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Fractional Integral Associated to Fractional Derivatives with Nonsingular Kernels

Jorge Losada[1] *and Juan J. Nieto*[2,*]

[1] Departamento de Matemática Aplicada a la Ingeniería Industrial, E.T.S. Ingenieros Industriales, Universidad Politécnica de Madrid, Spain

[2] Departamento de Estatística, Análise Matemática e Optimización, Instituto de Matemáticas, Universidade de Santiago de Compostela, Spain

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Abstract: This paper presents the fractional integral in the Caputo-Fabrizio sense. For the first order integral and the classical Riemann-Liouville fractional integral, the kernel of the corresponding operator is trivial, but it is not the case for the Caputo-Fabrizio integral operator. This is one of the misleading points in relation to this fractional derivative with nonsingular kernel and we present it here in a crystalline form.

Keywords: Fractional calculus, fractional derivative, fractional integral, nonsingular kernel.

1 Introduction

There are many generalizations of the classical integer-order derivatives and integrals.

For a function $f:\mathbb{R}\to\mathbb{R}$, the primitive or first order integral is $F(t)=[I^1 f](t)=\int_0^t f(s)ds$.

Integrating $n=1,2,3\ldots$ repeatedly and using the classical formula of repeated integration due to Cauchy:

$$I^n f(t)=I\cdot I^{n-1}f(t)=\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}f(s)ds$$

leading to the obvious definition of fractional integral of order $\alpha>0$ [1,2,3]

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds. \tag{1}$$

This is valid also for $\alpha\in\mathbb{C}$ with $Re(\alpha)>0$.

To define a fractional derivative, several ideas were given by great mathematicians, such as Euler [4], Laplace [5], Lacroix [6], Fourier [7], Liouville [8] and Riemann [9]. The derivative in the sense of Liouville and Riemann of fractional order $\beta\in(0,1)$ is set, as follows:

$$D^\beta\varphi(t)=\frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\int_0^t (t-s)^{-\beta}\varphi(s)ds \tag{2}$$

suggested by the relation

$$D^\beta f=[D^1\cdot I^{1-\beta}]f=\frac{d}{dt}I^{1-\beta}f.$$

If one uses the relation $D^\alpha=I^{1-\alpha}\cdot D^1$ to introduce the fractional derivative, then the fractional derivative in the sense of Caputo [1,10] is given by

* Corresponding author e-mail: juanjose.nieto.roig@usc.es

$$ {}^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{-\alpha}f'(s)ds. \tag{3}$$

For $a\le 0$ and $f:[a,\infty)\to\mathbb{R}$ a smooth function, Caputo and Fabrizio have recently defined the following integral trasnform or fractional derivative with exponential kernel [11]

$$ {}^{CF}D^{\beta}\varphi(t)=\frac{1}{1-\beta}\int_a^t e^{-\frac{\beta}{1-\beta}(t-s)}\varphi'(s)ds,\quad t\ge 0. \tag{4}$$

We point out that $a\in[-\infty,0]$. As indicated in a recent paper [12], for $a=0$, one obtains a classical operator with fading memory, but for $a<0$, one has

$$\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)}f'(s)ds=\int_a^0 e^{-\frac{\alpha}{1-\alpha}(t-s)}f'(s)ds+\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}f'(s)ds \tag{5}$$

and the last term

$$\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}f'(s)ds \tag{6}$$

is considered in some papers as the Caputo-Fabrizio fractional derivative. There is a substantial difference between definition (5) and (6) as developed in [12].

For some real applications of this operator, we refer the reader, for example, to [13,14,15,16].

In (5) we have a nonsingular kernel (the exponential $e^{-\frac{\alpha}{1-\alpha}(t-s)}, s\in[0,t]$ is always positive and continuous and so bounded). However, according to formula (6) of [12], we have

$$ {}^{CF}D^{\alpha}f(t)=\frac{1}{1-\alpha}\int_0^t\left[\frac{g_0'(s)}{f'(s)}\delta(t-s)+e^{-\frac{\alpha}{1-\alpha}(t-s)}\right]f'(s)ds,\quad t\ge 0$$

with

$$g_0'(t)=\int_a^0 e^{-\frac{\alpha}{1-\alpha}(t-s)}f'(s)$$

so a singular kernel $\delta(t-s)$ appears for $a<0$.

In the classical situation

$$D^1I^1f=f$$

and

$$I^1D^1f=f+c$$

c is an arbitrary real constant. Note that the subspace of constant functions is precisely the kernel of the operator D^1. Also, note that the kernel of the operator I^1 is trivial.

In the fractional case, we have an analogous situation depending on the type of fractional calculus used. For the Caputo fractional derivative and for $\alpha\in(0,1)$, it holds

$$ {}^{C}D^{\alpha}I^{\alpha}f=f$$

and

$$I^{\alpha}\,{}^{C}D^{\alpha}f=f+c$$

c is an arbitrary real constant. This is due to the fact that the Caputo fractional derivative of a constant is zero.

However, the Riemann-Liouville fractional derivative of a constant different from zero is not zero. Indeed, if $f(t)=c$, then

$$D^{\alpha}f(t)=\frac{c}{\Gamma(1-\alpha)}t^{-\alpha},$$

which is different from zero unless $c=0$.

Thus, for $\alpha \in (0,1)$

$$D^{\alpha} I^{\alpha} f(t) = f(t)$$

but

$$I^{\alpha} D^{\alpha} f(t) = f(t) + ct^{\alpha-1},$$

c is an arbitray real constant.

We see that $I^{\alpha} D^{\alpha} f$ coincides with f up to a term given by the function $t^{\alpha-1}$. For $\alpha = 1$, $I^1 D^1 f = f$ up to a constant since $t^{\alpha-1} = 1$.

Therefore, the general solution of the fractional differential equation

$$D^{\alpha} u = 0$$

is

$$u(t) = c \cdot t^{\alpha-1}, c \in \mathbb{R}.$$

In this note, we solve the question of what is the fractional integral corresponding to the Caputo-Fabrizio fractional derivative. Note that the classical fractional derivatives are introduced using integer derivatives and the original definition of fractional integral. Here, we start from definition (4) of a fractional derivative and will construct the associate fractional integral and the properties of the corresponding fractional operators.

We recall that ${}^{C}D^{\alpha} I^{\alpha} f = f$ and $I^{\alpha}\, {}^{C}D^{\alpha} f = f$ up to a constant times a function in the kernel of the fractional derivative operator. On the other hand, if we have that ${}^{CF}D^{\alpha} f = g$, then the fractional integral in the Caputo-Fabrizio sense, denoted by ${}^{CF}I^{\alpha} g$, should be the function f up to a multiple of a function and this is the case as we will show below. However, the fractional derivative of the fractional integral in the Caputo-Fabrizio sense of a given function is not, in general, the same function. This is a totally different aspect of this fractional calculus that has to be considered when solving fractional differential equations.

2 Kernel of the Caputo-Fabrizio fractional derivative

Consider the simple Caputo-Fabrizio fractional differential equation

$${}^{CF}D^{\alpha} f = 0. \tag{7}$$

We will see that f is constant. Indeed,

$$\frac{1}{1-\alpha} e^{-\frac{\alpha}{1-\alpha}t} \int_0^t e^{-\frac{\alpha}{1-\alpha}s} f'(s)ds = 0, \quad t \geq a,$$

and taking the derivative:

$$\frac{1}{1-\alpha} \cdot \frac{-\alpha}{1-\alpha} e^{-\frac{\alpha}{1-\alpha}t} \int_0^t e^{-\frac{\alpha}{1-\alpha}s} f'(s)ds + \frac{1}{1-\alpha} f'(s) = 0,$$

that is,

$$\frac{-\alpha}{1-\alpha}\, {}^{CF}D^{\alpha} f(t) + \frac{1}{1-\alpha} f'(t) = 0.$$

Therefore,

$$f'(t) = 0, \quad t \geq a,$$

and f is constant.

The solutions of (7) are only the constant functions or, in other words, the kernel of the Caputo-Fabrizio fractional derivative operator is the one-dimensional subspace of constant functions.

3 Caputo-Fabrizio fractional integral

Now, let us study the non-homogeneus equation associated to (7):

$$^{CF}D^{\alpha}f = g.$$

First of all, if we have two solutions f_1, f_2, then $^{CF}D^{\alpha}(f_1 - f_2) = 0$ and we conclude that $f_1 - f_2 = c$, i.e., f_1, f_2 coincide up to a constant.

Note that

$$\frac{1}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}t}\int_0^t e^{-\frac{\alpha}{1-\alpha}s}f'(s)ds = g(t), \quad t \geq a,$$

and then

$$\frac{d}{dt}\,^{CF}D^{\alpha}f(t) = \frac{-\alpha}{1-\alpha}\,^{CF}D^{\alpha}f(t) + \frac{1}{1-\alpha}f'(t) = g'(t).$$

This last relation is equation (5) of [17] with $a = 0$ to simplify expressions where the authors adopted a different perspective based on Laplace transform and Bode diagrams. It is standard in linear systems and describes the input/output of a highpass filter [18].

$$\frac{-\alpha}{1-\alpha}g(t) + \frac{1}{1-\alpha}f'(t) = g'(t).$$

Integrating on the interval $[a,t]$, we have

$$f(t) = (1-\alpha)[g(t) - g(a)] + \alpha\int_0^t g(s)ds + f(a), \quad t \geq a,$$

where $g(a) = \,^{CF}D^{\alpha}f = 0$ by the definition (4).

On the other hand, for $t \geq 0$, integrating on $[0,t]$, we obtain

$$f(t) = (1-\alpha)[g(t) - g(0)] + \alpha\int_0^t g(s)ds + f(0) \tag{8}$$

and now

$$g(0) = \int_a^0 e^{-\frac{\alpha}{1-\alpha}s}f'(s)ds.$$

In general, $g(0) \neq 0$ as it is incorrectly derived, in some cases, from [19].

Setting $g(0) = g_0, f(0) = f_0$ and using (8), we have

$$f = (1-\alpha)(g - g_0) + \alpha I^1 g + f_0$$

or

$$f - f_0 = (1-\alpha)(g - g_0) + \alpha I^1 g.$$

Therefore, the Caputo-Fabrizio fractional integral of order $\alpha \in (0,1)$ of the function g is $(1-\alpha)(g-g_0) + \alpha I^1 g$ up to a constant. Finally, we have derived the Caputo-Fabrizio fractional integral operator of order $\alpha \in (0,1)$. For a function $g : [a,\infty) \to \mathbb{R}$ smooth, the Caputo- Fabrizio integral of order $\alpha \in (0,1)$ is defined as

$$^{CF}I^{\alpha}g(t) = (1-\alpha)[g(t) - g(0)] + \alpha\int_0^t g(s)ds$$

with $g_0 = \int_a^0 e^{-\frac{\alpha}{1-\alpha}s}f'(s)ds$. Of course, if $a = 0$, then $g_0 = 0$, and $^{CF}I^{\alpha}g(t) = (1-\alpha)g(t) + \alpha\int_0^t g(s)ds$.

4 Kernel of the Caputo-Fabrizio fractional integral

Now, consider the integral equation

$$^{CF}I^{\alpha}g = 0.$$

Therefore,

$$(1-\alpha)[g(t)-g(a)]+\alpha\int_a^t g(s)ds = 0,$$

and $g'(t) = \frac{-\alpha}{1-\alpha}g(t)$, that is,

$$g(t) = g(a)e^{-\frac{\alpha}{1-\alpha}(t-a)}.$$

Let

$$c_\alpha(t) = e^{-\frac{\alpha}{1-\alpha}(t-a)}.$$

Note that fot $t > a$, $\lim_{\alpha\to 1^-} c_\alpha(t) = 0$, but $c_\alpha(a) = 1$.

The kernel of the operator $^{CF}I^{\alpha}$ is the one-dimensional subspace generated by c_α. This is quite different to the case of the usual fractional where $I^\alpha g = 0$ implies that $g = 0$.

We will now show that

$$^{CF}D^{\alpha}\,.\,^{CF}I^{\alpha}f = f - f(a)c_\alpha,$$

i.e., the fractional derivative of the fractional integral of a function is the same function up to a multiple of c_α. This is new and it is not noted in many papers.

Indeed, using that $^{CF}I^{\alpha}f(t) = (1-\alpha)[f(t)-f(0)]+\alpha\int_0^t f(s)ds$, we get

$$\frac{d}{dt}\,^{CF}I^{\alpha}f(t) = (1-\alpha)f'(t)+\alpha f(t)$$

and

$$^{CF}D^{\beta}\,.\,^{CF}I^{\beta}\varphi(t) = \frac{1}{1-\beta}\int_a^t e^{-\frac{\beta}{1-\beta}(t-s)}[(1-\beta)\varphi'(s)+\beta\varphi(s)]ds.$$

Integrating by parts the term corresponding to f', we arrive at

$$^{CF}D^{\alpha}\,.\,^{CF}I^{\alpha}f(t) = f(t)-f(a)e^{-\frac{\alpha}{1-\alpha}(t-a)} = f(t)-f(a)c_\alpha(t),$$

being $c_\alpha(t) = e^{-\frac{\alpha}{1-\alpha}(t-a)}$. To compute the value of $^{CF}I^{\alpha}\,^{CF}D^{\alpha}f$, note that

$$\frac{d}{dt}\,^{CF}D^{\alpha}f(t) = \frac{-\alpha}{1-\alpha}\,^{CF}D^{\alpha}f(t)+\frac{1}{1-\alpha}f'(t)$$

and

$$^{CF}I^{\alpha}\,^{CF}D^{\alpha}f(t) = (1-\alpha)[\,^{CF}D^{\alpha}f(t)-f_0]+\alpha\int_0^t\,^{CF}D^{\alpha}f(s)ds.$$

Taking derivatives,

$$\frac{d}{dt}\,^{CF}I^{\alpha}\,^{CF}D^{\alpha}f(t) = (1-\alpha)\frac{d}{dt}\,^{CF}D^{\alpha}f(t)+\alpha\,^{CF}D^{\alpha}f(t) = f'(t).$$

Therefore,

$$^{CF}I^{\alpha}\,.\,^{CF}D^{\alpha}f(t) = f(t)+c,$$

with c an arbitrary real constant.

These relations will be useful to solve fractional ordinary and partial differential equations.

5 Conclusion

The behavior of the classical fractional derivatives (Riemann-Liouville and Caputo) is similar to the classical integer order derivatives in the sense that they satisfy that the derivative of the integral is the same function. However, for the new Caputo-Fabrizio derivative, we have

$$^{CF}D^{\alpha} \cdot {}^{CF}I^{\alpha} f = f - f(a)c_{\alpha}, \qquad c_{\alpha}(t) = e^{-\frac{\alpha}{1-\alpha}(t-a)}.$$

Also, the kernel of the Riemann-Liouville integral operator is trivial. However, for the Caputo-Fabrizio integral, the kernel is not trivial:

$$\ker\left({}^{CF}I^{\alpha} \right) = < c_{\alpha} > .$$

This aspect has to be considered.

As a summary, we include the following table

Operator	Kernel	Subspace	Generator
D^1	constant	$< c >$	1
I^1	0	$\{0\}$	0
I^{α}	0	$\{0\}$	0
D^{α}	constant$\cdot t^{\alpha-1}$	$< c_{\alpha} >$	$c_{\alpha}(t) = t^{\alpha-1}$
${}^{C}D^{\alpha}$	constant	$< c >$	1
${}^{CF}I^{\alpha}$	constant $\cdot e^{-\frac{\alpha}{1-\alpha}(t-a)}$	$< c_{\alpha} >$	$c_{\alpha}(t) = e^{-\frac{\alpha}{1-\alpha}(t-a)}$
${}^{CF}D^{\alpha}$	constant	$< c >$	1

To conclude, the kernel of the Caputo-Fabrizio and Losada-Nieto fractional integral is not trivial.

Conflict of Interest

The authors declare that they have no conflict of interest.

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