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Exponentiated Inverse Flexible Weibull Extension Distribution

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Abstract: A new two parameter distribution is recently propoesd by El-Gohary et al. [8], called as the inverse flexible Weibull extension distribution. In this paper, we propose a new three parameter model by exponentiating the inverse flexible Weibull extension distribution [8]. We called it the exponentiated inverse flexible Weibull extension (EIFW) distribution. Several properties of this distribution have been discussed such as the probability density function, the survival function, the failure rate function and the moments. The maximum likelihood estimators of the parameters are derived. Two real data sets are analyzed using the new model, which show that the new model fits the data better than some other very well known models.

Keywords: Distribution theory, Hazard function, Moments, Maximum likelihood estimators, Median and mode.

1 Introduction

Many statisticians are interested in finding a new lifetime distributions, which have some properties that enable them to use these new lifetime distributions in predicting and describing the lifetime of some devices. The Weibull distribution [16] is always used in modeling the lifetimes of physical systems, engineering applications and many different fields. Therefore, the attention of many researchers in previous years turns to provided many extensions for the Weibull distribution and studied it. The exponentiated Weibull distribution is proposed by Mudholkar and et al. [14]. Sarhan et al. [15] introduced a four parameter distribution and called it the exponentiated modified Weibull extension distribution. The exponentiated generalized Weibull- Gompertz distribution is proposed by El-Bassiouny and et al. [5]. Many authors discussed the inverse Weibull distribution which is the reciprocal of a random variable has Weibull distribution such that Mudholkar and Kollia [13], Jiang et al. [10] and Drapella [4]. Bebbington et al. [2] has defined a new two parameter distribution referred to as a flexible Weibull extension distribution, which has a failure function that can be decreasing, increasing or bathtub shaped. El-Gohary et al. [7] introduced a three parameter distribution and referred to it as the exponentiated flexible Weibull extension distribution. Recently, the inverse flexible Weibull extension distribution is proposed by El-Gohary et al. [8], which is the reciprocal of a random variable has flexible Weibull extension distribution. The inverse flexible Weibull extension distribution has cumulative distribution function (CDF) given by

$$G(x) = e^{-e^{\alpha/x-\beta x}};\ \alpha,\ \beta > 0,\ x > 0, \tag{1}$$

and the probability density function (pdf) takes the following form

$$g(x) = (\beta + \frac{\alpha}{x^2})e^{\alpha/x-\beta x}e^{-e^{\alpha/x-\beta x}};\ \alpha,\ \beta > 0,\ \ x > 0. \tag{2}$$

The aim of this paper is to propose and studing a new three-parameter distribution by exponentiating the inverse flexible Weibull extension distribution [8]. We referred to it by the exponentiated inverse flexible Weibull extension distribution (EIFW).

The paper is organized as follows. In Section 2, we present the EIFW distribution, and provide its cumulative distribution function, the probability density function, the reliability function, the failure rate function and the reversed

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failure rate function. Sum statistical properties such as the quantile, the median, the mode and the moments are obtained in Section 3. Section 4 obtains the parameter estimation using MLE method. In Section 5, a numerical results are obtained by using two real data. Finally, a conclusion for the results is given in Section 6.

2 Exponentiated inverse flexible Weibull extension distribution

2.1 EIFW specifications

A non-negative random variable $X \sim EIFW$ distribution with three parameters $\Omega = (\alpha,\beta,\lambda)$, say $EIFW(\Omega)$ if its cumulative distribution function is given by the following form

$$F(x) = e^{-\lambda e^{\alpha/x-\beta x}};\ \alpha,\ \beta,\ \lambda > 0,\ x > 0. \tag{3}$$

The two parameters α and β are scale parameters but λ is the shape parameter. The density function corresponding to (3) is

$$f_X(x) = \lambda(\beta + \frac{\alpha}{x^2})e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}};\ \alpha,\ \beta,\ \lambda > 0,\ \ x > 0. \tag{4}$$

The inverse flexible Weibull extension distribution can be derived by putting the parameter λ equals one. Plots of the pdf for the $EIFW$ distribution at various values of α, β and λ are given in Figure 1. From this figure, it is clear that the pdf of the $EIFW$ distribution can be right skewed or unimodal.

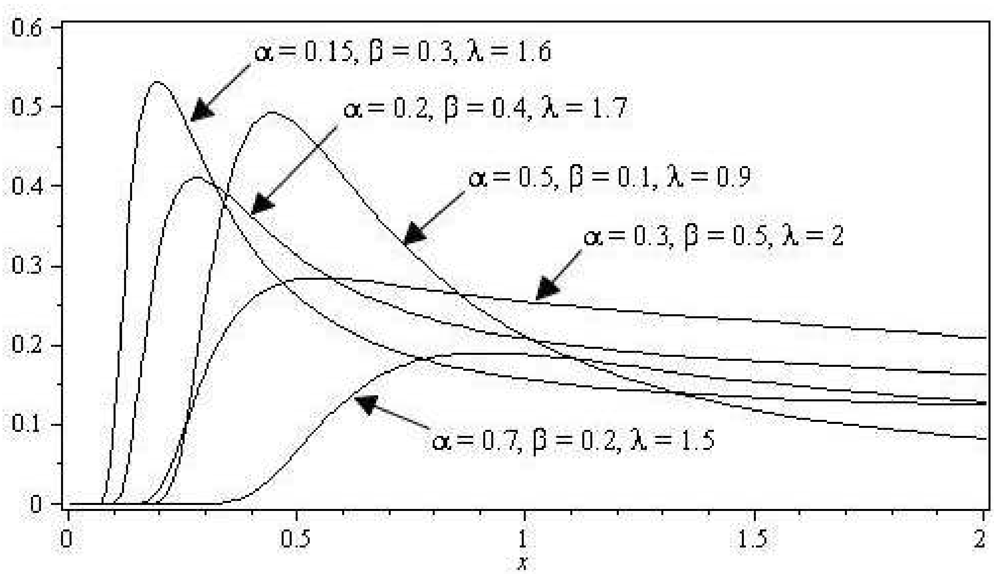

Fig. 1: The pdf of the $EIFW$ distribution at various values of α, β and λ.

2.2 Reliability analysis

If $X \sim EIFW(\Omega)$, then the reliability function of X is

$$R(x) = 1 - F(x) = 1 - e^{-\lambda e^{\alpha/x-\beta x}};\ \alpha,\beta,\lambda > 0,\ x > 0, \tag{5}$$

while its failure rate function is given by

$$h(x) = \frac{f(x)}{R(x)} = \frac{\lambda(\beta + \frac{\alpha}{x^2})e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}}{1 - e^{-\lambda e^{\alpha/x-\beta x}}};\ \alpha,\beta,\lambda > 0,\ x > 0. \tag{6}$$

Also, the reversed failure rate function of X is given by

$$r(x) = \frac{f(x)}{F(x)} = \frac{\lambda(\beta + \frac{\alpha}{x^2})e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}}{e^{-\lambda e^{\alpha/x-\beta x}}}$$
$$= \lambda(\beta + \frac{\alpha}{x^2})e^{\alpha/x-\beta x}. \quad (7)$$

Plots of the failure rate function of the $EIFW$ distribution for various values of its parameters are given in Figure 2. From this figure, it is clear that the failure rate function of the $EIFW$ distribution can take different shapes based on the values of α, β and λ, which makes the new model more flexible to fit different lifetime data sets.

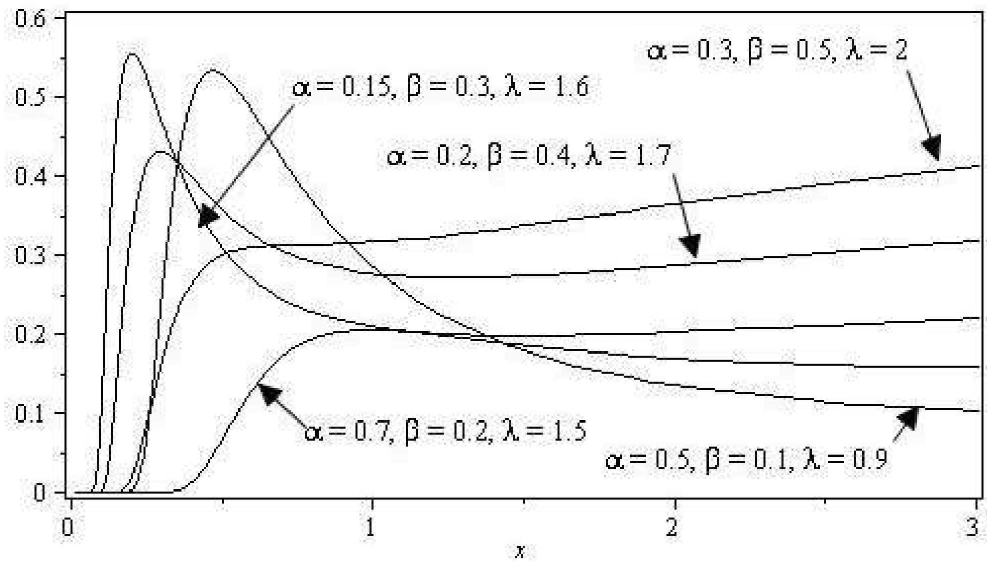

Fig. 2: The failure rate function of the $EIFW$ distribution at various values of α, β and λ.

3 Characteristics of EIFW

3.1 Quantile function and median

The quantile x_q of the $EIFW$ distribution can be easily given by

$$x_q = \frac{1}{2\beta}\left\{-\ln(-\frac{1}{\lambda}\ln(q)) + \sqrt{\left[\ln(-\frac{1}{\lambda}\ln(q))\right]^2 + 4\alpha\beta}\right\},\ 0 < q < 1. \quad (8)$$

Sitting $q = \frac{1}{2}$ in (8), we get the median of $EIFW$ distribution as

$$Med(X) = \frac{1}{2\beta}\left\{-\ln(-\frac{1}{\lambda}\ln(\frac{1}{2})) + \sqrt{\left[\ln(-\frac{1}{\lambda}\ln(\frac{1}{2}))\right]^2 + 4\alpha\beta}\right\}. \quad (9)$$

3.2 The mode

We will derive the mode of the $EIFW$ distribution by derivative (4) with respect to x and equate it to zero. The mode is the solution the following nonlinear equation with respect to x

$$(\frac{\alpha}{x^2}+\beta)\left[\lambda e^{\alpha/x-\beta x}-1\right]-\frac{2\alpha}{\alpha x+\beta x^3}=0. \tag{10}$$

In the general case, it is not possible to get an explicit solution in x to (10). So, we must use numerical methods such as bisection or fixed-point to find the solution of (10).

3.3 The moment

In this subsection, we will derive the r^{th} moment of the $EIFW$ distribution as infinite series expansion.

Theorem 1. If $X \sim EIFW(\Omega)$, then the r^{th} moment of X is given by

$$\mu^{(r)}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\lambda^{j+1}\alpha^k\Gamma(r-k-1)}{k!\;\; j!\;\; \beta^{r-k-1}(j+1)^{r-2k+1}}\left[\frac{(r-k)(r-k+1)}{\beta}+\alpha(j+1)^2\right]. \tag{11}$$

Proof: The r^{th} moment of the postive random variable X with pdf $f(x)$ is given by

$$\mu^{(r)}=\int_0^\infty x^r f(x;\alpha,\beta,\lambda)dx. \tag{12}$$

Substituting from (4) into (12), we get

$$\begin{aligned}\mu^{(r)}&=\int_0^\infty x^r\lambda(\beta+\frac{\alpha}{x^2})e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}dx\\ &=\lambda\beta\int_0^\infty x^r e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}dx+\lambda\alpha\int_0^\infty x^{r-2}e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}dx.\end{aligned}$$

Let

$$I_1=\int_0^\infty x^r e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}dx,\;\; I_2=\int_0^\infty x^{r-2}e^{\alpha/x-\beta x}e^{-\lambda e^{\alpha/x-\beta x}}dx,$$

then

$$\mu^{(r)}=\lambda\beta I_1+\lambda\alpha I_2. \tag{13}$$

Using the series expansion of $e^{-\lambda e^{\alpha/x-\beta x}}$, one gets

$$I_1=\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j}{j!}\int_0^\infty x^r e^{(j+1)\left[\frac{\alpha}{x}-\beta x\right]}dx.$$

Using the series expansion of $e^{(j+1)\alpha/x}$, we have

$$I_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\lambda^j\alpha^k(j+1)^k}{j!\;\; k!}\int_0^\infty x^{r-k}e^{-(j+1)\beta x}dx.$$

Using the substitution $y=(j+1)\beta x$ in the previous integral, then we can get

$$I_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\lambda^j\alpha^k\Gamma(r-k+1)}{k!\ \ j!\ \ \beta^{r-k+1}(j+1)^{r-2k+1}}. \tag{14}$$

Similary, we obtain

$$I_2=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\lambda^j\alpha^k\Gamma(r-k-1)}{k!\ \ j!\ \ \beta^{r-k-1}(j+1)^{r-2k-1}}. \tag{15}$$

Substituting from (14) and (15) into (13) we find (11), which completes the proof.

3.4 Moment generating function

In this subsection, we derived the moment generating function of $EIFW$ distribution as infinite series expansion according to the following theorem.

Theorem 2. If $X\sim EIFW(\Omega)$, then the moment generating function $M_X(t)$ is given by

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\lambda^{j+1}\alpha^k\Gamma(r-k-1)t^r}{r!\ k!\ \ j!\ \beta^{r-k-1}(j+1)^{r-2k+1}}\left[\frac{(r-k)(r-k+1)}{\beta}+\alpha(j+1)^2\right]. \tag{16}$$

Proof: We start with the well-known definition of the moment generating function given by

$$M_X(t)=\int_0^{\infty}e^{xt}f(x;\alpha,\beta,\lambda)dx.$$

Using the series expansion of e^{xt}, we have

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^rf(x;\alpha,\beta,\lambda)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu^{(r)}. \tag{17}$$

Substituting from (11) into (17), we find (16), which completes the proof.

4 Estimation and inference

In this section, we discuss the estimation of the model parameters by using the method of maximum likelihood. Also, the asymptotic confidence intervals of these parameters will be derived.

4.1 Maximum likelihood estimators

We will derive the maximum likelihood estimators (MLEs) of the unknown parameters α, β and λ. Let $X_1,X_2,...,X_n$ be a random sample of size n from $EIFW(\Omega)$, then the likelihood function l of this sample for the vector of parameters $\Omega=(\alpha,\beta,\lambda)$ is

$$l=\prod_{i=1}^{n}f(x_i;\alpha,\beta,\lambda). \tag{18}$$

Substituting from (4) into (18), we get

$$l=\prod_{i=1}^{n}\left\{\lambda(\beta+\frac{\alpha}{x_i^2})e^{\alpha/x_i-\beta x_i}e^{-\lambda e^{\alpha/x_i-\beta x_i}}\right\}.$$

The log-likelihood function $L = ln(l)$ can be written as

$$L = n\ln(\lambda) + \alpha \sum_{i=1}^{n} \frac{1}{x_i} - \beta \sum_{i=1}^{n} x_i - \lambda \sum_{i=1}^{n} e^{\alpha/x_i - \beta x_i} + \sum_{i=1}^{n} \ln(\beta + \frac{\alpha}{x_i^2}). \tag{19}$$

The log-likelihood function can be maximized either directly or by solving the normal equations of L. The normal equations can be obtained by setting the first partial derivatives of (19) with respect to α, β and λ to zero's. The first partial derivatives of (19) with respect to α, β and λ are obtained as follows

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} e^{\alpha/x_i - \beta x_i},$$

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{x_i} - \lambda \sum_{i=1}^{n} \frac{e^{\alpha/x_i - \beta x_i}}{x_i} + \sum_{i=1}^{n} \frac{1}{\beta x_i^2 + \alpha}$$

and

$$\frac{\partial L}{\partial \beta} = -\sum_{i=1}^{n} x_i + \lambda \sum_{i=1}^{n} x_i e^{\alpha/x_i - \beta x_i} + \sum_{i=1}^{n} \frac{x_i^2}{\beta x_i^2 + \alpha}.$$

The normal equations take the following form:

$$\frac{n}{\hat{\lambda}} - \sum_{i=1}^{n} e^{\hat{\alpha}/x_i - \hat{\beta} x_i} = 0, \tag{20}$$

$$\sum_{i=1}^{n} \frac{1}{x_i} - \hat{\lambda} \sum_{i=1}^{n} \frac{e^{\hat{\alpha}/x_i - \hat{\beta} x_i}}{x_i} + \sum_{i=1}^{n} \frac{1}{\hat{\beta} x_i^2 + \hat{\alpha}} = 0 \tag{21}$$

and

$$-\sum_{i=1}^{n} x_i + \hat{\lambda} \sum_{i=1}^{n} x_i e^{\hat{\alpha}/x_i - \hat{\beta} x_i} + \sum_{i=1}^{n} \frac{x_i^2}{\hat{\beta} x_i^2 + \hat{\alpha}} = 0. \tag{22}$$

The normal equations do not have explicit solutions and they have to be obtained numerically. From (20) we can be obtained the MLE of λ for a given $\hat{\alpha}$ and $\hat{\beta}$ as the following form

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} e^{\hat{\alpha}/x_i - \hat{\beta} x_i}}. \tag{23}$$

Substituting from (23) into (21) and (22), we get the MLEs of α and β by solving the following system of two non-linear equations:

$$\sum_{i=1}^{n} \frac{1}{x_i} - \hat{\lambda} \sum_{i=1}^{n} \frac{e^{\hat{\alpha}/x_i - \hat{\beta} x_i}}{x_i} + \sum_{i=1}^{n} \frac{1}{\hat{\beta} x_i^2 + \hat{\alpha}} = 0, \tag{24}$$

$$-\sum_{i=1}^{n} x_i + \hat{\lambda} \sum_{i=1}^{n} x_i e^{\hat{\alpha}/x_i - \hat{\beta} x_i} + \sum_{i=1}^{n} \frac{x_i^2}{\hat{\beta} x_i^2 + \hat{\alpha}} = 0. \tag{25}$$

Therefore, we have to use mathematical package such as MAPLE, MATHCAD, MATLAB and MATHEMATICA to get the MLEs of the unknown parameters.

4.2 Asymptotic confidence bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters α, β and λ when $\alpha, \beta, \lambda > 0$ [3]. The simplest large sample approach is to assume that the MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ are approximately multivariate normal with mean (α, β, λ) and covariance matrix I_0^{-1}, see [12], where I_0^{-1} is the inverse of the observed information matrix which defined by

$$I_0^{-1} = -\begin{pmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \beta} & \frac{\partial^2 L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 L}{\partial \beta \partial \alpha} & \frac{\partial^2 L}{\partial \beta^2} & \frac{\partial^2 L}{\partial \beta \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial \alpha} & \frac{\partial^2 L}{\partial \lambda \partial \beta} & \frac{\partial^2 L}{\partial \lambda^2} \end{pmatrix}^{-1} = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \beta) & Cov(\hat{\alpha}, \hat{\lambda}) \\ Cov(\hat{\beta}, \hat{\alpha}) & Var(\hat{\beta}) & Cov(\hat{\beta}, \hat{\lambda}) \\ Cov(\hat{\lambda}, \hat{\alpha}) & Cov(\hat{\lambda}, \hat{\beta}) & Var(\hat{\lambda}) \end{pmatrix}. \tag{26}$$

The second partial derivatives include in I_0 are given as follows

$$\frac{\partial^2 L}{\partial \lambda^2} = -\frac{n}{\lambda^2}, \qquad \frac{\partial^2 L}{\partial \lambda \partial \alpha} = -\sum_{i=1}^{n} \frac{e^{\alpha/x_i - \beta x_i}}{x_i},$$

$$\frac{\partial^2 L}{\partial \lambda \partial \beta} = \sum_{i=1}^{n} x_i e^{\alpha/x_i - \beta x_i},$$

$$\frac{\partial^2 L}{\partial \alpha^2} = -\lambda \sum_{i=1}^{n} \frac{e^{\alpha/x_i - \beta x_i}}{x_i^2} - \sum_{i=1}^{n} \frac{1}{(\beta x_i^2 + \alpha)^2},$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \lambda \sum_{i=1}^{n} e^{\alpha/x_i - \beta x_i} - \sum_{i=1}^{n} \frac{x_i^2}{(\beta x_i^2 + \alpha)^2}$$

and

$$\frac{\partial^2 L}{\partial \beta^2} = -\lambda \sum_{i=1}^{n} x_i^2 e^{\alpha/x_i - \beta x_i} - \sum_{i=1}^{n} \frac{x_i^4}{(\beta x_i^2 + \alpha)^2}.$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters α, β and λ by using variance covariance matrix as in the following forms

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})} \quad , \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\beta})} \text{ and } \hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\lambda})}$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$th percentile of the standard normal distribution.

5 Data analysis

In this section we analyze two real data sets to illustrate that the $EIFW$ can be a good lifetime model comparing with many known distributions such as flexible Weibull**,** inverse flexible Weibull**,** inverse Weibull, generalized inverse Weibull[9] and exponentiated generalized inverse Weibull[6] distributions (FW, IFW, IW, GIW, $EGIW$). We have fitted all selected distributions in each example. We calculated the Kolmogorov Smirnov (K-S) distance test statistic and its corresponding p-value, the log-likeihood (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Bayesian information criterion (BIC) values.

Example 6.1. The data set in Table 1, gives the lifetimes of 50 devices that were provided by (Aarset, 1987)[1]. The MLEs of the unknown parameters and the Kolmogorov-Smirnov (K-S) test statistic with its corresponding p-value for the six tested models are given in Table 2. The fitted survival and failure rate functions are shown in Figure 3 and Figure 4 respectively. The K-S test statistic value for $EIFW$ model is 0.1575, and the corresponding p-value is 0.15275. Depending on the K-S test statistic values and its corresponding p-values, which given in Table 2, we can deduce that: (i) both the IW and FW distributions are rejected at any level of significance $\delta \geq 6 \times 10^{-9}$, (ii) the GIW distribution must be rejected at $\delta \geq 3.8 \times 10^{-5}$, (iii) both the IFW and $EGIW$ distributions are rejected at $\delta \geq 2.5 \times 10^{-3}$, (iv) the $EIFW$ distribution is accepted at $\delta \leq 0.155$. and (v) the $EIFW$ model has the lowest K-S value and the highest p-value among all the models used here to fit the current data set, which means that the new model fits the data better than the FW, IW, GIW, $EGIW$ and IFW models. The log-likeihood, Akaike information criterion, correct Akaike information criterion and Bayesian information criterion values for the six tested models are given in Table 3. From Table 3 we find that, the $EIFW$

distribution has the lowest L, AIC, CAIC and BIC values. This confirms that the $EIFW$ model fits the data better than all models used here to fit the current data set.

Table 1.
Life time of 50 devices, see Aarset(1987)[1].

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18
18	21	32	36	40	45	46	47	50	55	60	63	63	67	67	67	67
72	75	79	82	82	83	84	84	84	85	85	85	85	85	86	86	

Table 2.
The MLEs, K-S and p-values for Aarset data.

The model	MLE of the parameters	K-S value	p-value
FW	$\hat{\alpha} = 0.0122, \hat{\beta} = 0.7002$	0.4386	4.29×10^{-9}
IW	$\hat{\alpha} = 1.043, \hat{\beta} = 0.397$	0.435	5.95×10^{-9}
GIW	$\hat{\alpha} = 0.596, \hat{\beta} = 0.274, \hat{\theta} = 1.273$	0.324	3.72×10^{-5}
EGIW	$\hat{\alpha} = 1.008, \hat{\beta} = 0.61, \hat{\theta} = 2.142, \hat{\lambda} = 0.75$	0.254	2.47×10^{-3}
IFW	$\hat{\alpha} = 0.165, \hat{\beta} = 0.024$	0.276	7.38×10^{-4}
EIFW	$\hat{\alpha} = 0.0988, \hat{\beta} = 0.02963, \hat{\lambda} = 2.1872$	**0.157**	**0.1528**

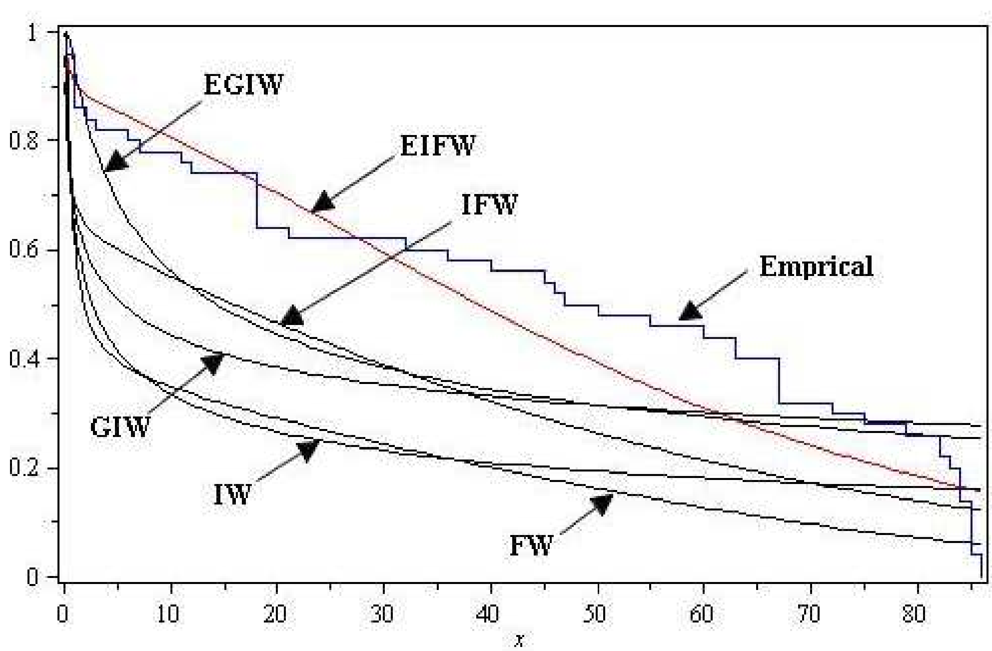

Fig. 3: The empirical and fitted survival functions of selected models for Aarset data.

Table 3.
The log-likelihood, AIC, CAIC and BIC values for Aarset data.

The model	L	AIC	CAIC	BIC
IW	−281.07	566.14	566.396	569.964
GIW	−287.48	580.951	581.473	586.687
EGIW	−254.92	517.839	518.727	525.487
FW	−250.81	505.620	505.88	509.448
IFW	−242.57	488.914	489.169	492.738
EIFW	**−233.52**	**473.029**	**473.551**	**478.765**

Substituting the MLEs of the unknown parameters into (26), we get estimation of the variance covariance matrix as the following:

$$I_0^{-1} = \begin{pmatrix} 0.000998 & -0.000026 & -0.00495 \\ -0.000026 & 0.0000139 & 0.00061 \\ -0.00495 & 0.00061 & 0.13745 \end{pmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters α, β and λ are given respectively as $[0.036919, 0.160760]$, $[0.022326, 0.036926]$ and $[1.460496, 2.913823]$. The profiles of the log-likelihood function of α,

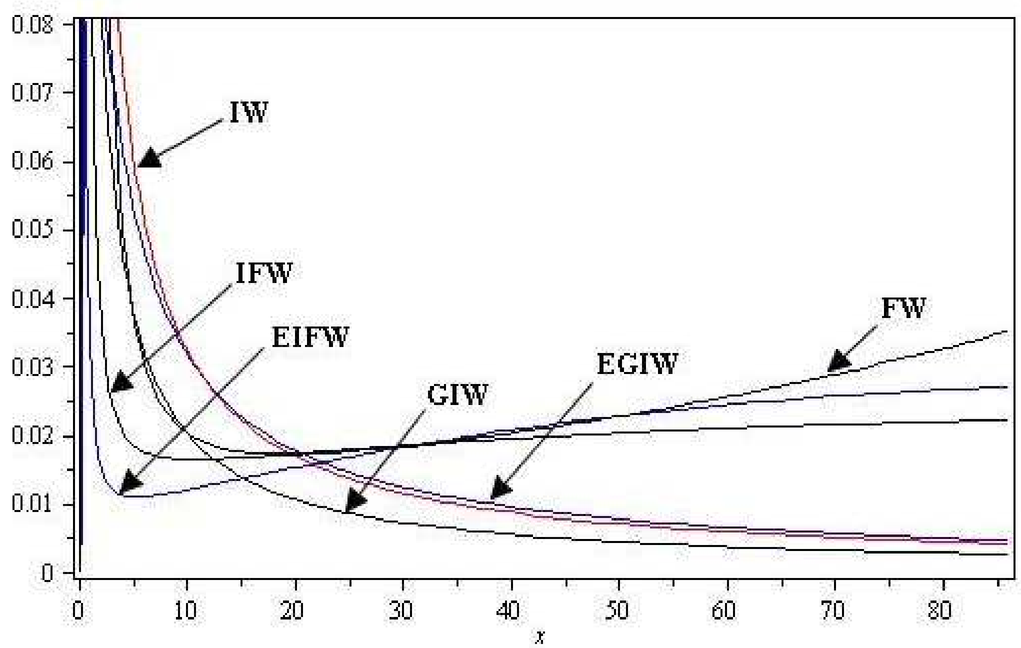

Fig. 4: The fitted hazard functions of selected models for Aarset data.

β and λ for Aarset data are ploted in Figure 5, Figure 6 and Figure 7 respectively. From the plots of the profiles of the log-likelihood function of α, β and λ, we show that the likelihood equations have a unique solution.

Example 6.2. Table 4, gives the data set corresponding to remission times (in months) of 128 bladder cancer patients reported in Lee and Wang (2003)[11]. The fitted survival and failure rate functions are shown in Figure 8 and Figure 9 respectively. From Figure 8 we can observed that, the $EIFW$ distribution fits the data set better than all other distributions considered here, because its fitted curve is closer to the empirical curve. The MLEs of the unknown parameters and the K-S test statistic with its corresponding p-value for the six tested models are given in Table 5. The K-S test statistic value for $EIFW$ model and the corresponding p-value are 0.179 and 4.312×10^{-4} respectively. In fact, based on the values of the K-S test statistic and its corresponding p-values , we can deduce that: (i) the IW, IFW, $EGIW$, GIW and FW distributions

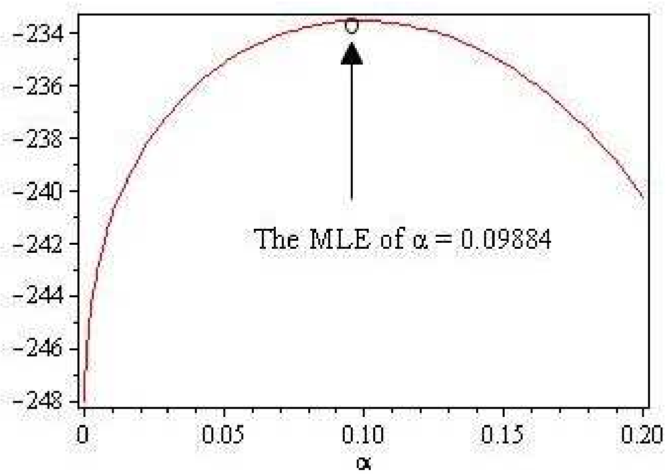

Fig. 5: The profile of the log-likelihood function ofα for Aarset data.

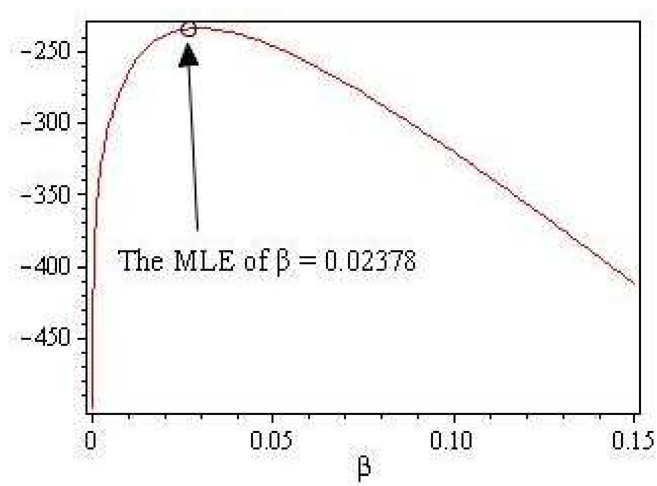

Fig. 6: The profile of the log-likelihood function of β for Aarset data.

are rejected at any level of significance $\delta \geq 5.3\times10^{-\mathbf{13}}$, (ii) the $EIFW$ distribution is accepted at $\delta \leq 4.5\times10^{-\mathbf{4}}$ and (iii) the $EIFW$ model has the lowest K-S value and the highest p-value among all the models used here to fit the current data set, which means that the new model fits the data better than the FW, IW, GIW, $EGIW$ and IFW models. In fact, based on the values of the L, AIC, CAIC and BIC given in Table 6, we observe that the $EIFW$ distribution has the lowest L, AIC, CAIC and BIC values. Therefore, the $EIFW$ model is the best fit for these data among all the models used here.

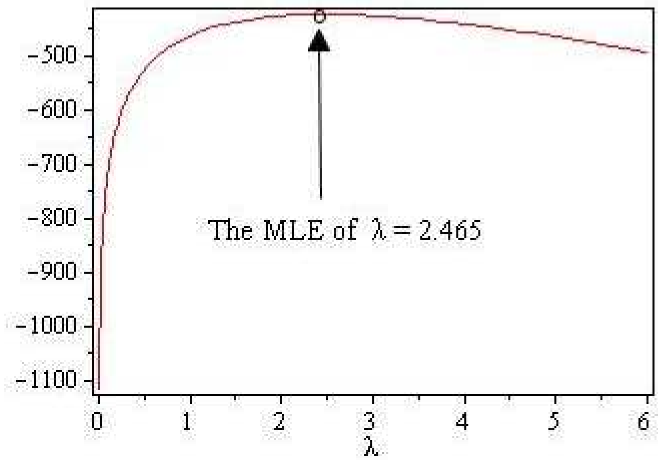

Fig. 7: The profile of the log-likelihood function of λ for Aarset data.

Table 4.
Remission times of 128 bladder cancer patients., see Lee and Wang (2003)[11].

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97
9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64
5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34
14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26
2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	13.31	4.51	6.54	8.53	12.03	20.28
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

Table 5.
The MLEs, K-S and p-values for Lee and Wang data.

The model	MLE of the parameters	K-S value	p-value
FW(α,β)	$\hat{\alpha}=0.0535$, $\hat{\beta}=0.915$	0.390	1.12×10^{-17}
IW	$\hat{\alpha}=16.14$, $\hat{\beta}=0.464$	0.503	4.02×10^{-29}
GIW	$\hat{\alpha}=0.75$, $\hat{\beta}=0.34$, $\hat{\theta}=1.79$	0.369	7.20×10^{-16}
EGIW	$\hat{\alpha}=1.006$, $\hat{\beta}=0.5$, $\hat{\theta}=1.05$, $\hat{\lambda}=2$	0.608	2.36×10^{-42}
IFW	$\hat{\alpha}=0.126$, $\hat{\beta}=0.143$	0.333	$5.294\times10^{\mathbf{-13}}$
EIFW	$\hat{\alpha}=0.0802$, $\hat{\beta}=0.1697$, $\hat{\lambda}=2.465$	**0.179**	$\mathbf{4.312\times10^{-4}}$

Table 6.
The log-likelihood, AIC, CAIC and BIC values for Lee and Wang data.

The model	L	AIC	CAIC	BIC
FW	−525.53	1055.07	1055.16	1060.77
IW	−500.12	1004.25	1004.33	1009.94
GIW	−495.18	996.362	996.56	1004.92
EGIW	−488.05	984.09	984.42	995.5
IFW	−453.61	911.22	911.31	916.92
EIFW	**−423.46**	**852.909**	**853.104**	**861.47**

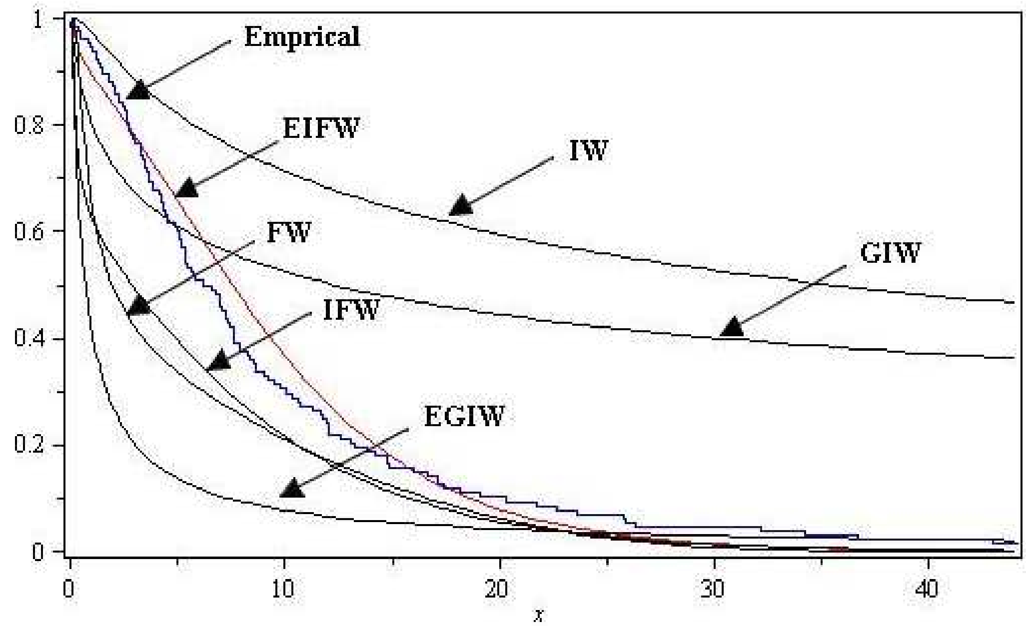

Fig. 8: The empirical and fitted survival functions of selected models for Lee and Wang data.

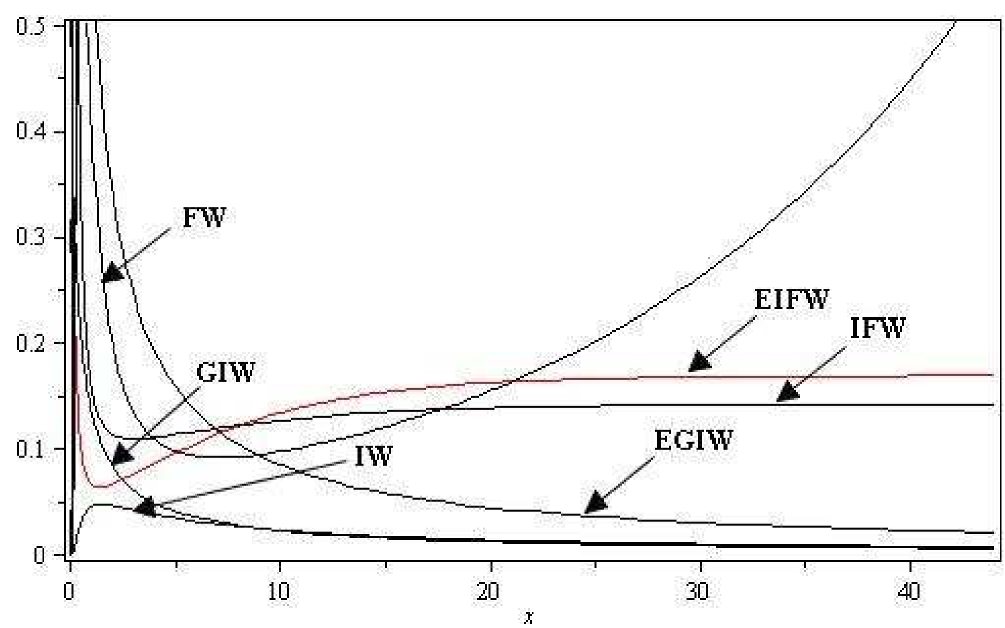

Fig. 9: The fitted hazard functions of selected models for Lee and Wang data data.

Substituting the MLEs of the unknown parameters into (26), we get estimation of the variance covariance matrix as the following

$$I_0^{-1} = \begin{pmatrix} 0.0005871 & -0.0001094 & -0.003909 \\ -0.0001094 & 0.0001639 & 0.0023999 \\ -0.003909 & 0.0023999 & 0.0929777 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters α, β and λ are given respectively as $[0.07840, 0.1734]$, $[0.11789, 0.16807]$ and $[1.8675, 3.06279]$.

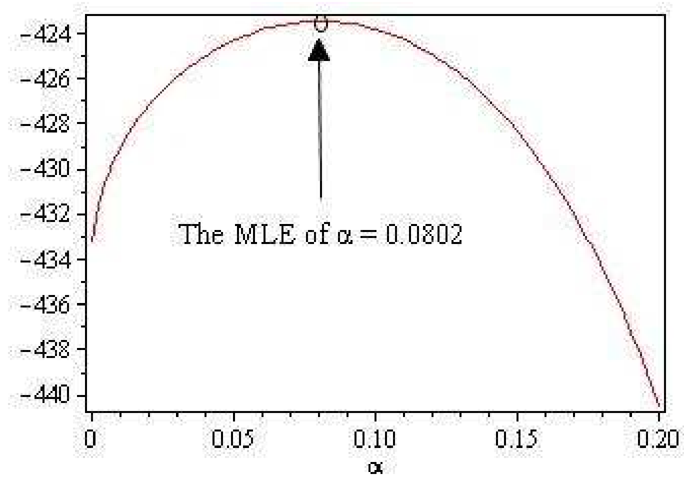

Fig. 10: The profile of the log-likelihood function ofα for Lee and Wang data.

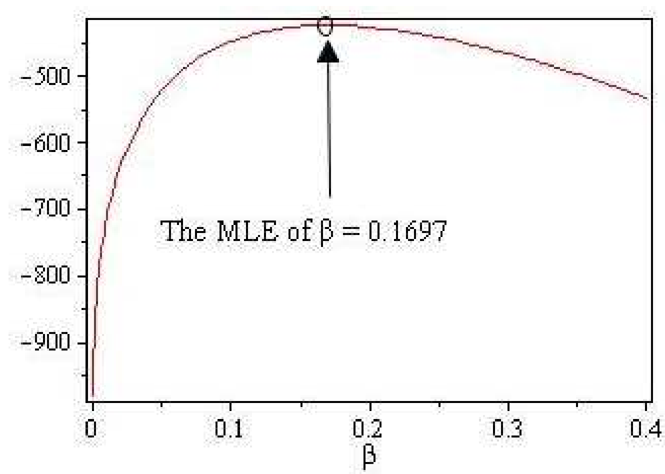

Fig. 11: The profile of the log-likelihood function ofβ for Lee and Wang data.

To show that the likelihood equations have a unique solution, we plot the profiles of the log-likelihood function of α, β and λ for Lee and Wang data. in Figure 10, Figure 11 and Figure 12 respectively.

6 Conclusions

In this paper, we propose a new three parameter model we called it the exponentiated inverse flexible Weibull extension distribution. Some statistical properties of this distribution have been derived and discussed. The quantile, median, mode and the moments of $EIFW$ are derived in closed forms. The maximum likelihood estimators of the parameters are derived and we obtained the observed Fisher information matrix. Two real data sets are analyzed using

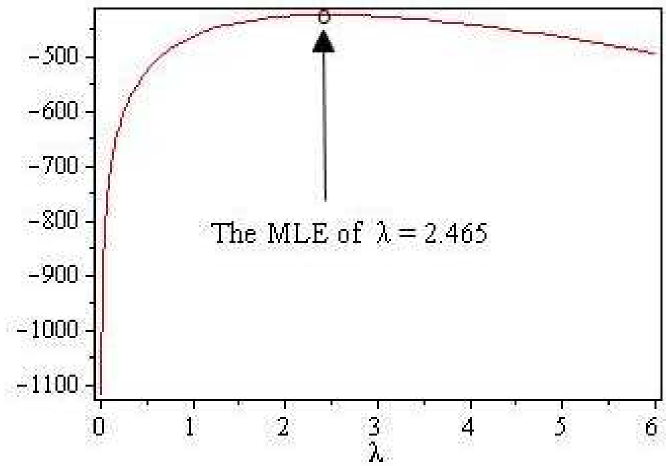

Fig. 12: The profile of the log-likelihood function of λ for Lee and Wang data.

the new distribution and it is compared with flexible Weibull**,** inverse flexible Weibull**,** inverse Weibull, generalized inverse Weibull and exponentiated generalized inverse Weibull distributions. It is evident from the comparisons that the new distribution is the best distribution for fitting these data sets compared to other distributions considered here.

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