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Bayesian Analysis of Generalized Gamma Distribution

Saima Naqash[1], S.P. Ahmad[1,] and A. Ahmed[2]*

[1]Department of Statistics, University of Kashmir, Srinagar, India.
[2]Department of Statistics and O.R., Aligarh Muslim University, Aligarh, India.

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Abstract: In this paper, we propose to obtain Bayesian estimators of unknown parameters of a three parameter generalized gamma distribution, based on different priors using different loss functions. These methods are compared by using mean square error for real life data as well as simulation study with varying sample sizes in R software.

Keywords: : Generalized gamma distribution, Bayesian estimators, loss function, Inverse-Gamma prior and R-software.

1. Introduction:

The generalized gamma (GG) distribution is a flexible distribution in the varieties of shapes and hazard functions for modelling duration. The GG family has exponential, gamma, and Weibull as subfamilies, and lognormal as a limiting distribution. It was introduced by Stacy (1962) who proposed a new generalized gamma model and gave its characteristics and applications to life testing. Stacy and Mihram (1965) derived estimators of the generalized gamma distribution. Upadhyay et. al. (2000) used Monte Carlo simulation technique for the Bayesian computation in life testing and reliability models. Pandey and Rao (2006) derived Bayes estimators of the scale parameter of generalized gamma distribution by taking quasi, inverted gamma and uniform prior distributions using precautionary loss function. Shukla and Kumar (2008) derived the Bayes estimators of the scale parameter of a generalized gamma type model for different priors. Ahmad et. al. (2011) also discussed the normal and Laplace methods of approximation for the posterior density of Gamma distribution. Reshi et. al. (2014) considered a new class of size-biased generalized gamma distribution. The estimation of the parameters of this model was obtained by employing the Bayesian method of estimation.
Let $x_{1,} x_2,...,x_n$ be independently and identically distributed Generalized Gamma (GG) random variables with α as scale parameter and β and p as shape parameters, then the P.D.F of x is:

$$f(x,\alpha\,\beta,p)=\frac{p}{\alpha^{\beta}\Gamma(\beta)}\,x^{\beta p-1}\exp(-\frac{x^{p}}{\alpha}), \qquad x>\text{o};\alpha>0,\beta>0,\text{p}>0 \tag{1.1}$$

For $p=1$, the GG distribution reduces to two- parameter gamma distribution.
The C.D.F is given by

$$F(x;\alpha,\beta,p)=1-\frac{\Gamma\left(\beta,\frac{x^{p}}{\alpha}\right)}{\Gamma(\beta)} \tag{1.2}$$

where $\Gamma\left(\beta,\frac{x^{p}}{\alpha}\right)$ is an incomplete upper gamma function and is equal to $\Gamma(\beta)-\int_{\frac{x^{p}}{\alpha}}^{\infty}e^{-y}\,y^{\beta-1}\,dy$.

2. Maximum Likelihood Estimation:

We assume that $X=(x_1,x_2,...,x_n)$ is a random sample from GE distribution. The likelihood function of α for the given sample observation is:

*Corresponding author e-mail: sprvz@yahoo.com

$$L(\alpha,x)=\left[\frac{p}{\Gamma(\beta)}\right]^n\frac{1}{\alpha^{n\beta}}\prod_{i=1}^{n}x_i^{\beta p-1}\exp\left(-\frac{1}{\alpha}\sum_{i=1}^{n}x_i^{\ p}\right) \qquad (2.1)$$

The MLE of α (keeping β and p as fixed) is given by solving $\frac{\partial}{\partial\alpha}\log L=0$ which gives

$$\hat{\alpha}=\frac{\sum_{i=1}^{n}x_i^{\ p}}{n\beta}=\frac{T}{n\beta} \qquad ,\text{where } T=\sum_{i=1}^{n}x_i^{\ p} \qquad (2.2)$$

3. Bayesian Method of Estimation:

Using the Bayesian method of estimation, we will use two non-informative prior distributions, viz. Jeffrey's prior, extension of Jeffrey's prior; and its conjugate prior called Inverse-Gamma prior distribution and obtain the posterior distributions thereof.

3.1. Jeffrey’s prior information:

Consider that the parameter α has the non-informative Jeffrey’s prior and is given by:

$$g(\alpha)\propto\sqrt{\det(I(\alpha))},\quad \alpha>0$$

where $I(\alpha)$ is the Fisher Information Matrix given by

$$I(\alpha)=-nE\left[\frac{\partial^2}{\partial\alpha^2}\log f(x,\alpha)\right]=\frac{n\beta}{\alpha^2}$$

and Jeffrey’s prior distribution becomes

$$g(\alpha)\propto\frac{1}{\alpha} \qquad (3.1.1)$$

The posterior distribution is given by

$$\pi(\alpha|x)\propto g(\alpha)L(\alpha)$$

$$\Rightarrow\pi(\alpha|x)\propto\frac{1}{\alpha}\left[\frac{p}{\Gamma(\beta)}\right]^n\frac{1}{\alpha^{n\beta}}\prod_{i=1}^{n}x_i^{\beta p-1}\exp\left(-\frac{T}{\alpha}\right)$$

$$=k\left[\frac{p}{\Gamma(\beta)}\right]^n\frac{1}{\alpha^{n\beta+1}}\prod_{i=1}^{n}x_i^{\beta p-1}\exp\left(-\frac{T}{\alpha}\right)$$

The constant ‘ k ’ is determined such that

$$\int_0^\infty\pi(\alpha\,|\,x)d\alpha=1 \qquad \Rightarrow k^{-1}=\frac{\Gamma(n\beta)}{T^{n\beta}}\left[\frac{p}{\Gamma(\beta)}\right]^n\prod_{i=1}^{n}x_i^{\beta p-1}$$

Hence the posterior distribution of α becomes

$$\pi(\alpha\,|\,x)=\frac{T^{n\beta}}{\Gamma(n\beta)}\frac{1}{\alpha^{\beta n+1}}\exp\left(-\frac{T}{\alpha}\right) \qquad (3.1.2)$$

which is the P. D. F. of Inverse-Gamma distribution with parameters $n\beta$ and T, where $T=\sum_{i=1}^{n}x_i^{\ p}$ i.e.,

$$\alpha \mid x \sim IG(n\beta, \sum_{i=1}^{n} x_i^{\,p})$$

The expected value (mean) and variance of the posterior distribution is given by

$$E(\alpha|x) = \frac{T}{n\beta - 1} \tag{3.1.3}$$

and $$V(\alpha|x) = \frac{T^2}{(n\beta-1)^2\,(n\beta-2)}, \qquad \text{where } T = \sum_{i=1}^{n} x_i^{\,p}$$

3.1.1 Bayes estimator under jeffrey's prior using different loss functions

a) Squared Error Loss Function (SELF):

We consider the following SELF:

$$l(\hat{\alpha},\alpha) = c\,(\hat{\alpha}-\alpha)^2$$

and obtain the Risk function as:

$$R(\hat{\alpha},\alpha) = \int_0^\infty l(\hat{\alpha},\alpha)\,\pi(\alpha|x)d\alpha = c\frac{T^{\beta n}}{\Gamma(\beta n)}\int_0^\infty (\hat{\alpha}-\alpha)^2\,\frac{1}{\alpha^{\beta n+1}}\exp(-\frac{T}{\alpha})\,d\alpha$$

$$= c\left[\hat{\alpha}^2 - 2\hat{\alpha}\frac{T}{\beta n - 1} + \frac{T^2}{(\beta n-1)(\beta n-2)}\right], \qquad T = \sum_{i=1}^{n} x_i^{\,p}$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}R(\hat{\alpha},\alpha) = 0$ will give the Bayes estimator:

$$\hat{\alpha}_{B_1} = \frac{T}{n\beta-1}, \quad T = \sum_{i=1}^{n} x_i^{\,p} \tag{3.1.4}$$

b) Al-Bayyati's Loss Function:

Al-Bayyati's loss function is of the form:

$$l(\hat{\alpha},\alpha) = \alpha^{c_2}(\hat{\alpha}-\alpha)^2, c_2 \in R^+$$

This loss function is used to obtain the estimator of the parameter of GE distribution. The risk function is obtained as:

$$R(\hat{\alpha},\alpha) = \int_0^\infty l(\hat{\alpha},\alpha)\,\pi(\alpha|x)d\alpha = \int_0^\infty \alpha^{c_2}(\hat{\alpha}-\alpha)^2\,\frac{T^{n\beta}}{\Gamma(n\beta)}\frac{1}{\alpha^{n\beta+1}}\exp\left(-\frac{T}{\alpha}\right)d\alpha$$

$$= \frac{T^{c_2}}{\Gamma(n\beta)}\left[\hat{\alpha}^2\Gamma(n\beta-c_2) - 2\hat{\alpha}T\,\Gamma(n\beta-c_2-1) + T^2\,\Gamma(n\beta-c_2-2)\right]$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}R(\hat{\alpha},\alpha) = 0$, we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_3} = \frac{T}{n\beta - c_2 - 1} \tag{3.1.5}$$

where $T = \sum_{i=1}^{n} x_i^{\,p}$

Remark 1:

For $c_2 = 0$ in (3.1.5), we get $\hat{\alpha} = \frac{T}{n\beta-1}$ which gives the Bayes estimator under SELF using Jeffrey's prior.

c) Linex Loss Function:

The Linex Loss function is defined as:

$l(\hat{\alpha},\alpha)=\exp(a\sigma)-a\sigma-1,\quad \sigma=\left(\dfrac{\hat{\alpha}}{\alpha}-1\right), a\neq 0$ we obtain the Risk function as:

$$R(\hat{\alpha},\alpha)=\int_0^\infty l(\hat{\alpha},\alpha)\,\pi(\alpha|x)d\alpha$$

$$=\int_0^\infty[\exp(a\sigma)-a\sigma-1]\frac{T^{n\beta}}{\Gamma(n\beta)}\frac{1}{\alpha^{n\beta+1}}\exp\left(-\frac{T}{\alpha}\right)d\alpha$$

$$=\frac{T^{n\beta}}{n\beta}\left[\int_0^\infty \exp(a\sigma)\,\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+1}}d\alpha-a\int_0^\infty \sigma\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+1}}d\alpha-\int_0^\infty\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+1}}d\alpha\right]$$

$$=\frac{T^{n\beta}}{n\beta}\left[e^{-a}\int_0^\infty\frac{e^{-\frac{(T-a\hat{\alpha})}{\alpha}}}{\alpha^{n\beta+1}}d\alpha-a\hat{\alpha}\int_0^\infty\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2}}d\alpha+(a-1)\int_0^\infty\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+1}}d\alpha\right]$$

$$=e^{-a}\left(1-\frac{a\hat{\alpha}}{T}\right)^{-n\beta}-a\frac{n\beta}{T}\hat{\alpha}+(a-1)$$

Solving the equation $\dfrac{\partial}{\partial\hat{\alpha}}R(\hat{\alpha},\alpha)=0,$ we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_4}=\frac{T}{a}\left\{1-e^{\frac{-a}{n\beta+1}}\right\} \tag{3.1.6}$$

where $T=\sum_{i=1}^{n}x_i^{\,p}$

3.2 New extension of Jeffrey's prior information:

The new extension of Jeffrey's prior information is given by:

$$g(\alpha)\propto[I(\alpha)]^{c_1}, c_1\in R^+$$

$$\Rightarrow\qquad g(\alpha)\propto\left[\frac{n}{\alpha^2}\right]^{c_1}=\frac{n^{c_1}}{\alpha^{2c_1}}$$

$$\Rightarrow\qquad g(\alpha)\propto\frac{1}{\alpha^{2c_1}} \tag{3.2.1}$$

The posterior distribution is obtained in a similar way as in case of Jeffrey's prior information and is given by

$$\pi_1(\alpha|x)=\frac{T^{n\beta+2c_1-1}}{\Gamma(n\beta+2c_1-1)}\frac{1}{\alpha^{n\beta+2c_1}}\exp\left(-\frac{T}{\alpha}\right) \tag{3.2.2}$$

where $T=\sum_{i=1}^{n}x_i^{\,p}$

which is the Inverse Gamma distribution with parameters $(\beta n+2c_1-1)$ and T, where $T=\sum_{i=1}^{n}x_i^{\,p}$, i.e.,

$$\alpha \sim IG\left(n\beta + 2c_1 - 1, \sum_{i=1}^{n} x_i^{\,p}\right)$$

The expected value (mean) and variance of the distribution is given by

$$E(\alpha|x) = \frac{T}{n\beta + 2c_1 - 2} \tag{3.3.3}$$

and $$V(\alpha|x) = \frac{T^2}{(n\beta + 2c_1 - 2)(n\beta + 2c_1 - 3)}, \quad \text{where } T = \sum_{i=1}^{n} x_i^{\,p}$$

Remark 2:

1. For $c_1 = \frac{1}{2}$ in (3.2.2), the posterior distribution under the extension of Jeffrey's prior reduces to the posterior distribution under the Jeffrey's prior.
2. For $c_1 = \frac{3}{2}$ in (3.2.2), the posterior distribution under the extension of Jeffrey's prior reduces to the posterior distribution under the Hartigan's prior.

3.2.1 Bayes estimator under the extension of Jeffrey's prior using different loss functions

a) Squared Error Loss Function (SELF):

The risk function under SELF is obtained as:

$$R(\hat{\alpha}, \alpha) = \int_0^\infty l(\hat{\alpha}, \alpha)\, \pi_1(\alpha|x)\, d\alpha$$

$$= c\frac{T^{n\beta+2c_1-1}}{\Gamma(n\beta + 2c_1 - 1)} \int_0^\infty (\hat{\alpha} - \alpha)^2 \frac{e^{-\frac{T}{\hat{\alpha}}}}{\alpha^{n\beta+2c_1}} d\alpha$$

$$= c\left[\hat{\alpha}^2 - 2\hat{\alpha}\frac{T}{n\beta + 2c_1 - 2} + \frac{T^2}{(n\beta + 2c_1 - 2)(n\beta + 2c_1 - 3)}\right], \qquad T = \sum_{i=1}^{n} x_i^{\,p}$$

Solving the equation $\frac{\partial}{\partial \hat{\alpha}} R(\hat{\alpha}, \alpha) = 0$ we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_5} = \frac{T}{n\beta + 2c_1 - 2} \tag{3.3.4}$$

where $T = \sum_{i=1}^{n} x_i^{\,p}$

Remark 3:

For $c_1 = \frac{1}{2}$ in (14), $\hat{\alpha} = \frac{T}{\beta n - 1}$ which gives the Jeffrey's estimator under SELF.

b) Al-Bayyati's Loss Function:

Al-Bayyati's loss function is of the form:

$$l(\hat{\alpha}, \alpha) = \alpha^{c_2}(\hat{\alpha} - \alpha)^2; c_2 \in R^+$$

The risk function is given by

$$R(\hat{\alpha},\alpha)=\int_0^\infty l(\hat{\alpha},\alpha)\,\pi_1(\alpha|x)d\alpha$$

$$=\frac{T^{n\beta+2c_1-1}}{\Gamma(n\beta+2c_1-1)}\int_0^\infty \alpha^{c_2}(\hat{\alpha}-\alpha)^2\frac{e^{-\frac{T}{\hat{\alpha}}}}{\alpha^{n\beta+2c_1}}d\alpha$$

$$=\frac{1}{T^{c_2}\Gamma(n\beta-2c_1+1)}\left[\hat{\alpha}^2\Gamma(n\beta+c_2-2c_1+1)-2\hat{\alpha}\frac{\Gamma(n\beta+c_2-2c_1+1)}{T}+\frac{\Gamma(n\beta+c_2-2c_1+3)}{T^2}\right]$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}R(\hat{\alpha},\alpha)=0$ we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_7}=\frac{T}{n\beta-c_2+2c_1-2} \tag{3.3.5}$$

where $T=\sum_{i=1}^{n}x_i^{\,p}$

Remark 4:

1. For $c_1=\frac{1}{2}$ in (3.3.5), $\hat{\alpha}=\frac{T}{n\beta-c_2-1}$ which gives the Bayes estimator using ABLF under Jeffrey's prior.
2. For $c_1=\frac{1}{2}$ & $c_2=0$ in (3.3.5), $\hat{\alpha}=\frac{T}{n\beta-1}$ which gives the Bayes estimator using SELF under Jeffrey's prior.

c) Linex Loss Function:

Using the Linex Loss function:

$$l(\hat{\alpha},\alpha)=\exp(a\sigma)-a\sigma-1,\quad \sigma=\left(\frac{\hat{\alpha}}{\alpha}-1\right),a\neq 0$$

we obtain the Risk function as:

$$R(\hat{\alpha},\alpha)=\int_0^\infty l(\hat{\alpha},\alpha)\,\pi_1(\alpha|x)d\alpha$$

$$=\int_0^\infty\left[\exp(a\sigma)-a\sigma-1\right]\frac{T^{n\beta+2c_1-1}}{\Gamma(n\beta+2c_1-1)}\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2c_1}}d\alpha$$

$$=\frac{T^{n\beta+2c_1-1}}{\Gamma(n\beta+2c_1-1)}\left[\int_0^\infty\exp(a\sigma)\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2c_1}}d\alpha-a\int_0^\infty\sigma\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2c_1}}d\alpha-\int_0^\infty\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2c_1}}d\alpha\right]$$

$$=\frac{T^{n\beta+2c_1-1}}{\Gamma(n\beta+2c_1-1)}\left[e^{-a}\int_0^\infty\frac{e^{-\frac{(T-a\hat{\alpha})}{\alpha}}}{\alpha^{n\beta+2c_1}}d\alpha-a\hat{\alpha}\int_0^\infty\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2c_1+1}}d\alpha+(a-1)\int_0^\infty\frac{e^{-\frac{T}{\alpha}}}{\alpha^{n\beta+2c_1}}d\alpha\right]$$

$$= e^{-a}\left(1-\frac{a\hat{\alpha}}{T}\right)^{-(n\beta+2c_1)} - a\frac{n\beta+2c_1-1}{T}\hat{\alpha}+(a-1)$$

Solving the equation $\frac{\partial}{\partial\hat{\alpha}}R(\hat{\alpha},\alpha)=0$ we get the Bayes estimator of α as:

$$\hat{\alpha}_{B_4} = \frac{T}{a}\left\{1-e^{\frac{-a}{n\beta+2c_1}}\right\} \tag{3.3.6}$$

where $T=\sum_{i=1}^{n} x_i^{\,p}$

Remark 5:

For $c_1=\frac{1}{2}$ in (3.3.6), $\alpha=\frac{T}{a}\left\{1-\left(\frac{a}{e^{-a}}\right)^{\frac{-1}{n\beta+1}}\right\}$ which gives the Bayes estimator under Linex Loss function using Jeffrey's prior.

4. Worked example:

In this section, we have studied the data from Lawless (2002).The observations involves the number of million revolutions between failures for each of 23 ball bearings, the individual bearings were inspected periodically to determine whether "failure" had occurred. Treating the failure times as continuous, the 23 failure times are:
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.02, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

The MLE of the above data works out to be $\hat{\alpha}_{ML}=16.4693$ and its variance is $Var(\hat{\alpha}_{ML})=0.0418$

The tables below provides the Bayes estimators under the three priors, viz. Jeffrey's prior, extension of Jeffrey's prior and Inverse-Gamma prior using three loss functions, viz. Squared Error loss function, Al-Bayyati's loss function and Linex loss function. We choose (β,p) as(0.5,0.5).

Table 1: Posterior Mean and Variance of $\hat{\alpha}$ under Jeffrey's prior using different loss functions:

Loss Functions			Posterior Mean	Posterior Variance
SL			18.0378	0.0418
AB	c_2	0.5	18.9397	0.0461
		-0.5	17.2179	0.0381
		1	19.9366	0.0510
		-1	16.4693	**0.0348**
LL	a	0.5	14.8527	**0.0283**
		-0.5	15.4589	0.0307

Table 2: Posterior Mean and Variance of $\hat{\alpha}$ under extension of Jeffrey's prior using different loss functions:

Loss Function	C_1, C_2, a	Posterior Mean	Posterior Variance
SL	C1=0.5	18.0378	0.0418
	C1=1.0	16.4693	**0.0348**
	C1=1.5	18.0378	0.0418
AB	C1=0.5, C2= 0.5	18.9397	0.0461
	C1=0.5,C2= -0.5	17.2179	0.0381
	C1=1.0,C2= 0.5	17.2179	0.0381
	C1=1.0, C2=-0.5	15.7831	**0.0320**
	C1=1.5,C2= 0.5	18.9397	0.0461
	C1=1.5,C2= 0.5	17.2179	0.0381
	C1=0.5, C2=1.0	19.9366	0.0510
	C1=0.5,C2= 1.0	16.4693	0.0348
	C1=1.0,C2= 1.0	18.0378	0.0418
	C1=1.0,C2= -1.0	15.1518	0.0295
	C1=1.5,C2= 1.0	19.9366	0.0510
	C1=1.5,C2= -1.0	16.4693	0.0348
LL	C1=0.5,a= 0.5	14.8527	0.0283
	C1=0.5, a= -0.5	15.4589	0.0307
	C1=1.0, a= 0.5	13.7728	**0.0244**
	C1=1.0 ,a= 0.5	14.2925	0.0262
	C1=1.5, a= -0.5	14.8527	0.0283
	C1=1.5, a= -0.5	15.4589	0.0307

Table 3: Posterior Mean and Variance under Inverse-Gamma prior using different loss functions:

Loss Function	Loss Parameter (a_1, b_1),(C_2,a)	Posterior Mean	Posterior Variance
SL	(0,0)	18.0378	0.0418
	(0,1)	16.4693	**0.0348**
	(1,0)	18.1331	0.0418
	(1,1)	16.5563	**0.0348**
AB	(0,0), (0.5, -)	18.9397	0.0461
	(0,0), (-0.5, -)	17.2179	0.0381
	(0,1), (0.5, -)	17.2179	0.0381
	(0,1), (-0.5, -)	15.7831	0.0320

	(1,0), (0.5, -)	19.0397	0.0461
	(1,0), (-0.5, -)	17.3088	0.0381
	(1,1), (0.5, -)	17.3088	0.0381
	(1,1), (-0.5, -)	15.8664	0.0320
	(0,0), (1, -)	19.9366	0.0510
	(0,0), (-1, -)	16.4693	0.0348
	(0,1), (1, -)	18.0378	0.0418
	(0,1), (1, -)	15.1518	**0.0295**
	(1,0), (1, -)	20.0418	0.0510
	(1,0), (-1, -)	16.5563	0.0348
	(1,1), (1, -)	18.1331	0.0418
	(1,1), (-1, -)	15.2318	**0.0295**
LL	(0,0), (-, 0.5)	14.8527	0.0283
	(0,0), (-, -0.5)	15.4589	0.0307
	(0,1), (-, 0.5)	13.7728	**0.0244**
	(0,1), (-, -0.5)	14.2925	0.0262
	(1,0), (-, 0.5)	14.9312	0.0283
	(1,0), (-, -0.5)	15.5405	0.0307
	(1,1), (-, 0.5)	13.8455	**0.0244**
	(1,1), (-, -0.5)	14.3679	0.0262

SL=squared error loss function, AB=Albayyti's loss function, ,LL=Linex loss function

5. Simulation study:

To compare the performance of the estimates, a simulation study was conducted using R-software. Samples of size 25, 50 and 100 were taken to generate the data set of GGD. The simulation study was iterated 5,000 times for each pairs of (α, β, p) where $(\alpha = 0.5, 1.0, 1.5; \beta = 0.5, 1.5, 2.0; p = 0.5, 1.0, 1.5)$. The scale parameter is estimated for GGD with MLE and Bayesian method of estimation using three priors viz. non-informative Jeffrey's prior and its extension, and its conjugate Inverse-Gamma prior under different loss functions. The values for Jeffrey's extension were taken as (C_1 = 0.5, 1.0, 1.5) and the values for the loss parameter were taken as (C_2=0.5,-0.5, 1,-1). The hyper-parameters for inverse gamma prior are taken as 0 and 1.The results are presented in the following tables.

Table 4: Posterior Mean of $\hat{\alpha}$ under Jeffrey's prior:

n	α	β	p	α_{ML}	α_{SL}	α_{AB}				α_{LL}	
						c_2=0.5	c_2=0.5	c_2=1	c_2=-1	a=0.5	a=-0.5
25	0.5	0.5	0.5	0.7617	0.8279	0.8656	0.7934	0.9068	0.7617	0.6924	0.7185
	1.0	1.5	1.0	0.5544	0.5696	0.5775	0.5619	0.5856	0.5544	0.5365	0.5435
	1.5	2.0	1.5	1.3096	1.3363	1.3501	1.3228	1.3641	1.3096	1.2776	1.2902
50	0.5	0.5	0.5	0.5061	0.5271	0.5384	0.5164	0.5501	0.5061	0.4819	0.4913
	1.0	1.5	1.0	0.5822	0.5901	0.5941	0.5861	0.5982	0.5822	0.5727	0.5764
	1.5	2.0	1.5	1.0900	1.1010	1.1066	1.0955	1.1123	1.0900	1.0766	1.0819
100	0.5	0.5	0.5	0.6975	0.7117	0.7190	0.7045	0.7265	0.6975	0.6804	0.6872

	1.0	1.5	1.0	0.6306	0.6349	0.6370	0.6328	0.6392	0.6307	0.6254	0.6275
	1.5	2.0	1.5	1.1767	1.1827	1.1856	1.1797	1.1886	1.1767	1.1694	1.1723

Table 5:Posterior Mean of $\hat{\alpha}$ under extension of Jeffrey's prior:

n	α	β	p	C_1	α_{ML}	α_{SL}	α_{AB}				α_{LL}	
							c_2=0.5	c_2=-0.5	c_2=1	c_2=-1	a=0.5	a=-0.5
25	0.5	0.5	0.5	0.5	0.7617	0.8279	0.8656	0.7934	0.9068	0.7617	0.6924	0.7185
				1.0	0.7617	0.7617	0.7934	0.7323	0.8279	0.7053	0.6454	0.6681
				1.5	0.7617	0.7053	0.7324	0.6801	0.7617	0.6566	0.6045	0.6243
	1.0	1.5	1.0	0.5	0.5544	0.5696	0.5775	0.5619	0.5856	0.5544	0.5365	0.5435
				1.0	0.5544	0.5544	0.5619	0.5471	0.5696	0.5400	0.5230	0.5297
				1.5	0.5544	0.5400	0.5471	0.5331	0.5544	0.5263	0.5102	0.5165
	1.5	2.0	1.5	0.5	1.3096	1.3363	1.3501	1.3228	1.3641	1.3096	1.2776	1.2902
				1.0	1.3096	1.3096	1.3228	1.2966	1.3363	12839	1.2532	1.2653
				1.5	1.3096	1.2839	1.2966	1.2714	1.3096	1.2592	1.2296	1.2413
50	0.5	0.5	0.5	0.5	0.5061	0.5271	0.5384	0.5164	0.5501	0.5061	0.4819	0.4913
				1.0	0.5061	0.5061	0.5164	0.4961	0.5271	0.4866	0.4643	0.4729
				1.5	0.5061	0.4866	0.4961	0.4774	0. 5061	0.4686	0.4478	0.4559
	1.0	1.5	1.0	0.5	0.5822	0.5901	0.5941	0.5861	0.5982	0.5822	0.5727	0.5764
				1.0	0.5822	0.5822	0.5861	0.5783	0.5901	0.5745	0.5652	0.5689
				1.5	0.5822	0.5745	0.5783	0.5708	0.5822	0.5671	0.5580	0.5616
	1.5	2.0	1.5	0.5	1.0900	1.1010	1.1066	1.0955	1.1123	1.0900	1.0766	1.0819
				1.0	1.0900	1.0900	1.0955	1.0846	1.1010	1.0792	1.0660	1.0713
				1.5	1.0900	1.0792	1.0846	1.0739	1.0900	1.0687	1.0557	1.0609
100	0.5	0.5	0.5	0.5	0.6975	0.7117	0.7190	0.7045	0.7265	0.6975	0.6804	0.6872
				1.0	0.6975	0.6975	0.7045	0.6906	0.7117	0.6838	0.6674	0.6739
				1.5	0.6975	0.6838	0.6906	0.6771	0.6975	0.6706	0.6549	0.6611
	1.0	1.5	1.0	0.5	0.6306	0.6349	0.6370	0.6328	0.6392	0.6307	0.6254	0.6275
				1.0	0.6307	0.6052	0.6133	0.5972	0.6217	0.5894	0.5709	0.5782
				1.5	0.6307	0.6264	0.6286	0.6244	0.6307	0.6224	0.6173	0.6193
	1.5	2.0	1.5	0.5	1.1767	1.1827	1.1856	1. 1797	1.1886	1.1767	1.1694	1.1723
				1.0	1.1767	1.1767	1.1797	1.1738	1.1827	1.1709	1.1637	1.1665
				1.5	1.1767	1.1709	1.1738	1.1780	1.1767	1.1651	1.1579	1.1608

Table 6: Posterior Mean of $\hat{\alpha}$ under Inverse-Gamma prior:

n	α	β	p	a1	b1	α_{ML}	α_{SL}	α_{AB}				α_{LL}	
								c_2=0.5	c_2=-0.5	c_2=1	c_2=-1	a=0.5	a=-0.5
25	0.5	0.5	0.5	0	0	0.7617	0.8279	0.8656	0.7934	0.9068	0.7617	0.6924	0.7185
				1.0	1.0	0.7617	0.8417	0.8768	0.8093	0.9149	0.7793	0.7132	0.7382
				0	1.0	0.7617	0.7617	0.7934	0.7323	0.8279	0.7053	0.6454	0.6681

				1.0	0	0.7617	0.9149	0.9565	0.8768	1.0020	0.8417	0.7651	0.7940
	1.0	1.5	1.0	0	0	0.5544	0.5696	0.5775	0.5619	0.5856	0.5544	0.5365	0.5435
				1.0	1.0	0.5544	0.5811	0.5889	0.5734	0.5970	0.5660	0.5482	0.5551
				0	1.0	0.5544	0.5544	0.5619	0.5471	0.5696	0.5400	0.5230	0.5297
				1.0	0	0.5544	0.5970	0.6053	0.5890	0.6138	0.5811	0.5623	0.5697
	1.5	2.0	1.5	0	0	1.3096	1.3363	1.3501	1.3228	1.3641	1.3096	1.2776	1.2902
				1.0	1.0	1.3096	1.3296	1.3430	1.3164	1.3567	1.3035	1.2723	1.2846
				0	1.0	1.3096	1.3096	1.3228	1.2966	1.3363	1.2839	1.2531	1.2653
				1.0	0	1.3096	1.3567	1.3707	1.3430	1.3850	1.3296	1.2971	1.3099
50	0.5	0.5	0.5	0	0	0.5061	0.5271	0.5384	0.5164	0.5501	0.5061	0.4819	0.4913
				1.0	1.0	0.5061	0.5461	0.5572	0.4354	0.5688	0.5251	0.5010	0.5103
				0	1.0	0.5061	0.5061	0.5164	0.4961	0.5271	0.4866	0.4643	0.4729
				1.0	0	0.5061	0.5688	0.5809	0.5572	0.5935	0.5460	0.5200	0.5301
	1.0	1.5	1.0	0	0	0.5822	0.5901	0.5941	0.5861	0.5982	0.5822	0.5727	0.5764
				1.0	1.0	0.5822	0.5955	0.5995	0.5916	0.6036	0.5877	0.5782	0.5820
				0	1.0	0.5822	0.5822	0.5861	0.5783	0.5901	0.5745	0.5652	0.5689
				1.0	0	0.5822	0.6036	0.6077	0.5995	0.6118	0.5955	0.5858	0.5896
	1.5	2.0	1.5	0	0	1.0900	1.1010	1.1066	1.0955	1.1123	1.0900	1.0766	1.0819
				1.0	1.0	1.0900	1.1000	1.1056	1.0946	1.1111	1.0891	1.0758	1.0811
				0	1.0	1.0900	1.0900	1.0955	1.0846	1.1010	1.0792	1.0660	1.0712
				1.0	0	1.0900	1.1111	1.1168	1.1056	1.1225	1.1000	1.0864	1.0918
100	0.5	0.5	0.5	0	0	0.6975	0.7117	0.7190	0.7045	0.7265	0.6975	0.6804	0.6872
				1.0	1.0	0.6975	0.7175	0.7247	0.7104	0.7321	0.7034	0.6866	0.6932
				0	1.0	0.6975	0.6975	0.7045	0.6906	0.7117	0.6838	0.6674	0.6739
				1.0	0	0.6975	0.7321	0.7397	0.7247	0.7474	0.7175	0.7000	0.7068
	1.0	1.5	1.0	0	0	0.6306	0.6349	0.6370	0.6328	0.6392	0.6307	0.6254	0.62752
				1.0	1.0	0.6306	0.6373	0.6395	0.6352	0.6416	0.6331	0.6279	0.6300
				0	1.0	0.6306	0.6307	0.6328	0.6286	0.6349	0.6265	0.6213	0.6234
				1.0	0	0.6306	0.6416	0.6438	0.6395	0.6459	0.6373	0.6321	0.6341
	1.5	2.0	1.5	0	0	1.1767	1.1827	1.1856	1. 1797	1.1886	1.1767	1.1694	1.1723
				1.0	1.0	1.1767	1.1817	1.1847	1.1788	1.1877	1.1759	1.1686	1.1715
				0	1.0	1.1767	1.1767	1.1797	1.1738	1.1827	1.1709	1.1637	1.1665
				1.0	0	1.1767	1.1877	1.1907	1.1847	1.1937	1.1817	1.1744	1.1773

Table 7: Mean Squared Error for $\hat{\alpha}$ under Jeffrey's prior:

n	α	β	p	α_{ML}	α_{SL}	α_{AB}				α_{LL}	
						c_2=0.5	c_2=-.5	c_2=1	c_2=-1	a=0.5	a=-0.5
25	0.5	0.5	0.5	0.0713	0.1104	0.1368	0.0887	0.1689	0.0071	0.0039	0.0050
	1.0	1.5	1.0	0.1987	0.1853	0.1786	0.1920	0.1719	0.1987	0.2149	0.2085
	1.5	2.0	1.5	0.0374	0.0279	0.0236	0.0325	0.0196	0.0373	0.0505	0.0450
50	0.5	0.5	0.5	0.0003	0.0010	0.0018	0.0006	0.0028	0.0003	0.0005	0.0003
	1.0	1.5	1.0	0.1747	0.1681	0.1649	0.1714	0.1615	0.1747	0.1827	0.1795
	1.5	2.0	1.5	0.1683	0.1594	0.1550	0.1638	0.1505	0.1683	0.1795	0.1750

100	0.5	0.5	0.5	0.0391	0.0449	0.0481	0.0419	0.0514	0.0391	0.0326	0.0351
	1.0	1.5	1.0	0.1365	0.1333	0.1318	0.1349	0.1302	0.1364	0.1403	0.1388
	1.5	2.0	1.5	0.1046	0.1008	0.0989	0.1027	0.0971	0.1046	0.1094	0.1075

Table 8: Mean Squared Error for $\hat{\alpha}$ under the extension of Jeffrey's prior:

n	α	β	p	C_1	α_{ML}	α_{SL}	α_{AB}				α_{LL}	
							c_2=0.5	c_2=-0.5	c_2=1	c_2=-1	a=0.5	a=-0.5
25	0.5	0.5	0.5	0.5	0.0713	0.1104	0.1368	0.0887	0.1689	0.0071	0.0039	0.0050
				1.0	0.0713	0.0071	0.0887	0.0562	0.1104	0.0442	0.0228	0.0302
				1.5	0.0713	0.0442	0.0562	0.0343	0.0709	0.0263	0.0124	0.0171
	1.0	1.5	1.0	0.5	0.1987	0.1853	0.1807	0.1920	0.1719	0.1987	0.2149	0.2045
				1.0	0.1987	0.1987	0.1920	0.2052	0.1853	0.2117	0.2276	0.2213
				1.5	0.1987	0.2117	0.2052	0.2181	0.1987	0.2245	0.2400	0.2339
	1.5	2.0	1.5	0.5	0.0374	0.0279	0.0236	0.0325	0.0196	0.0373	0.0505	0.0450
				1.0	0.0374	0.0373	0.0325	0.0424	0.0279	0.4770	0.0619	0.0561
				1.5	0.0374	0.0477	0.0424	0.0533	0.0373	0.0590	0.0741	0.0679
50	0.5	0.5	0.5	0.5	0.0003	0.0010	0.0018	0.0006	0.0028	0.0003	0.0005	0.0003
				1.0	0.0003	0.0003	0.0006	0.0002	0.0001	0.0004	0.0015	0.0009
				1.5	0.0003	0.0004	0.0002	0.0007	0.0003	0.0012	0.0029	0.0021
	1.0	1.5	1.0	0.5	0.1747	0.1681	0.1649	0.1714	0.1615	0.1747	0.1827	0.1795
				1.0	0.1747	0.1747	0.1714	0.1779	0.1681	0.1812	0.1892	0.1859
				1.5	0.1747	0.1812	0.1779	0.1843	0.1747	0.1875	0.1955	0.1923
	1.5	2.0	1.5	0.5	0.1683	0.1594	0.1550	0.1638	0.1505	0.1683	0.1795	0.1750
				1.0	0.1683	0.1683	0.1638	0.1726	0.1594	0.1773	0.1886	0.1840
				1.5	0.1683	0.1773	0.1728	0.1818	0.1683	0.1862	0.1976	0.1930
100	0.5	0.5	0.5	0.5	0.0391	0.0449	0.0481	0.0419	0.0514	0.0391	0.0326	0.0351
				1.0	0.0391	0.0391	0.0419	0.0365	0.0449	0.0339	0.0281	0.0303
				1.5	0.0391	0.0339	0.0364	0.0315	0.0391	0.0292	0.0241	0.0261
	1.0	1.5	1.0	0.5	0.1364	0.1333	0.1318	0.1349	0.1302	0.1364	0.1403	0.1388
				1.0	0.1364	0.1364	0.1349	0.1380	0.1333	0.1395	0.1434	0.1418
				1.5	0.1364	0.1396	0.1379	0.1411	0.1364	0.1426	0.1465	0.1450
	1.5	2.0	1.5	0.5	0.1046	0.1008	0.0989	0.1027	0.0971	0.1046	0.1094	0.1075
				1.0	0.1046	0.1046	0.1027	0.1065	0.1008	0.1084	0.1132	0.1113
				1.5	0.1046	0.1084	0.1065	0.1038	0.1046	0.1123	0.1171	0.1547

Table 9: Mean Squared Error for $\hat{\alpha}$ under Inverse-Gamma prior:

n	α	β	p	a1	b1	α_{ML}	α_{SL}	α_{AB}				α_{LL}	
								c_2=0.5	c_2=0.5	c_2=1	c_2=-1	a=0.5	a=-0.5
25	0.5	0.5	0.5	0	0	0.0713	0.1104	0.1368	0.0887	0.1689	0.0071	0.0390	0.0050
				1.0	1.0	0.0713	0.1192	0.1446	0.0979	0.1750	0.0080	0.0472	0.0586
				0	1.0	0.0713	0.0071	0.0887	0.0562	0.1104	0.0442	0.0228	0.0302
				1.0	0	0.0713	0.1750	0.2143	0.1446	0.2554	0.1192	0.0723	0.0886
	1.0	1.5	1.0	0	0	0.1987	0.1853	0.1786	0.1920	0.1719	0.1987	0.2149	0.2085
				1.0	1.0	0.1987	0.1756	0.1691	0.1821	0.1625	0.1885	0.2042	0.1980

				0	1.0	0.1987	0.1987	0.1920	0.2052	0.1853	0.2117	0.2276	0.2213
				1.0	0	0.1987	0.1625	0.1559	0.1690	0.1494	0.1756	0.1917	0.1853
	1.5	2.0	1.5	0	0	0.0374	0.0279	0.0236	0.0325	0.0196	0.0373	0.0505	0.0450
				1.0	1.0	0.0374	0.0300	0.0257	0.0347	0.0216	0.0396	0.0528	0.0474
				0	1.0	0.0374	0.0373	0.0324	0.0424	0.0279	0.0477	0.0620	0.0561
				1.0	0	0.0374	0.0216	0.0178	0.0257	0.0143	0.0300	0.0422	0.0371
50	0.5	0.5	0.5	0	0	0.0003	0.0010	0.0018	0.0006	0.0028	0.0003	0.0005	0.0003
				1.0	1.0	0.0003	0.0024	0.0036	0.0044	0.0050	0.0008	0.0002	0.0003
				0	1.0	0.0003	0.0003	0.0005	0.0002	0.0010	0.0004	0.0015	0.0009
				1.0	0	0.0003	0.0050	0.0068	0.0036	0.0090	0.0024	0.0006	0.0011
	1.0	1.5	1.0	0	0	0.1747	0.1681	0.1649	0.1714	0.1615	0.1747	0.1827	0.1795
				1.0	1.0	0.1747	0.1637	0.1605	0.1669	0.1572	0.1701	0.1780	0.4181
				0	1.0	0.1747	0.1747	0.1714	0.1779	0.1681	0.1812	0.1892	0.1859
				1.0	0	0.1747	0.1572	0.1540	0.1605	0.1508	0.1637	0.1717	0.1685
	1.5	2.0	1.5	0	0	0.1683	0.1594	0.1550	0.1638	0.1505	0.1683	0.1795	0.1750
				1.0	1.0	0.1683	0.1602	0.1558	0.1645	0.1514	0.1690	0.1801	0.1757
				0	1.0	0.1683	0.1683	0.1638	0.1728	0.1594	0.1773	0.1886	0.1841
				1.0	0	0.1683	0.1514	0.1470	0.1558	0.1427	0.1620	0.1713	0.1668
100	0.5	0.5	0.5	0	0	0.0391	0.0449	0.0481	0.0419	0.0514	0.0391	0.0326	0.0351
				1.0	1.0	0.0391	0.0474	0.0506	0.0444	0.0540	0.0415	0.0349	0.0374
				0	1.0	0.0391	0.0391	0.0419	0.0364	0.0449	0.0339	0.0281	0.0303
				1.0	0	0.0391	0.0540	0.0576	0.0506	0.0613	0.0474	0.0401	0.0429
	1.0	1.5	1.0	0	0	0.1365	0.1333	0.1318	0.1349	0.1302	0.1365	0.1403	0.1388
				1.0	1.0	0.1365	0.1316	0.1300	0.1331	0.1285	0.1346	0.1385	0.1369
				0	1.0	0.1365	0.1364	0.1349	0.1380	0.1333	0.1395	0.1434	0.1418
				1.0	0	0.1365	0.1285	0.1269	0.1300	0.1254	0.1316	0.1354	0.1339
	1.5	2.0	1.5	0	0	0.1046	1.1008	0.0989	0.1027	0.0971	0.1046	0.1094	0.1075
				1.0	1.0	0.1046	0.1014	0.0995	1.1033	0.0976	0.1051	0.1099	0.1080
				0	1.0	0.1046	0.1046	0.1027	0.1074	0.1008	0.1084	0.1132	0.1113
				1.0	0	0.1046	0.0976	0.0958	0.0995	0.0939	0.1014	0.1061	0.104

ML=Maximum Likelihood, SL=squared error loss function, AB=Albayyti's loss function, LL=Linex loss function

6. Results:

i. From tables 1, 2 and 3, we conclude that the Linex Loss function provides the minimum variance to $\hat{\alpha}$ using the Jeffrey's extension prior when C_1= 1, a=0.5 and Inverse-Gamma prior when (a_1, b_1)=(0,1) & (1,1); a=0.5.

ii. In table 7, Bayes estimation with Al-Bayyati's new loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter (C_2= 1).

iii. In table 8, Bayes estimation with Al-Bayyati's new loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter (C_2= 1) and (C_1=1.0, 1.5).

iv. In table 9, Bayes estimation with Al-Bayyati's new loss function under its conjugate Inverse-Gamma prior provides the smallest values in most cases especially when loss parameter (a_1 =1, b_1=0) and (C_2=1.0).

7. Conclusion:

We observe that Bayesian method of estimation is better than classical method of estimation. By comparing the results of our study, we observe that the Linex loss function has the least MSE under all the priors by taking β and p as 0.5 each in the worked example as well as in the simulation study.

Also, Bayesian estimator under Al-Bayyati's new loss function provides the smallest MSE values under Jeffrey's prior and its extension prior as well as the Inverse-Gamma prior compared to other loss functions and the classical estimator when the loss parameter is $C_{1}=1.5$, and Al-Bayyati's loss parameter is $C_2=1$.Thus we can say that Al-Bayyati's loss is better than other loss functions.

It is also observed that among all the priors, the Inverse-Gamma prior provides the Bayes estimators with least MSE.

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