivors, Louzada *et al.*(2011) introduced the Poisson-exponential distribution a bayesian approach, Lu *et al.*(2012) gave the Weibull poisson distribution, Mahmoudi *et al.* (2013) gave a exponentiated Weibull Poisson distribution, Nadarajh *et al.* (2013) introduced geometric exponential Poisson distribution, Oluyede *et al.* (2016) introduced a new compound class of log-logistic Weibull Poisson distribution, Hassan *et al.* (2015) proposed complementary Burr III Poisson distribution, Pararai *et al.* (2015) introduced an extended Lindley Poisson distribution, Pararai *et al.*(2015) introduced Kumaraswamy Lindley Poisson distribution, Ramos *et al.* (2013)

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gave the exponetiated Lomax Poisson distribution with application to lifetime data, Ramos *et al.* (2015) introduced the Kumaraswamy-G Poisson family of distributions.
In this paper, we propose the Odd Burr-G Poisson (OBGP) family of distribution by compounding the Odd Burr XII (OB) distribution and the Poisson distribution. This family has a clear physical interpretation (Section 2). We expect that it will attract wider applications in biology, medicine and reliability, and other areas of research. Furthemore, the basic motivations for using the OBGP family in practice are the following:(i) to make the kurtosis more flexible compared to the baseline model,(ii) to produce a skewness for symmetrical distributions,(iii) to generate distributions with symmetric, left-skewed, right-skewed, reversed-J and U-shaped,(iv) to define special models with increasing, decreasing, bathtub and upside-down bathtub hazard rate function,(v) to provide consistently better fits than other generated models under the same baseline distribution.
This paper is organized as follows. In Section 2, we gave a new family of distributions called Odd Burr XII Poisson family of distributions. In section 3, we considered four special models odd Burr Weibull Poisson, odd Burr Lomax Poisson (OBLxP), odd Burr Gamma Poisson and odd Burr beta Poisson and also gave the plots of probability density function (pdf) and hazard rate function (hrf). In section 4, we demonstrate that its cdf and pdf is given as mixture linear of baseline distribution, shapes of the density function and hazard rate function are given. Besides, the methods to compute rth moment, sth incomplete moment, moment generating function, mean deviation, two entropies Reyni and Shannon, stochastic ordering and the expression of ith order statistic are discussed. In section 5, estimation of parameters is carried out using maximum likelihood method. In Section 6, a special sub model OBLxP is discussed in detail. Two real data sets are used in Section 7 to illustrate the usefulness of the OBLxP distribution. Concluding remarks are given in Section 8.

2 The new Model

We introduce Odd Burr XII (OB) family of distributions with its cumulative distribution function (cdf) defined as

$$B_{c,k}(x,\xi) = 1 - \left\{1 + \left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k}, \tag{2.1}$$

where F_X denotes the cdf of a random variable, $c>0$, $k>0$ and ξ denotes the vector of unknown parameters in F_X. The pdf corresponding is

$$b_{c,k}(x,\xi) = c\,k\,f_X(x,\xi)\,\frac{F_X^{c-1}(x,\xi)}{\bar{F_X}^{c+1}(x,\xi)}\left\{1 + \left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k-1},$$

where $f_X(x,\xi) = \partial F_X(x,\xi)/\partial x$ and $\bar{F_X}(x,\xi) = 1 - F_X(x,\xi)$. For convenience let $F(x,\xi) = F(x)$ and $f(x,\xi) = f(x)$. Gomes *et al.* introduced exponentiated-G Poisson family of distributions, the cdf is defined as

$$F(x;\lambda,\alpha) = \frac{1-\exp[-\lambda G^{\alpha}(x)]}{1-e^{-\lambda}},$$

where $\lambda>0$, $\alpha>0$ and $G(x)$ is the cdf of a random variable. Let $\alpha=1$, then the cdf and pdf are given by

$$F(x;\lambda,\alpha) = \frac{1-\exp[-\lambda G(x)]}{1-e^{-\lambda}} \tag{2.2}$$

and

$$f(x;\lambda,\alpha) = \lambda g(x)\frac{\exp[-\lambda G(x)]}{1-e^{-\lambda}},$$

respectively. Now we introduce the OBGP family of distributions by taking $G(x)$ in (2.2) to be the cdf (2.1) of the OB distribution.
We now provide a physical interpretation of the proposed model. Suppose that a system has N subsystems functioning independently at a given time, where N is a truncated Poisson random variable with probability mass function (pmf)

$$P(N=n) = \frac{\lambda^n}{(e^{\lambda}-1)\,n!}$$

for $n = 1,2,....$ Let X denotes the time of failure of the first out of the N functioning systems defined by the independent random variable $Y_1 \sim OB,...,Y_N \sim OB$ given by the cdf (2.1). Then $X = Min(Y_1,...,Y_N)$. So the conditional cdf of X (for $x > 0$) given N is

$$\begin{aligned} F(x|N) &= 1 - P(X > x|N) = 1 - P(Y_1 > x,...,Y_N > x) \\ &= 1 - P^N(Y_1 > x) = 1 - [1 - P(Y_1 \leq x)]^N \\ &= 1 - \left[\left\{1 + \left(\frac{F_X(x,\xi)}{1 - F_X(x,\xi)}\right)^c\right\}^{-k}\right]^N, \end{aligned}$$

where $c,k > 0$. Hence the unconditional cdf of X is

$$F(x) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \left\{1 - \left[\left\{1 + \left(\frac{F_X(x,\xi)}{1 - F_X(x,\xi)}\right)^c\right\}^{-k}\right]^n\right\} \frac{\lambda^n}{n!}.$$

$$F(x) = \frac{1}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \left\{1 - \left[1 - B_{c,k}(x)\right]^n\right\} \frac{\lambda^n}{n!}.$$

The above expression simplify to

$$F(x) = \frac{1 - \exp\left\{-\lambda B_{c,k}(x)\right\}}{1 - e^{-\lambda}}. \tag{2.3}$$

The associated density function, survival function, hazard rate function and quantile function are given below

$$f(x) = \frac{\lambda\, b_{c,k}(x)}{1 - e^{-\lambda}} \exp\left\{-\lambda B_{c,k}(x)\right\}. \tag{2.4}$$

$$\bar{F}(x) = \frac{\exp\left\{-\lambda B_{c,k}(x)\right\} - e^{-\lambda}}{1 - e^{-\lambda}}. \tag{2.5}$$

$$h(x) = \frac{\lambda\, b_{c,k}(x) \exp\left\{-\lambda B_{c,k}(x)\right\}}{\exp\left\{-\lambda B_{c,k}(x)\right\} - e^{-\lambda}}.$$

and

$$Q_X(u) = F_X^{-1}\left[\frac{\left[(1+z)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{1 + \left[(1+z)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}\right], \tag{2.6}$$

where $z = -\frac{1}{\lambda}\ln\left\{1 - (1 - e^{-\lambda})u\right\}$ and $u \sim Uniform(0,1)$.

3 Special models of the family

Here, we will consider four special model of the OBGP family of distributions along with their plots of density and hazard rate functions. In the following models, λ, c, k are the parameters of the proposed family.

3.1 Odd Burr Weibull Poisson distribution (OBWP)

Taking the Weibull distribution as the parent distribution with cdf $F_X(x) = 1 - \exp\left[-\alpha x^\beta\right]$, with $x > 0$ and $\alpha > 0$, $\beta > 0$ be the scale and shape parameters, respectively. Then cdf of OBWP distribution is given as

$$F(x) = \frac{1 - \exp\left\{-\lambda\left[1 - \left(1 + \left[e^{\alpha x^\beta} - 1\right]^c\right)^{-k}\right]\right\}}{1 - e^{-\lambda}}. \tag{3.1}$$

The pdf corresponding to (3.1) is

$$f(x) = \frac{\lambda\, c\, k\, \alpha\, \beta\, x^{\beta-1}\, e^{-\alpha x^{\beta}} \left[1 - e^{-\alpha x^{\beta}}\right]^{c-1}}{\left(1 - e^{-\lambda}\right) \left[e^{-\alpha x^{\beta}}\right]^{c+1} \left(1 + \left[e^{\alpha x^{\beta}} - 1\right]^{c}\right)^{k+1}} \times \exp\left\{-\lambda \left[1 - \left(1 + \left[e^{\alpha x^{\beta}} - 1\right]^{c}\right)^{-k}\right]\right\}.$$

If $\beta = 1$, then cdf in (3.1) reduces to Odd Burr exponential Poisson distribution and if $c = 1$ and $k = 1$, then cdf in (3.1) reduces to Weibull Poisson distribution. If $c = k = \beta = 1$, then cdf in (3.1) reduces to exponential Poisson distribution. A random variable in (3.1) is denoted by $X \sim OBWP(\lambda, c, k, \alpha, \beta)$.
In Figure 1, the plots of pdf and hrf of OBWP distribution are given. The pdf gives the negatively and positively skewed, symmetrical and reverse J-shapes. While hrf gives increasing, decreasing and bathtub shapes.

(a) (b)

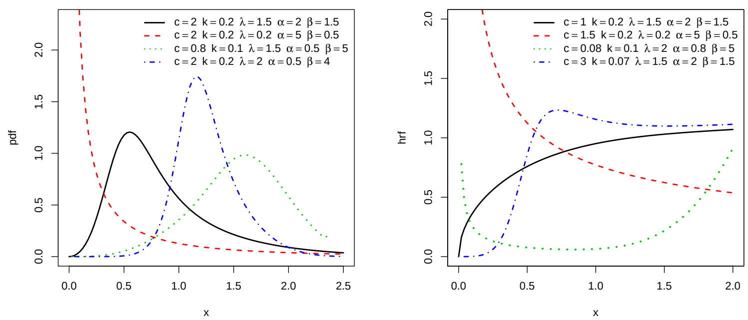

Fig. 1: Plots of pdf and hrf of OBWP distribution.

3.2 Odd Burr Lomax Poisson distribution (OBLxP)

Let the random variable X follows the Lomax distribution as the parent distribution with cdf $F_X(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}$ with $x > 0$ and $\alpha > 0$ and $\beta > 0$ be the shape and scale parameter respectively. Then cdf of BLxP distribution is given as under.

$$F(x) = \frac{1 - \exp\left\{-\lambda \left[1 - \left(1 + \left[\left(1 + \frac{x}{\beta}\right)^{\alpha} - 1\right]^{c}\right)^{-k}\right]\right\}}{1 - e^{-\lambda}}. \tag{3.2}$$

The corresponding pdf to (3.2) is

$$f(x) = \frac{\lambda\, c\, k\, \alpha \left(1 + \frac{x}{\beta}\right)^{\alpha c - 1} \left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]^{c-1}}{\left(1 - e^{-\lambda}\right) \left[\left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]^{c+1} \left(1 + \left[\left(1 + \frac{x}{\beta}\right)^{\alpha} - 1\right]^{c}\right)^{k+1}} \times \exp\left\{-\lambda \left[1 - \left(1 + \left[\left(1 + \frac{x}{\beta}\right)^{\alpha} - 1\right]^{c}\right)^{-k}\right]\right\}. \tag{3.3}$$

If we put $c = 1$ and $k = 1$, then cdf in (3.2) reduces to Lomax poisson distribution and if $k = 1$, then cdf in (3.2) reduces to Log-logistic Lomax poisson distribution. A random variable in (3.2) is denoted by $X \sim OBLxP(\lambda, c, k, \alpha, \beta)$.
In Figure 2, the plots of pdf and hrf of OBLxP distribution are given. The pdf gives only positively skewed and reverse J-shapes. While hrf gives decreasing and upside down bathtub shapes.

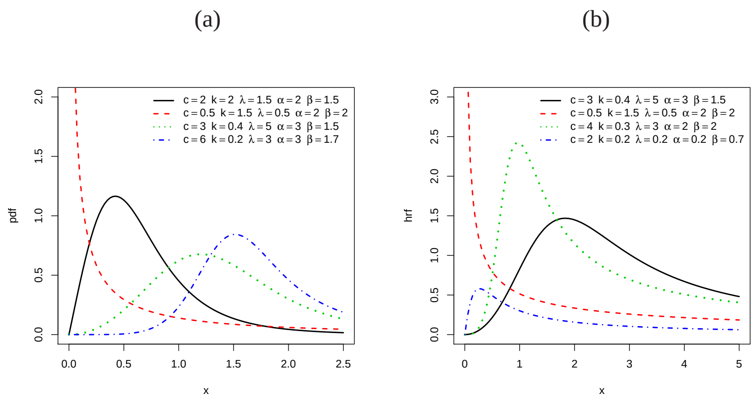

Fig. 2: Plots of pdf and hrf of OBLxP distribution

3.3 Odd Burr gamma Poisson distribution (OBGaP)

Let the random variable X follows the gamma distribution as the parent distribution with cdf $F_X(x) = \frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)} = P\left(\alpha, \frac{x}{\beta}\right)$ with $x > 0$ and $\alpha > 0$ and $\beta > 0$ be the shape parameters. Then cdf of BGaP distribution is given as

$$F(x) = \frac{1 - \exp\left\{-\lambda\left[1 - \left(1 + \left[\frac{P\left(\alpha, \frac{x}{\beta}\right)}{1 - P\left(\alpha, \frac{x}{\beta}\right)}\right]^c\right)^{-k}\right]\right\}}{1 - e^{-\lambda}} \tag{3.4}$$

The pdf corresponding to (3.4) is

$$f(x) = \frac{\lambda c k \beta^{\alpha} x^{\alpha-1} e^{-\beta x}\left[P\left(\alpha, \frac{x}{\beta}\right)\right]^{c-1}}{\Gamma(\alpha)(1 - e^{-\lambda})\left[1 - P\left(\alpha, \frac{x}{\beta}\right)\right]^{c+1}\left(1 + \left[\frac{P\left(\alpha, \frac{x}{\beta}\right)}{1 - P\left(\alpha, \frac{x}{\beta}\right)}\right]^c\right)^{k+1}}$$
$$\times \exp\left\{-\lambda\left[1 - \left(1 + \left[\frac{P\left(\alpha, \frac{x}{\beta}\right)}{1 - P\left(\alpha, \frac{x}{\beta}\right)}\right]^c\right)^{-k}\right]\right\}$$

If we put $c = 1$, then cdf in (3.4) reduces to Odd Lomax gamma Poisson distribution and if $k = 1$, then cdf in (3.4) reduces to Odd Log-logistic gamma Poisson distribution. If $c = k = 1$, then cdf in (3.4) reduces to Exponential Poisson distribution. A random variable in (3.4) is denoted by $X \sim OBGaP(\lambda, c, k, \alpha, \beta)$.
In Figure 3, the plots of pdf and hrf of OBGaP distribution are given. The pdf gives only positively skewed, symmetrical and reverse J-shapes. While hrf gives decreasing, bathtub and upside down bathtub shapes.

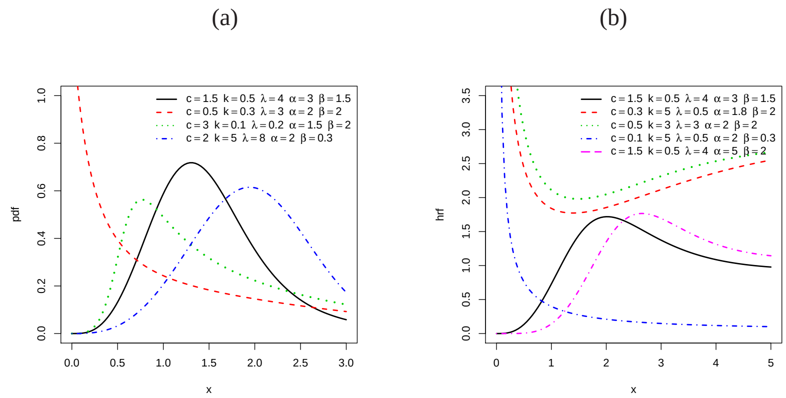

Fig. 3: Plots of pdf and hrf of OBGaP distribution.

3.4 Odd Burr beta Poisson distribution (OBBP)

Let the random variable X follows the beta distribution as the parent distribution with cdf $F_X(x) = \frac{B_X(\alpha,\beta)}{B(\alpha,\beta)} = I_X(\alpha,\beta)$, with $0 < x < 1$ $\alpha > 0$ and $\beta > 0$ be the shape parameters. Then cdf of OBBP distribution is given as

$$F(x) = \frac{1-\exp\left\{-\lambda\left[1-\left(1+\left[\frac{I_X(\alpha,\beta)}{1-I_X(\alpha,\beta)}\right]^c\right)^{-k}\right]\right\}}{1-e^{-\lambda}} \tag{3.5}$$

The pdf corresponding to (3.5) is

$$f(x) = \frac{\lambda\, c\, k\, [I_X(\alpha,\beta)]^{c-1}}{B(\alpha,\beta)\,(1-e^{-\lambda})\,[1-I_X(\alpha,\beta)]^{c+1}\left(1+\left[\frac{I_X(\alpha,\beta)}{1-I_X(\alpha,\beta)}\right]^c\right)^{k+1}}$$
$$\times \exp\left\{-\lambda\left[1-\left(1+\left[\frac{I_X(\alpha,\beta)}{1-I_X(\alpha,\beta)}\right]^c\right)^{-k}\right]\right\}$$

If we put $c = 1$, then cdf in (3.5) reduces to Odd Lomax beta Poisson distribution and if $k = 1$, then cdf in (3.5) reduces to Odd Log-logistic beta Poisson distribution. If $c = k = 1$, then cdf in (3.5) reduces to beta Poisson distribution. A random variable in (3.4) is denoted by $X \sim OBBP(\lambda, c, k, \alpha, \beta)$.
In Figure 4, the plots of pdf and hrf of OBBP distribution are given. The pdf gives left skewed, right skewed, symmetrical and U-shapes. While hrf gives increasing, decreasing and bathtub.

4 Some mathematical properties

In this section, we will discuss the expansion, shapes, ordinary moments, incomplete moments, moment generating function and mean deviation of the OBGP family of distribution.

4.1 Useful expansion

In this section, we will give a linear combination cdf and pdf of OBGP in terms of cdf and pdf of base line distribution.

Theorem 1. *If $X \sim OBGP(\lambda, c, k, \xi)$, we have the following approximation.*
1. *For $\lambda, c, k > 0$ be the real non-integer, we have following linear combination form.*

$$F(x) = \sum_{q=0}^{\infty} a_q H_q(x), \tag{4.1}$$

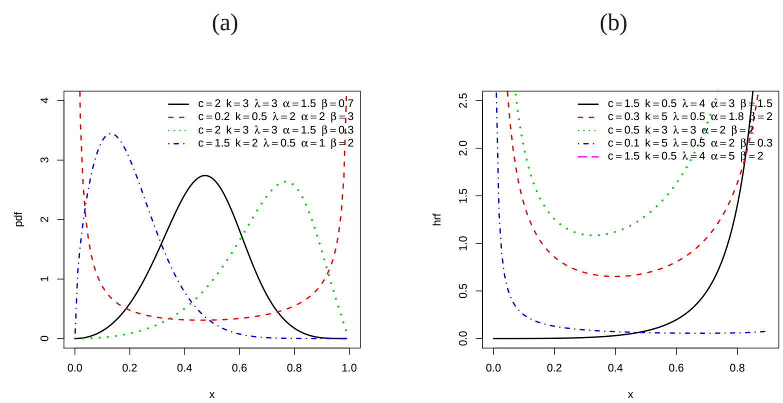

Fig. 4: Plots of pdf and hrf of OBBP distribution.

where $H_q(x) = F_X^q(x;\xi)$ represents the exp-G distribution with power parameter q, and the coefficients are given by

$$a_q = \sum_{i=1}^{\infty} \sum_{j,l,m=0}^{\infty} \frac{(-1)^{i+j+1}\lambda^i}{i!(1-e^{-\lambda})} \binom{i}{j} \binom{kj+l-1}{l} \binom{cl+m-1}{m} S_q(m+cl)$$

$$S_q(m+cl) = \sum_{r=q}^{\infty} \binom{m+cl}{r} \binom{r}{q} (-1)^{r+q} \tag{4.2}$$

Equation (4.1) reveals that the OBGP distribution can be expressed as the infinite mixture combination of the base line density functions. ***2.*** *For $\lambda, c, k > 0$ be the real non-integer, we have*

$$f(x) = \sum_{q=0}^{\infty} a_{q+1} h_{q+1}(x), \tag{4.3}$$

where a_{q+1} are given in (4.2)

*Proof.*First, if $b > 0$ is real number, we have generalized binomial theorem as

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \binom{k+i-1}{i} z^i. \tag{4.4}$$

And Taylor series expansion as

$$1 - e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^i}{i!} \tag{4.5}$$

Using (4.4) and (4.4), OBGP cdf for $\lambda, c, > 0$ real non-integer. Finally, we can obtain

$$F(x) = \sum_{q=0}^{\infty} a_q H_q(x),$$

where a_q is given in (4.2) and $H_q(x) = f_X^q(x;\xi)$ is the exp$-G$ density function with ξ parametric space.

4.2 Shapes

The shapes of the density and hazard rate functions can be described analytically. The critical points of the OBGP density function are the roots of the equation: This equation may have more than one root.

$$\frac{f_X'(x)}{f_X(x)} - (c-1)\frac{f_X(x)}{1-F_X(x)} - (c+1)\frac{f_X(x)}{F_X(x)} - (k+1)\frac{z_i'}{z_i} - \lambda \left[k(1+z_i)^{-k-1} z_i' \right] = 0 \tag{4.6}$$

The critical point of h(x) are obtained as

$$\frac{f'_X(x)}{f_X(x)}+(c-1)\frac{f_X(x)}{F_X(x)}+(c+1)\frac{f_X(x)}{1-F_X(x)}-(k+1)\frac{z'_i}{1+z_i}-\lambda\left[k(1+z_i)^{-k-1}z'_i\right]$$
$$+\left[\frac{\exp\left[-\lambda\{1-(1+z_i)^{-k}\}\right]\lambda k(1+z_i)^{-k-1}z'_i}{\exp\left[-\lambda\{1-(1+z_i)^{-k}\}\right]-e^{-\lambda}}\right]=0,$$

where $z_i=\left(\frac{F_X(x_i)}{1-F_X(x_i)}\right)$ and $z'_i=\frac{d}{dx}\left(\frac{F_X(x_i)}{1-F_X(x_i)}\right)$

4.3 Moments

The moments of the OBGP family of distributions can be obtained by using the following expression

$$E(X^r)=\sum_{q=0}^{\infty}a_{q+1}\int_0^{\infty}x^r h_{q+1}(x)dx \tag{4.7}$$

where a_{q+1} is given in (4.2) , $h_{q+1}(x)=(q+1)f_X(x)F_X^q(x)$ and $q+1$ is the power parameter.
Similarly, the m^{th} incomplete moment of the OBGP family of distributions can be obtained as

$$\mu^m(x)=\sum_{q=0}^{\infty}a_{q+1}T'_m(x), \tag{4.8}$$

where $T'_m(x)=\int_0^x x^r h_{q+1}(x)dx$.
The moment generating function of the OBGP family of distributions can be obtained as

$$M(t)=\sum_{q=0}^{\infty}a_{q+1}M_{q+1}(t), \tag{4.9}$$

where $M_{q+1}(t)=\int_0^{\infty}e^{tx}h_{q+1}(x)dx$.
The mean deviations of the OBGP family of distributions about the mean and median, respectively, can be put as

$$D_\mu=2\mu F(\mu)-2\mu^1(\mu). \tag{4.10}$$
$$D_M=\mu-2\mu^1(M), \tag{4.11}$$

where $\mu=E(X)$ comes from the equation (4.7), $M=Median(X)$ is the median given in equation (2.6), $F(\mu)$ is easily calculated from equation (2.3) and $\mu^1(.)$ is obtained by (4.8). Other applications of the equations above are obtaining the Bonferroni and Lorenz curves defined for a given probability π as

$$B(\pi)=\frac{\mu^1(q)}{\pi\mu}\qquad L(\pi)=\frac{\mu^1(q)}{\mu} \tag{4.12}$$

respectively, where $q=F^{-1}(\pi)$ is the OBGP quantile function at π determined from equation (2.6).

4.4 Entropies

Here, we will study two entropies Reyni and Shannon entropy.
By definition of Reyni entropy

$$I_R=\frac{1}{1-\delta}\log\left[\int_0^{\infty}f^{\delta}(x)dx\right] \tag{4.13}$$

from equation (2.4) we get

$$I_R = \frac{1}{1-\delta}\left[\log(K) + \log\left\{\sum_{m=0}^{\infty} V_{m+c(l+\delta)-\delta}\int_0^{\infty} f_X^{\delta}(x)F_X^{m+c(l+\delta)-\delta}(x)\,dx\right\}\right],$$

where

$$V_{m+c(l+\delta)-\delta} = \sum_{i=0}^{\infty}\frac{(-1)^i(\lambda\,\delta)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{l=0}^{\infty}\binom{k(\delta+j)+\delta+l-1}{l}(-1)^l \times \sum_{m=0}^{\infty}\binom{c\,l+\delta(c+1)+m-1}{m} \tag{4.14}$$

Detail is given in appendix A.
By definition of Shannon entropy

$$\eta_x = -E\left(\log f(x)\right)$$

From equation (2.4) we get

$$\eta_x = -\log(\lambda\,c\,k) + \log(1-e^{\lambda}) - E(\log f_X(x_i)) - (c-1)E(\log F_X(x_i)) + (c-1)E(\log \bar{F}_X(x_i)) + (k+1)E\left[\log\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}\right] + \lambda\,E\left[1-\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k}\right]. \tag{4.15}$$

By using log power expansion $\log\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}$ and using generalized binomial expansion $\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k}$ one can find the shannon entropy of the OBGP distribution. See Appendix B.

4.5 Stochastic ordering

The concept of stochastic ordering are frequently used to show the ordering mechanism in life time distributions. For more detail about stochastic ordering see (Shaked *et al.* (1994)). A random variable is said to be stochastically greater ($X \leq_{st} Y$) than Y if $F_X(x) \leq F_Y(x)$ for all x. In the similar way, X is said to be stochastically greater ($X \leq_{st} Y$) than Y in the

1. Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x.
2. Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x.
3. Mean residual order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x.
4. Likelihood ratio order ($X \leq_{hr} Y$) if $f_X(x) \geq f_Y(x)$ for all x.
5. Reversed hazard rate order ($X \leq_{rhr} Y$) if $\frac{F_X(x)}{F_Y(x)}$ is decreasing for all x.

The stochastic orders defined above are related to each other, as the following implications.

$$X \leq_{rhr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y \tag{4.16}$$

Let $X_1 \sim OBGP(c,k,\beta,\lambda_1)$ and $X_2 \sim OBGP(c,k,\beta,\lambda_2)$. Then according to the definition of likelihood ratio ordering $\left[\frac{f(x)}{g(x)}\right]$.

$$\frac{f(x)}{g(x)} = \exp\{-(\lambda_1-\lambda_2)(1-w_i)\},$$

$$\text{where } w_i = \left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k}.$$

Therefore,

$$\frac{d}{dx}\frac{f(x)}{g(x)} = (\lambda_1-\lambda_2)\exp\{-(\lambda_1-\lambda_2)(1-w_i)\}\,w_i',$$

where $w_i' = -c\,k\,f_X(x)\,\frac{F_X^{c-1}(x)}{\bar{F}_X^{c+1}(x)}\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k-1}$ from the above equation, we observe that, if $\lambda_1 < \lambda_2 \Rightarrow \frac{d}{dx}\frac{f(x)}{g(x)} < 0$, hence $X \leq_{lr} Y$. The remaining statements follow form the equation (4.16).

4.6 Order statistics

The density function $f_{i:n}(x)$ of the i-th order statistic, for $i=1,..,n$, from i.i.d random variables $X_1,...,X_2$ following OBGP distribution is simply given by.

$$f_{i:n}(x)=\frac{n!}{(i-1)!\times(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^{i}f(x)[F(x)]^{i+j-1}.$$

Using mixture representation in (4.3), (4.1) and power series expansion(see Granshteyn-Ryzhik (2007) pages [17,18])

$$\left(\sum_{i=0}^{\infty}a_i x^i\right)^n=\sum_{i=0}^{\infty}c_i x^i$$

$c_0=a_0^i$ and $c_m=(m\,a_0)^{-1}\sum_{k=0}^{m}\big(k(n+1)-m\big)\,a_k\,c_{m-k}$

$$f_{i:n}(x)=\sum_{j=0}^{n-i}\sum_{q,t=0}^{\infty}m_{j,q,t}\,h_{q+t}(x), \tag{4.17}$$

where

$$m_{j,q,t}=\frac{n!(-1)^{j}\,a_{q+1}\,d_{t:j+i-1}}{(i-1)!(n-i-j)!\,j!(q+t+1}.$$

and

$$h_m(x)=(m+1)\,f_X(x)\,F_X^m(x).$$

$h_{q+t}(x)$ the exp-G is density function with power parameter $q+t$.

5 Maximum Likelihood method

In this section, we will use the maximum likelihood method to estimate the unknown parameters of the new model from complete samples only. let $X_1,X_2,...,X_n$ be a random sample of size n from the OBGP family given in equation (2.4) distribution. The log-likelihood function for the vector of parameter $\Theta=(c,k,\beta,\xi)^T$ can be expressed as.

$$\begin{aligned} l(\Theta)=&\;n\log(\lambda\,c\,k)-n\log(1-e^{-\lambda})+\sum_{i=1}^{n}\log f_X(x_i)+(c-1)\sum_{i=1}^{n}\log F_X(x_i)\\ &-(c+1)\sum_{i=1}^{n}\log\bar{F}_X(x_i)-(k+1)\sum_{i=1}^{n}\log z_i-\lambda\sum_{i=1}^{n}\{1-z_i^{-k}\}, \end{aligned}\tag{5.1}$$

where $z_i=\left\{1+\left(\frac{1-F_X(x,\xi)}{F_X(x,\xi)}\right)^c\right\}$.
The components of score vector $\Theta=(c,k,\beta,\xi)^T$, are given by

$$\begin{aligned} U_\lambda&=\frac{n}{\lambda}+\left[\frac{n\,e^{-\lambda}}{1-e^{-\lambda}}\right]-\sum_{i=1}^{n}\{1-z_i^{-k}\}\\ U_k&=\frac{n}{k}-\sum_{i=1}^{n}\log z_i-\lambda\,k\sum_{i=1}^{n}z_i^{-k-1}z_i'\\ U_c&=\frac{n}{c}+\sum_{i=1}^{n}\log F_X(x_i)-\sum_{i=1}^{n}\log\bar{F}_X(x_i)-(k+1)\sum_{i=1}^{n}\left[\frac{z_{i:c}'}{z_i}\right]-\lambda\,k\sum_{i=1}^{n}z_i^{-k-1}z_{i:c}'\\ U_\xi&=\sum_{i=1}^{n}\left[\frac{f_X^{\xi}(x_i)}{f_X(x_i)}\right]+(c-1)\sum_{i=1}^{n}\left[\frac{F_X^{\xi}(x_i)}{F_X(x_i)}\right]+(c-1)\sum_{i=1}^{n}\left[\frac{F_X^{\xi}(x_i)}{1-F_X(x_i)}\right]-(k+1)\sum_{i=1}^{n}\left[\frac{z_{i:\xi}'}{z_i}\right]\\ &\quad-\lambda\,k\sum_{i=1}^{n}z_i^{-k-1}z_{i:\xi}' \end{aligned}$$

Setting U_λ, U_c, U_k and U_ξ equal to zero and solving these equations simultaneously yields the the maximum likelihood estimates. These equations cannot be solved analytically, and analytical softwares are required to solve them numerically. For interval estimation of the parameters, we obtain the 3×3 observed information matrix $J(\Theta)=U_{rs}$ (for $r,s=\lambda,c,k,\xi$). Under standard regularity conditions, the multivariate normal $N_3(0,J(\widehat{\Theta})^{-1})$ distribution is used to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\alpha)$ confidence intervals for c, k and ξ are given by $\hat{c}\pm z_{\gamma^*/2}\times\sqrt{var(\hat{c})}$, $\hat{k}\pm z_{\gamma^*/2}\times\sqrt{var(\hat{k})}$ and $\hat{\xi}\pm z_{\gamma^*/2}\times\sqrt{var(\hat{\xi})}$, respectively, where the var()s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\gamma^*/2}$ is the quantile $(1-\gamma^*/2)$ of the standard normal distribution.

6 Some Properties of OBLxP distribution

In this section, we will discuss a special sub model Odd Burr Lomax Poisson distribution in detail for illustrative purpose.

6.1 Odd Burr Lomax Poisson distribution(OBLxP)

Let $X\sim Lomax(1,\alpha)$ then the pdf and cdf of Lomax distribution is $F_X(x)=1-\left(1+\frac{x}{\beta}\right)^{-\alpha}$ and $f_X(x)=\alpha\left(1+\frac{x}{\beta}\right)^{-\alpha-1}$. Then the PDF and CDF of OBLxP distribution is given in equations (3.2) and (3.3) we have. The mixture representation of OBLxP from equations (4.1) and (4.3) is

$$F(x)=\sum_{q=0}^{\infty}a_q\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^q.$$

$$f(x)=\sum_{m=0}^{\infty}a_{q+1}(q+1)\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}.\right\}^q$$

The quantile function of OBLxP is given by

$$Q_x(u)=\beta\left\{\left[\frac{\left\{(1+z)^{-\frac{1}{k}}-1\right\}^{\frac{1}{c}}}{1+\left\{(1+z)^{-\frac{1}{k}}-1\right\}^{\frac{1}{c}}}\right]^{-\frac{1}{\alpha}}-1\right\},$$

where $z=-\frac{1}{\lambda}\ln\left\{1-(1-e^{-\lambda})u\right\}$.
The r^{th} moment of OBLxP distribution is

$$\mu_r'=\sum_{m=0}^{\infty}a_{q+1}\,(q+1)\sum_{s=0}^{q}\binom{q}{s}(-1)^s\,\alpha\,\beta^r\mathbf{B}\left(\alpha(s+1)-r;r+1\right).$$

The m^{th} moment of OBLxP distribution is

$$\mu^m=\sum_{m=0}^{\infty}a_{q+1}\,(q+1)\sum_{s=0}^{q}\binom{q}{s}(-1)^s\,\alpha\,\beta^m\mathbf{B}_{\frac{x}{\beta}}\left(\alpha(s+1)-m;m+1\right),$$

where $\int_0^x x^{a-1}(1-x)^{b-1}=\mathbf{B}_x(a,b)$ is the incomplete beta function.
The mgf of OBLxP distribution is

$$M_0(t)=\sum_{m=0}^{\infty}a_{q+1}\,(q+1)\sum_{s=0}^{q}\binom{q}{s}(-1)^s e^{-t}\,\Gamma\left(-\alpha(s+1)\right)(-t\beta)^{\alpha(s+1)}.$$

Put $m=1$ in μ^m we get the first incomplete moment of OBLxP distribution

$$\mu^1=\sum_{m=0}^{\infty}a_{q+1}\,(q+1)\sum_{s=0}^{q}\binom{q}{s}(-1)^s\,\alpha\,\beta^1\mathbf{B}_{\frac{x}{\beta}}\,(\alpha(s+1)-2;2).$$

that can be used to obtain mean deviation about mean and median respectively from equations (4.10) and (4.11).
Let $x_1,\ldots,x_n$ be a sample of size n from the OBLxP distribution, then the log-likelihood function can be expressed as

$$\begin{aligned}l(\Theta)&=n\log(\lambda\,c\,k\,\alpha)-n\log\left(1-e^{-\lambda}\right)+(c\alpha-1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right)+(c-1)\\&\times\sum_{i=1}^{n}\log\left\{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right\}-(k+1)\sum_{i=1}^{n}\log\left[1+\left\{\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right\}^{c}\right]\\&-\lambda\sum_{i=1}^{n}\left\{1-\left[1+\left\{\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right\}^{c}\right]^{-k}\right\},\end{aligned}$$

where $z_i=\left\{(1+\frac{x_i}{\beta})^{\alpha}-1\right\}^{c}$ The components of score vector are

$$\begin{aligned}U_\lambda&=\frac{n}{\lambda}+\frac{n\,e^{-\lambda}}{1-e^{-\lambda}}-\sum_{i=1}^{n}\left\{1-[1+z_i]^{-k}\right\}\\U_k&=\frac{n}{k}-\sum_{i=1}^{n}\log(1+z_i)-\lambda\sum_{i=1}^{n}\left[\{1+z_i\}^{-k}\log\{1+z_i\}\right]\\U_c&=\frac{n}{c}+b\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right)+\sum_{i=1}^{n}\log\left\{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right\}-(k+1)\sum_{i=1}^{n}\left[\frac{z'_{i;c}}{1+z_i}\right]\\&-\lambda\sum_{i=1}^{n}\left[k\,(1+z_i)^{-k-1}z'_{i;c}\right]\\U_\alpha&=\frac{n}{\alpha}+c\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right)+(c-1)\sum_{i=1}^{n}\left[\frac{\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\log\left(1+\frac{x_i}{\beta}\right)}{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}}\right]\\&-(k+1)\sum_{i=1}^{n}\left[\frac{z'_{i;\alpha}}{1+z_i}\right]+k\lambda\sum_{i=1}^{n}\left[(1+z_i)^{-k-1}z'_{i;\alpha}\right]\\U_\beta&=-\frac{n}{\beta}-(c\,\alpha-1)\sum_{i=1}^{n}\left[\frac{\frac{x_i}{\beta^2}}{1+\frac{x_i}{\beta}}\right]-(c-1)\sum_{i=1}^{n}\left[\frac{\alpha\left(1+\frac{x_i}{\beta}\right)^{\alpha-1}\left[\frac{x_i}{\beta^2}\right]}{\left(1+\frac{x_i}{\beta}\right)^{\alpha}}\right]\\&+(k+1)\sum_{i=1}^{n}\left[\frac{c\left\{\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right\}^{c-1}\alpha\left(1+\frac{x_i}{\beta}\right)^{\alpha-1}\left[\frac{x_i}{\beta^2}\right]}{1+\left\{\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right\}^{c}}\right]-\lambda\sum_{i=1}^{n}\left[z^{-k-1}z'_{i:\beta}\right]\end{aligned}$$

We have the pdf of i^{th} order statistic from equation (4.17)

$$f_{i:n}(x)=\sum_{j=0}^{n-i}\sum_{q,\delta=0}^{\infty}m_{j,q,\delta}\,(q+\delta+1)\,\frac{\alpha}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right]^{q+\delta}.$$

6.2 Simulations study

The mean, variance and the mean squared error (MSE)of the maximum likelihood estimative were calculated for simulated samples. We performed various simulation studies for different settings of n and combination of parameter

values, generating 1000 random samples simulated with the support of the *Software R*.
The observations denoted by $X_1, \ldots, X_n$ were generated from the OBLxP distribution given in (3.3), where they were generated from the inverse transformation method.
From the simulation results, the data of which are shown in Tables 1 and 4,it was observed that the estimates of the parameters were close to the true value of the parameters for n is bigger than 100. Besides, it was observed *MSE* decreased when n increased. In relation to the bias,their values remained close in all the scenarios. The greatest impact of the bias occurred with the parameters α, except when k assumes big values. Also, higher values of the bias occurred in the situation that the size of n was smaller than 100 independent of the combinations. The results are better as k is decreased. We can verify that the mean squared errors of the MLEs of c, k, λ, α and β decay as the sample size increases.

Table 1: Mean, bias and *MSE* (Mean Square Error) of the estimates of the parameters of OBLxP with $c = 10$, $k = 0.07$, $\lambda = 4$, $\alpha = 0.6$ and $\beta = 6.14$.

n	Parameters	Mean	Bias	*M.S.E*
20	c	21.001	11.001	3472.6
	k	0.1326	0.06256	50.835
	λ	7.709	3.709	760.45
	α	0.993	0.3934	14.5328
	β	12.866	6.726	4648.0
50	c	13.840	3.840	588.36
	k	0.0745	0.0045	0.0305
	λ	5.966	1.9664	195.129
	α	0.8459	0.2459	2811.5
	β	10.239	4.099	32.854
100	c	11.875	1.8752	134.391
	k	0.0794	0.0094	0.0140
	λ	4.7986	0.7985	78.714
	α	0.7051	0.10510	3.7860
	β	8.020	1.8799	1170.00
150	c	11.519	0.7778	1.5187
	k	0.0807	0.0106	0.0107
	λ	4.2711	0.4049	0.2711
	α	0.6564	0.0457	0.0564
	β	7.1986	-0.04056	1.0586

Table 2: Mean, bias and *MSE* (Mean Square Error) of the estimates of the parameters of OBLxP model with $c = 10$, $k = 0.07$, $\lambda = 4$, $\alpha = 7.0$ and $\beta = 0.7$.

n	Parameters	Mean	Bias	*M.S.E*
20	c	23.679	13.696	3689.3
	k	0.16384	0.1303	1965
	λ	7.914	3.9406	941.06
	α	10.1283	3.1283	8342.0
	β	1.0124	0.3142	102.058
50	c	14.247	4.2473	361.912
	k	0.0882	0.0203	0.01812
	λ	5.608	1.616	147.853
	α	9.045	2.045	2865.8
	β	0.9035	0.2035	31.2901
100	c	11.416	1.4163	32.1373
	k	0.0871	0.01711	0.01228
	λ	5.001	1.0011	88.211
	α	8.7085	1.7085	0.413603
	β	0.8726	0.1726	2844.7
150	c	11.154	1.1536	26.908
	k	0.0887	0.01864	0.0112
	λ	4.3687	0.3687	37.105
	α	9.1183	2.1183	2198.58
	β	0.9155	0.2155	24.0969

Table 3: Mean, bias and *MSE* (Mean Square Error) of the estimates of the parameters of OBLxP with $c = 0.5$, $k = 0.07$, $\lambda = 4$, $\alpha = 7$ and $\beta = 0.7$.

n	Parameters	Mean	Bias	*M.S.E*
20	c	1.1450	0.6450	26.291
	k	0.22313	0.1531	13.157
	λ	5.4870	1.4869	788.48
	α	11.8426	4.843	1770.6
	β	5.1005	4.4005	2030.42
50	c	0.5701	0.0701	1.0774
	k	0.1957	0.1257	0.5795
	λ	3.467	−0.5327	105.269
	α	9.939	2.9396	274.06
	β	1.9307	1.2307	262.03
100	c	0.5044	0.0044	0.0235
	k	0.1382	0.0682	0.1410
	λ	3.324	−0.6760	51.754
	α	8.5996	1.5996	101.484
	β	1.1155	0.4155	69.951
150	c	0.5000	0.0000457	0.0152
	k	0.106	0.04039	0.0293
	λ	3.3787	−0.6203	14.455
	α	8.2819	1.2819	66.405
	β	0.8880	0.0797	1.4103

Table 4: Mean, bias and *MSE* (Mean Square Error) of the estimates of the parameters of OBLxP with $c = 10$, $k = 1.5$, $\lambda = 0.4$, $\alpha = 7$ and $\beta = 0.7$.

n	Parameters	Mean	Bias	*M.S.E*
20	c	13.486	3.4857	414.67
	k	6.2523	4.7523	19942.0
	λ	15.894	11.8936	9628.47
	α	10.076	3.0764	10542.05
	β	1.2282	0.5282	216.045
50	c	10.777	0.7768	21.2080
	k	4.5357	3.0357	18527
	λ	8.668	4.668	3307.33
	α	8.794	1.7937	4385.32
	β	0.9976	0.2976	75.306
100	c	10.290	0.2900	6.6031
	k	3.3822	1.8821	3.81e+03
	λ	5.2151	1.2151	610.91
	α	8.595	1.59458	2348.64
	β	0.9114	0.2203	34.480
150	c	10.207	0.2065	3.4840
	k	2.7502	1.2502	436.42
	λ	3.992	−0.0084	112.018
	α	8.546	1.5459	1405.19
	β	0.880	0.1880	17.948

7 Application

In this section, we provide two applications to real data sets to illustrate the importance of the OBLxP distribution. The model parameters are estimated by the method of maximum likelihood and well-recognized goodness-of-fit statistics are calculated.

7.1 Complete/ Uncensored Data set

The data refers the failure times of 20 mechanical components and taken from the book ”Weibull models, series in probability and statistics” by Murthy DNP *et al.*(2004).
The goodness of fit measure Anderson-Darling (A^*) and Cramer-von Mises (W^*) are computed and given in table 6. The lower the values of these criteria, the better the fit. We compare the OBLxP distribution with Kw-Weibull Poisson (Kw-WP)(Ramos, 2015), Beta Lomax (B-Lx) , Kumaraswamy Lomax (K-Lx) and Lomax distributions. The computations were performed using the package Adequacy Model in *R* developed. **Remarks**: Ramos (2015) compare Kumaraswamy-Weibull Poisson (Kw-WP) to Kumaraswamy-Weibull (Kw-W), beta Weibull (BW), exponentiated Weibull (EW) and Weibull (W) models for the remission times data (Lee and Wang, 2003). He computed Statistic A^*=0.14942 and W^*=0.02250 for Kw-WP while OBLxP gives minimum A^*=0.1188 and W^*=0.0175 for same data set (Table 6) . We can say OBLP is best fit then Kw-WP, Kw-W, BW and Weibull models for this data set.

Table 5: MLEs and their standard errors (in parentheses)

Distribution	c	k	λ	β	α
OBLxP	9.8829	0.0658	3.9548	6.1425	0.6561
	(5.2939)	(0.1119)	(5.1296)	(32.9497)	(3.6866)
Kw-WP	1.3435	25.8359	5.1352	19.6074	0.1512
	(0.0151)	(0.1612)	(2.0805)	(6.3839)	(0.0667)
B-Lx	67.5047	0.8771	0.1044	6.8834	-
	(58.3961)	(0.7190)	(0.1250)	(7.2679)	-
K-Lx	47.5001	1.2606	0.0755	4.6266	-
	39.4774	0.9579	0.0783	3.7908	-
Lx	5.4148	45.2542	-	-	-
	(11.2841)	(93.2701)	-	-	-

Table 6: Statistics $A*$ and $W*$.

Distribution	W*	A*
OBLxP	0.0430	0.2846
Kw-WP	0.0638	0.4918
B-Lx	0.0769	0.5981
K-Lx	0.0851	0.6562
Lx	0.2818	1.8519

(a) (b)

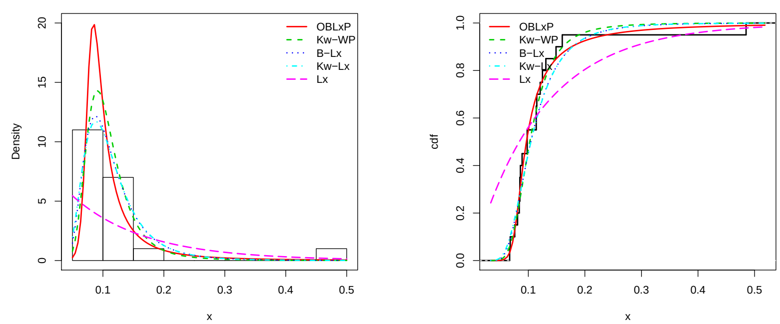

Fig. 5: Plots of estimated pdf and cdf of OBLxP distribution for uncensored data sets.

7.2 Censored data set.

In this sub section, we provide application of the OBLxP model to censored data set. We provide application of the OBLxP model to censored data set to compare with Lomax distributions. Noting that goodness-of-fit statistics computations have not been developed for censored data but the quality of fit can be checked by Akaike and Bayesian information criteria (AIC and BIC), see (Delignette-Muller and Dutang, 2015). The data considered data on the times to failure of 20 aluminum reduction cells. Failure times, in units of 1000, quoted in Lawless(2003).

Consider a data set $D = (x,r)$, where $x = (x_1,x_2,...,x_n)^T$ are the observed failure times and $r_i = (r_1,r_2,...,r_n)^T$ are the censored failure times. The r_i is equal to 1 if a failure is observed and 0 otherwise. Suppose that the data are independently and identically distributed and come from a distribution with pdf given in Eq. (11). Let $T = (c,k,\lambda,\beta,\alpha)^T$ denote the vector of parameters. Then the likelihood of Θ can be expressed as

$$\ell(D;\Theta) = \prod_{i=1}^{n}[f(x_i;\Theta)]^{r_i}[1-F(x_i;\Theta)]^{1-r_i}.$$

The log-likelihood reduces to

$$\ell(\Theta) = r_i \sum_{i=1}^{n} \log f(x_i;\Theta) + (1-r_i) \sum_{i=1}^{n} \log[1-F(x_i;\Theta)],$$

where

$$\log[1-F(x_i;\Theta)] = -\log\left(1-e^{-\lambda}\right) - \lambda\left[1-\left(1+\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{c}\right)^{-k}\right]$$

and

$$\log f(x_i;\Theta) = \log(\lambda\, c\, k\, \alpha) + (\alpha c-1)\log\left(1+\frac{x}{\beta}\right) + (c-1)\log\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]$$
$$-\log\left(1-e^{-\lambda}\right) + \alpha(c+1)\log\left(1+\frac{x}{\beta}\right) - (k+1)\log\left(1+\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{c}\right)$$
$$-\lambda\left[1-\left(1+\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{c}\right)^{-k}\right].$$

The log likelihood function can be maximized numerically to obtained the MLEs. There are various routines available for numerical maximization of l. We use the routine optimum in the R software.

Table 7: Censored data set

Model	Parameters	MLE	Standard error	Log-Likelihood	AIC	BIC
OBLxP	c	3.3414	2.4257	-19.0912	48.1825	53.1612
	k	9.9500	41.4452			
	λ	2.2978	24.8598			
	β	0.4805	1.4625			
	α	1.6996	4.8124			
Lx	c	33.7550	48.5211	-26.8868	57.7737	59.7651
	k	19.5786	27.8037			

8 Conclusions and Results

In this article, we propose a new family of Burr XII distribution called OBGP family of distribution, we study most of its mathematical properties including moments, incomplete moments,moment generating function, mean deviation, stochastic ordering, Reyni and Shannon entropies, order statistics and estimation of parameters by ML method are carried out. We also compare a sub-model as an example and discuss its properties, also fit on two real life data sets to show the usefulness of proposed family, the model provides consistently better fit than other lifetime models. The proposed family may attract great application in the areas such as Economics, Actuarial sciences, Finance, Engineering, survival and lifetime data.

9 Appendix

9.1 Appendix A

Consider

$$f^{\delta}(x) = \left[\lambda\, b_{c,k}(x)\,\frac{\exp\left[-\lambda\, B_{c,k}(x)\right]}{1-e^{-\lambda}}\right] \tag{9.1}$$

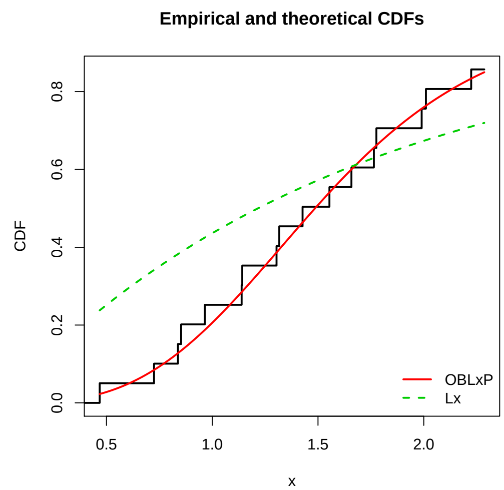

Fig. 6: Plots of estimated cdf of OBLxP for censored data set.

Using (4.5) we get

$$\exp\left[-\lambda\,\delta\,B_{c,k}(x)\right]=\sum_{i=0}^{\infty}\frac{(-1)^{i}\,(\lambda\,\delta)^{i}}{i!}\,B_{c,k}^{i}(x) \tag{9.2}$$

$$b_{c,k}^{\delta}(x)=(c\,k)^{\delta}\,f_X^{\delta}(x)\,\frac{F_X^{\delta(c-1)}(x)}{\bar{F_X}^{\delta(c+1)}(x)}\left\{1+\left(\frac{F_X(x)}{1-F_X(x)}\right)^{c}\right\}^{-\delta(k+1)} \tag{9.3}$$

$$B_{c,k}^{i}(x)=\left[1-\left\{1+\left(\frac{F_X(x)}{1-F_X(x)}\right)^{c}\right\}^{-k}\right]^{i}=\sum_{j=0}^{i}\binom{i}{j}(-1)^{j}\left\{1+\left(\frac{F_X(x)}{1-F_X(x)}\right)^{c}\right\}^{-kj} \tag{9.4}$$

$$\left\{1+\left(\frac{F_X(x)}{1-F_X(x)}\right)^{c}\right\}^{-k(\delta+j)-\delta}=\sum_{l=0}^{i}\binom{k(\delta+j)+\delta+l-1}{l}(-1)^{l}\left(\frac{F_X(x)}{1-F_X(x)}\right)^{cl} \tag{9.5}$$

$$\left(\frac{F_X(x)}{1-F_X(x)}\right)^{cl}\frac{F_X^{\delta(c-1)}(x)}{\bar{F_X}^{\delta(c+1)}(x)}=F_X^{cl+\delta(c-1)}(x)\,\bar{F_X}^{-cl-\delta(c+1)}(x) \tag{9.6}$$

$$\bar{F_X}^{-cl-\delta(c+1)}(x)=\sum_{m=0}^{\infty}\binom{cl+\delta(c+1)+m-1}{m}F_X^{m}(x) \tag{9.7}$$

substituting equations (9.7) to (9.2) in (9.1), we can get the final result

9.2 Appendix B

we know that $\log(1+z)=\sum\limits_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}z^{j}$.

Consider

$$\log\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^{c}\right\}=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^{cj} \tag{9.8}$$

we know that $(1-z)^{-n}=\sum\limits_{j=0}^{\infty}\binom{n+j-1}{j}z^{j}$.

$$\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^{cj}=\sum_{i=1}^{\infty}\binom{n+i-1}{i}F_X^{cj+i}(x) \tag{9.9}$$

submitting (9.9) in (9.8) we get following result

$$\log\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sum_{i=1}^{\infty}\binom{n+i-1}{i}F_X^{cj+i}(x) \tag{9.10}$$

we know that $1-(1+z)^{-n}=1-\sum_{j=0}^{\infty}\binom{n+j-1}{j}(-1)^j z^j$.

opening the sum for $j=0$ we get $1-(1+z)^{-n}=-\sum_{j=1}^{\infty}\binom{n+j-1}{j}(-1)^j z^j$ Now Consider

$$1-\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k}=-\sum_{j=1}^{\infty}\binom{n+j-1}{j}(-1)^j\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^{cj} \tag{9.11}$$

$$\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^{cj}=\sum_{i=1}^{\infty}\binom{cj+i-1}{i}F_X^{cj+i}(x) \tag{9.12}$$

submitting (9.12) in (9.11) we get the final result

$$1-\left\{1+\left(\frac{F_X(x,\xi)}{1-F_X(x,\xi)}\right)^c\right\}^{-k}=-\sum_{j=1}^{\infty}\binom{n+j-1}{j}(-1)^j\sum_{i=1}^{\infty}\binom{cj+i-1}{i}F_X^{cj+i}(x) \tag{9.13}$$

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