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A Computational Tool for Some Boolean Partial Maps

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Abstract: This work is devoted to constructing a deterministic finite automaton whose states are particular types of order-preserving Boolean partial maps introduced by Bisi and Chiaselotti. The domains of such maps are subsets of a finite poset equipped with an idempotent and antitone map. These maps can be identified with certain linear systems of real inequalities and this automaton provides a computational model useful for building the global extensions of such maps.

Keywords: Antitone Maps, Boolean Maps, Systems of Linear Inequalities, Deterministic Finite Automata.

1 Introduction

Let $\mathbf{2}$ be the Boolean lattice composed of a chain with 2 elements which will be denoted by N (the minimal element) and P (the maximal element). In this paper, $(X,\leq)$ represents an arbitrary finite poset. The set of all the partial maps from $(X,\leq)$ to $\mathbf{2}$, here denoted by $(X \rightsquigarrow \mathbf{2})$, is a poset with the following order (see [13]):
if $(A,\mathrm{dom}(A)),(B,\mathrm{dom}(B)) \in (X \rightsquigarrow \mathbf{2})$,

$$(A,\mathrm{dom}(A)) \trianglelefteq (B,\mathrm{dom}(B))$$
$$\Updownarrow$$
$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B),\ B_{|\mathrm{dom}(A)} = A.$$

A *Boolean partial map* (BPM) on X is an element $(A,\mathrm{dom}(A))$ of $(X \rightsquigarrow \mathbf{2})$, (which in the following will be denoted only by A). If $\mathrm{dom}(A) = X$, it is said that A is a *Boolean total map* (BTM) on X. $\mathscr{OP}(X,\mathbf{2})$ will denote the family of all BTM's on X which are order-preserving.

It is said that a BPM A on X is: *up-positive* if $A^{-1}(P)$ is an up-set of X, i.e., if for all $z \in A^{-1}(P)$ and $x \in X$ with $z \leq x$, then $x \in A^{-1}(P)$; *down-negative* if $A^{-1}(N)$ is a down-set of X, i.e., for all $z \in A^{-1}(N)$ and $x \in X$ with $z \geq x$, then $x \in A^{-1}(N)$. An *up-down* map A on X is a BPM A on X which is up-positive and down-negative. Let us denote by $\mathscr{UD}(X,\mathbf{2})$ the set of all the up-down maps on X. Then, $\mathscr{UD}(X,\mathbf{2})$ is a sub-poset of $((X \rightsquigarrow \mathbf{2}),\trianglelefteq)$.

An *involution poset* (IP) is a poset $(X,\leq,c)$ with a unary operation $c : x \in X \mapsto x^c \in X$, such that:
$I1)(x^c)^c = x$, for all $x \in X$;
$I2)$if $x,y \in X$ and if $x \leq y$, then $y^c \leq x^c$.

The map c is called *complementation* of X and x^c the *complement* of x. Let us observe that if X is an involution poset, from $I1$), it follows that c is bijective. Therefore c can be considered an anti-automorphism of X into itself and this is equivalent to say that the complementation is an isomorphism between X and its dual poset X^*. If X is an IP, it is said that a BPM A on X is *complemented positive*, if $A^{-1}(N)^c \subseteq A^{-1}(P)$. If X is an IP, a BPM A on X is called *weighted Boolean partial map* (WBPM), if it is up-positive, down-negative and complemented-positive. In particular, if A is also total on X, it is called *weighted Boolean total map* (WBTM). Let us denote by $\mathscr{WP}(X,\mathbf{2})$ the subset of all the WBPM's on X and by $\mathscr{W}(X,\mathbf{2})$ the subset of all the WBTM's on X. In both cases, $\mathscr{WP}(X,\mathbf{2})$ and $\mathscr{W}(X,\mathbf{2})$ are sub-posets of $(X \rightsquigarrow \mathbf{2})$.

In [4], it was introduced a particular IP, $S(n,r)$, in order to study some extremal combinatorial sum problems (see [7,8,9,17]). $S(n,r)$ is a finite distributive lattice with 2^n elements and its construction, depending from two integers $0 \leq r \leq n$, is recalled in the section 3 of this paper. The structure of $S(n,r)$ is also related to interesting aspects concerning the sequential and parallel dynamics (see [1,2,3]) in certain discrete dynamical systems (see [5,10,11]). For a generalization of the partial order of $S(n,r)$ to a wider class of lattices see [12]. In this context, the families of maps $\mathscr{UD}(S(n,r),\mathbf{2})$ and $\mathscr{WP}(S(n,r),\mathbf{2})$ can be identified with particular types of systems of linear real inequalities.

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In [4], Bisi and Chiaselotti began a research line oriented to study the links between such types of Boolean maps and their associated linear systems inequalities. In particular, in their paper, the authors raise a relevant and difficult problem: to establish necessary and sufficient conditions on the Boolean map A so that the corresponding linear system is compatible. In the study of this problem, a crucial aspect is to have a model of computation to build all WBTM's in $\mathscr{W}(S(n,r),\mathbf{2})$ which extend a fixed WBPM in $\mathscr{WP}(S(n,r),\mathbf{2})$. In particular, if it is possible to select the minimal WBPM, B, which have a unique extension, A, in $\mathscr{W}(S(n,r),\mathbf{2})$, and if $\mathscr{S}_B$ and $\mathscr{S}_A$ are respectively the corresponding linear system associated to B and A, the compatibility of the system $\mathscr{S}_B$ implies the compatibility of the system $\mathscr{S}_A$ (see [4] for details). In this context, it raises the natural necessity to build a computational model which allows us to pass from a "local" map to a "global" map. This problem can be studied in a more abstract context, because the only required properties are the monotonicity and complemented positivity of the map A with respect to the antitone map c. Therefore, this problem is examined in the context of a generic finite involution poset $(X,\leq)$. For recent studies concerning the involution posets see [6]. Notice that the class of the involution posets is very large, because it includes the *orthocomplemented lattices*, which are involution bounded lattices such that $c(x)\vee x = 1$ and $c(x)\wedge x = 0$. For the relevance of these order structures in quantum logic see the classical book [18].

This work is devoted to constructing a deterministic finite automaton that models the computational way through which a WBPM on an IP becomes "global" (i.e. defined on the whole poset), continuing to maintain its properties, i.e., becomes a WBTM. The states of this automaton are therefore the elements of $\mathscr{WP}(X,\mathbf{2})$ and the symbols of transitions are the pairs of the form (w,ξ), where w is an element of the poset X and ξ is a Boolean value (N, as "negative", or P, as "positive").

This paper is structured as follows. In section 2, the general results used in the sequel are described. In section 3, the definition of the lattice $(S(n,r),\sqsubseteq)$ is recalled and it is proved how the partial order $\sqsubseteq$ characterizes the partial sums on indeterminate real numbers. In this section, it is also recalled the way through which the Boolean maps are connected to particular types of linear systems inequalities. Section 4 is devoted to constructing an automaton whose states are the partial maps of $\mathscr{UD}(X,\mathbf{2})$ and whose final states are the maps of $\mathscr{OP}(X,\mathbf{2})$, when X is an arbitrary finite poset. Finally, in section 5, the construction given in the previous section is refined, in order to have an automaton whose states are the partial maps of $\mathscr{WP}(X,\mathbf{2})$ and whose final states are the maps of $\mathscr{W}(X,\mathbf{2})$, when X is a finite IP.

2 Preliminaries and notations

For $Z\subseteq X$, it can be considered $\downarrow Z = \{x\in X : \exists z\in Z \text{ s.t. } z\geq x\}$ and $\uparrow Z = \{x\in X : \exists z\in Z \text{ s.t. } z\leq x\}$. In particular, for $z\in X$, it can be considered $\downarrow z = \downarrow\{z\} = \{x\in X : z\geq x\}$ and $\uparrow z = \uparrow\{z\} = \{x\in X : z\leq x\}$.

Definition 2.1. Z is said to be a *down-set* of X if $z\in Z$ and $x\in X$ with $z\geq x$, then $x\in Z$. Z is said to be an *up-set* of X if $z\in Z$ and $x\in X$ with $z\leq x$, then $x\in Z$.

Observe that $\downarrow Z$ is the smallest down-set of X which contains Z and Z is a down-set in X if and only if $Z=\downarrow Z$. Similarly $\uparrow Z$ is the smallest up-set of X which contains Z and Z is an up-set in X if and only if $Z=\uparrow Z$.

The following proposition shows that the concepts of up-positivity, down-negativity and of order-preserving are equivalent for Boolean total maps.

Proposition 2.1. Let A be a BTM on X. Then the following conditions are equivalent:
$i)$ A is order-preserving (op);
$ii)$ A is up-positive (up);
$iii)$ A is down-negative (dn).

Proof. In the following, it will be proved that $i)$ and $ii)$ are equivalent. The equivalence of $i)$ and $iii)$ follows similarly.
$i)\Rightarrow ii)$: Suppose that $x_1,x_2\in X$ and $A(x_1)=P$. Since A is order-preserving, we have that $A(x_1)\leq A(x_2)$ and this in $\mathbf{2}$ implies that $A(x_2)=P$.
$ii)\Rightarrow i)$: Let $x_1,x_2\in X$ such that $x_1\leq x_2$ and suppose, by reduction to the absurd, that $A(x_1)\not\leq A(x_2)$. Since $\mathbf{2}$ is totally ordered, this means that $A(x_1)>A(x_2)$. Hence $A(x_1)=P$ and $A(x_2)=N$. But, since $x_1\leq x_2$ and $A(x_1)=P$, and A is up-positive, it follows that $A(x_2)=P$, which is a contradiction because $A(x_2)=N$. $\square$

Obviously, if X is an IP, thanks to Proposition 2.1, it follows that $\mathscr{W}(X,\mathbf{2})$ is the sub-family of all the maps in $\mathscr{OP}(X,\mathbf{2})$ which are also complemented positive.

From Proposition 2.1, it also follows that if A is a BTM on X, then A is a WBTM if and only if A is up-positive and complemented positive. Let us note that each orthocomplemented lattice $(L,\wedge,\vee,0,1,')$ is also an IP and that, in this case, a lattices morphism $A: L\to\mathbf{2}$ is a WBTM.

The following result will be essential in the sequel.

Proposition 2.2. Let X be an IP and A a WBPM on X.
$i)$ If w is a minimal positive of A such that $A(w^c)=N$, then w^c is a maximal negative of A.
$ii)$ If $x,\ x^c\in dom(A)$ and $x^c\leq x$, then $A(x)=P$.

Proof.
$i)$ Suppose, by contradiction, that w^c is not a maximal negative of A. Then there exists an element $w'\in A^{-1}(N)$ such that $w'>w^c$. Since A is complemented positive, we have that $A^{-1}(N)^c\subseteq A^{-1}(P)$ and hence $(w')^c\in A^{-1}(P)$. Furthermore, since $w'>w^c$, we have also that $w=(w^c)^c>(w')^c$, but this is a contradiction by the minimality of w in $A^{-1}(P)$.
$ii)$ Suppose, by contradiction, that $A(x)=N$. Since A is complemented positive, we have that $x^c\in dom(A)$ and

$A(x^c) = P$. Since $x^c \le x$ and A is up-positive, we have that $A(x) = P$ and this is a contradiction. □

The part *ii*) of Proposition 2.2 motivates the following definition:

Definition 2.2. The elements $w \in X$ such that $w^c \le w$ are called *complemented*.

3 Partial Sums of Real Numbers and Boolean Maps

Let n and r be two fixed non-negative integers such that $r \le n$. We call (n,r)-*string* a n-pla of integers

$$a_r \dots a_1 | b_1 \dots b_{n-r}, \tag{1}$$

such that:
i) $a_1, \dots, a_r \in \{1, \dots, r, 0\}$;
ii) $b_1, \dots, b_{n-r} \in \{-1, \dots, -(n-r), 0\}$;
iii) $a_r \ge \dots \ge a_1 \ge 0 \ge b_1 \ge \dots \ge b_{n-r}$;
iv) the unique element in (1) which can be repeated is 0.

By $S(n,r)$ it is denoted the set of all the (n,r)-strings. On $S(n,r)$, it can be considered the partial order on the components, which is denoted by $\sqsubseteq$. To simplify the notations, in all the numerical examples, the integers on the right of the vertical bar | will be written without minus sign.

Since $(S(n,r), \sqsubseteq)$ is a finite distributive lattice, it is also graded, with minimal element $0 \cdots 0 | 12 \cdots (n-r)$ and maximal element $r(r-1) \cdots 21 | 0 \cdots 0$. The lattice $S(n,r)$ was introduced in [4] in order to study some combinatorial extremal sum problems.

In [4], it is shown that $(S(n,r), \sqsubseteq)$ is an IP and its complementation map c is such that the complement

$$(a_k \dots a_1\, 0 \dots 0 | 0 \dots 0\, b_1 \dots b_l)^c$$

is given by

$$a'_{r-k} \dots a'_1\, 0 \dots 0 | 0 \dots 0\, b'_1 \dots b'_{n-r-l}$$

where $\{a'_1, \dots, a'_{r-k}\}$ is the usual complement of $\{a_1, \dots, a_k\}$ in $\{1, \dots, r\}$ and $\{b'_1, \dots, b'_{n-r-l}\}$ is the usual complement of $\{b_1, \dots, b_l\}$ in $\{-1, \dots, -(n-r)\}$ (for example, in the distributive lattice $S(7,4)$, we have that $(4310|001)^c = 2000|023)$.

Consider now

$$I(n,r) := \{r, \dots, 1, 0, -1, \dots, -(n-r)\}.$$

Then, $F(n,r)$ will denote the set of all the functions $f : I(n,r) \to \mathbb{R}$ such that

$$f(r) \ge \dots \ge f(1) \ge f(0) = 0 > f(-1) \ge \dots \ge f(-(n-r)) \tag{2}$$

and $WF(n,r)$ the subset of all the functions $f \in F(n,r)$ such that

$$f(r) + \dots + f(1) + f(-1) + \dots + f(-(n-r)) \ge 0. \tag{3}$$

If $f \in F(n,r)$, we can consider the map $\sigma_f : S(n,r) \to \mathbb{R}$ such that

$$\sigma_f(a_r \dots a_1 | b_1 \dots b_{n-r}) := \sum_{i=1}^{r} f(a_i) + \sum_{j=1}^{n-r} f(b_j) \tag{4}$$

The next result shows how the order structure in $S(n,r)$ is strictly related to the properties of the family of maps $\{\sigma_f : f \in F(n,r)\}$. Recall first the definition of valuation on an arbitrary lattice X.

Definition 3.1. If X is a lattice, a map $v : X \to \mathbb{R}$ is called a *valuation* on X if for all $a, b \in X$: $v(a \wedge b) + v(a \vee b) = v(a) + v(b)$.

Fundamentals studies concerning the valuations on distributive lattices were carried out in [14,15,16].

Proposition 3.1. According to the definitions above:
i) If $f \in F(n,r)$, the map σ_f is a valuation on $S(n,r)$.
ii) If $w, w' \in S(n,r)$, then $w \sqsubseteq w'$ if and only if $\sigma_f(w) \le \sigma_f(w')$ for each $f \in F(n,r)$.

Proof. Statement *i*) follows directly from (4) and the definition of $\sqsubseteq$.
To prove *ii*), let us consider $w = a_r \dots a_1 | b_1 \dots, b_{n-r}$ and $w = a'_r \dots a'_1 | b'_1 \dots, b'_{n-r}$ two elements in $S(n,r)$. If $w \sqsubseteq w'$, it is immediate that $\sigma_f(w) \le \sigma_f(w')$ for each $f \in F(n,r)$. We assume now that $\sigma_f(w) \le \sigma_f(w')$ for each $f \in F(n,r)$ and that the condition $w \sqsubseteq w'$ is false. This means that there exists some $i \in \{1, \dots, r\}$ such that $a_i > a'_i$ or some $j \in \{1, \dots, n-r\}$ such that $b_j > b'_j$. Let us suppose at this point that there exists $i \in \{1, \dots, r\}$ such that $a_i > a'_i$ and assume that i is maximal among all the positive integers $l \in \{1, \dots, r\}$ such that $a_l > a'_l$. Thus,

$$a_r \ge \dots \ge a_{i+1} \ge a_i > a'_i \ge a'_{i-1} \ge \dots \ge a'_1 \tag{5}$$

and

$$a'_r \ge a_r, \dots, a'_{i+1} \ge a_{i+1}. \tag{6}$$

Consider now the following function

$$f(\alpha) := \begin{cases} -1 \text{ if } \alpha \in \{-1, \dots, -(n-r)\} \\ 0 \quad \text{if } \alpha \in \{0, 1, \dots, a_i - 1\} \\ +1 \text{ if } \alpha \in \{a_i, \dots, r\} \end{cases}$$

Then, $f \in F(n,r)$ and from (5) and (6), it follows that

$$\begin{aligned} \sigma_f(w) &\ge (r - i + 1) + \textstyle\sum_{1 \le j \le n-r} f(b_j) \\ &> (r - i) + \textstyle\sum_{1 \le j \le n-r} f(b_j) \\ &= (r - i) + \textstyle\sum_{1 \le j \le n-r} f(b'_j) = \sigma_f(w'), \end{aligned}$$

which is a contradiction.

We can suppose then that $a_i \le a'_i$ for all $i = 1, \dots, r$, so that there exists $j \in \{1, \dots, n-r\}$ such that $b_j > b'_j$, and assume that j is minimal among all the positive integers $l \in \{1, \dots, n-r\}$ such that $b_l > b'_l$. Thus

$$b_1 \ge \dots \ge b_{j-1} \ge b_j > b'_j \ge b'_{j+1} \ge \dots \ge b'_{n-r} \tag{7}$$

and

$$b'_1 \geq b_1, \ldots, b'_{j-1} \geq b_{j-1} \tag{8}$$

We must now distinguish two cases. First, suppose that $b_j = 0$. In this case, consider the following function:

$$h(\alpha) := \begin{cases} 0 & \text{if } \alpha \in \{0, 1, \ldots, r\} \\ -1 & \text{if } \alpha \in \{-1, \ldots, -(n-r)\} \end{cases}$$

Then, $h \in F(n,r)$ and from (7) and (8), it follows that

$$\begin{aligned} \sigma_h(w) &\geq (-1)(n-r-j) \\ &> (-1)(n-r-j+1) = \sigma_h(w'), \end{aligned}$$

which is a contradiction.

Assume now that $b_j < 0$. In this case, consider the following function

$$g(\alpha) := \begin{cases} 0 & \text{if } \alpha = 0 \\ +1 & \text{if } \alpha \in \{1, \ldots, r\} \\ -1 & \text{if } \alpha \in \{-1, \ldots, b_j\} \\ -2 & \text{if } \alpha \in \{b_j - 1, \ldots, -(n-r)\} \end{cases}$$

Then, $g \in F(n,r)$ and from (7) and (8), it follows that

$$\begin{aligned} \sigma_g(w) &\geq \sum_{1 \leq l \leq j-1} g(b_l) + (-1) + \sum_{j+1 \leq l \leq n-r} g(b_l) \\ &> \sum_{1 \leq l \leq j-1} g(b_l) + (-2) + \sum_{j+1 \leq l \leq n-r} g(b_l) \\ &= \sum_{1 \leq l \leq j-1} g(b'_l) + g(b'_j) + \sum_{j+1 \leq l \leq n-r} g(b_l) \\ &\geq \sum_{1 \leq l \leq j-1} g(b'_l) + (-2) + (-2)(n-r-j) \\ &= \sigma_g(w') \end{aligned}$$

which is again a contradiction. This complete the proof of ii). □

From the previous proposition, it can be deduced that the map σ_f is an order-preserving valuation on $S(n,r)$ for all $f \in F(n,r)$. In particular, if σ_f is one-to-one, then σ_f is also a linear extension of $S(n,r)$. In [19] Rota showed that a valuation on a distributive lattice is uniquely determined by the values that it takes on the join-irreducible elements of the lattice. Therefore, in our case, this means that $\sum_f$ is uniquely determined by the values that it takes on the join-irreducible elements of the distributive lattice $S(n,r)$.

Let now $x_r, \ldots, x_1, y_1, \ldots, y_{n-r}$ be n real variables that satisfy the following inequalities:

$$x_r \geq \cdots \geq x_1 \geq 0 > y_1 \geq \cdots \geq y_{n-r} \tag{9}$$

If $w = a_r \ldots a_1 | b_1 \ldots b_{n-r} \in S(n,r)$, we can set $\sum(w) = x_{a_r} + \cdots + x_{a_1} + y_{b_1} + \cdots + y_{b_{n-r}}$. Then, from Proposition 3.1, we can think the partial order $\sqsubseteq$ on $S(n,r)$ as the natural order induced from the linear systems inequalities (9) on the partial sum of the real variables $x_r, \ldots, x_1, y_1, \ldots, y_{n-r}$. In other terms, if we formally identify the signed partitions w and w' respectively with the indeterminate real partial sums $\sum(w)$ and $\sum(w')$, then the result of Proposition 3.1 allows us to infer that $w \sqsubseteq w'$ if and only if the real inequality $\sum(w) \leq \sum(w')$ holds. It can be deduced by using only the inequalities in (9). Therefore, we can use a more suggestive terminology and think the lattice $(S(n,r), \sqsubseteq)$ as a lattice of indeterminate partial sums taken over the indeterminate real variables $x_r, \ldots, x_1, y_1, \ldots, y_{n-r}$ which satisfy (9).

We call (n,r)-*system* of *size* p a system $\mathscr{S}$ of linear inequalities having the following form:

$$\mathscr{S}: \begin{cases} x_r \geq \cdots \geq x_1 \geq 0 > y_1 \geq \cdots \geq y_{n-r} \\ \sum(w_1) \geq 0 \text{ (or } < 0) \\ \sum(w_2) \geq 0 \text{ (or } < 0) \\ \cdots \\ \cdots \\ \sum(w_p) \geq 0 \text{ (or } < 0) \end{cases} \tag{10}$$

where $w_1, \ldots, w_p \in S(n,r)$. It is clear that the (n,r)-system $\mathscr{S}$ can be uniquely identified with the Boolean partial map $A_{\mathscr{S}} \in (S(n,r) \rightsquigarrow \mathbf{2})$, with $dom(A) = \{w_1, \ldots, w_s\}$, defined as

$$A_{\mathscr{S}}(w_j) = \begin{cases} P & \text{if } \sum(w_j) \geq 0 \\ N & \text{if } \sum(w_j) < 0 \end{cases}$$

for $j = 1, \ldots, p$. At this point, one can check that the Boolean partial map $A_{\mathscr{S}}$ is up-positive and down-negative. Furthermore, when the inequality

$$x_r + \cdots + x_1 + y_1 + \cdots + y_{n-r} \geq 0 \text{ (or } < 0) \tag{11}$$

appears in (10), it is said that it is a (n,r)-*positively weighted* system (or a (n,r)-*negatively weighted* system). Then, if the (n,r)-system $\mathscr{S}$ is positively weighted, the map $A_{\mathscr{S}}$ is also complemented positive. Therefore, we have two interesting families of BPM on $S(n,r)$: the family of the up-positive and down-negative BPM and its sub-family of the complemented positive BPM (related to the (n,r)- positively weighted systems).

The important point is that there are several open combinatorial problems related to the links between the properties of these types of Boolean maps and the properties of the (n,r)-systems (see [4]).

4 The DFA $op - \mathscr{AUT}(X, \mathbf{2})$

In this section, we will study the possibility of determining a computational model which allows us to build total order-preserving complemented positive Boolean maps through additions of single elements to their domains. From Proposition 2.1, the subset of all the total maps of $\mathscr{UD}(X, \mathbf{2})$ is exactly $\mathscr{OP}(X, \mathbf{2})$. We will define a structure of deterministic finite automaton, which will be denoted by $op - \mathscr{AUT}(X, \mathbf{2})$.

The set $\mathscr{ST}_2(X)$ of the states of $op - \mathscr{AUT}(X, \mathbf{2})$ is exactly $\mathscr{UD}(X, \mathbf{2})$ by adding a new state EXIT. Formally, $\mathscr{ST}_2(X)$ is the linear sum of the poset $\mathscr{UD}(X, \mathbf{2})$ and of $\mathbf{1}$(=EXIT) (therefore EXIT is an isolate element in this poset). However, the partial order on $\mathscr{ST}_2(X)$ will be denoted again by $\trianglelefteq$. The set of the final states of $op - \mathscr{AUT}(X, \mathbf{2})$ is $\mathscr{OP}(X, \mathbf{2})$.

If $X = \{x_1, \cdots, x_m\}$, the set $\mathscr{TR}_2(X)$ of the transitions symbols of $op - \mathscr{AUT}(X, \mathbf{2})$ is $\{x_1P, x_1N \cdots, x_mP, x_mN\}$.

The initial state $\mathfrak{I}_{2,X}$ is the empty partial map of $\mathscr{UD}(X, \mathbf{2})$, i.e., the Boolean map having empty domain, which is also the minimum of the poset $((X \rightsquigarrow \mathbf{2}), \trianglelefteq)$.

Before defining how the transition symbols act on the automaton's states, we fix some preliminary concepts. Let $w \in X$ be fixed, and let $u_1, \cdots, u_p, v_1, \cdots, v_q$ be the elements of X such that $\{u_1, \cdots, u_p\} = \uparrow w$, $\{v_1, \cdots v_q\} = \downarrow w$.

Let ξ be an arbitrary Boolean value in $\mathbf{2}$. The following sequence of transition symbols

$$w\xi[\cdots] = \begin{cases} u_1P \cdots u_pP \text{ if } \xi = P \\ v_1N \cdots v_qN \text{ if } \xi = N \end{cases} \tag{12}$$

is called *production of the transition symbol* $w\xi$, and it is denoted by $w\xi[\cdots]$.

Let $(A, \Omega_A) \in \mathscr{UD}(X, \mathbf{2})$ and let $\Omega_A = \{w_1, \cdots, w_s\}$. If $A(w_i) = \xi_i$, $(i = 1, \cdots, s)$, where $\xi_i \in \mathbf{2}$, we will identify A with the sequence $w_1\xi_1 \cdots w_s\xi_s$ and write

$$A \equiv w_1\xi_1 \cdots w_s\xi_s. \tag{13}$$

When thinking of A in the form (13), we will write A instead of (A, Ω_A).

We define now how the transition symbols act on the states. Given a non-exit state $A \equiv w_1\xi_1 \cdots w_s\xi_s$ and a transition symbol $w\xi$, then:

s_1)If $w \notin \{w_1, \cdots, w_s\}$, then $w\xi$ transforms the state A into the state

$$A' \equiv w_1\xi_1 \cdots w_s\xi_s w\xi[\cdots]; \tag{14}$$

s_2)If $w \in \{w_1, \cdots, w_s\}$ and $w = w_i$, for some $i = 1, \cdots, s$, then:
- if $\xi \neq \xi_i$, $w\xi$ sends the state A into the exit state;
- if $\xi = \xi_i$, $w\xi$ sends the state A into itself.

s_3)Each transition symbol $w\xi$ sends the state exit into itself.

In order to prove that the constructed automaton $op - \mathscr{AUT}(X, \mathbf{2})$ is well-defined, we need to prove that the sequences of transition symbols in (14) define effectively an up-down map on X, i.e., an element of $\mathscr{UD}(X, \mathbf{2})$. This is established in the following result.

Theorem 4.1. Let (A, Ω_A) be an up-down map on X, and let $w \in X \setminus \Omega_A$. Let $\xi \in \mathbf{2}$ and suppose that $w\xi[\cdots] = \alpha_1\xi \cdots \alpha_m\xi$ is the production of $w\xi$. Let $\Omega_A[w] = \Omega_A \cup \{\alpha_1, \cdots, \alpha_m\}$ and $A_w : \Omega_A[w] \to \mathbf{2}$ defined by

$$A_w(u) = \begin{cases} A(u) \text{ if } u \in \Omega_A \\ \xi \quad\ \ \text{ if } u = \alpha_i \ (i = 1, \cdots, m). \end{cases}$$

Then the couple $(A_w, \Omega_A[w])$ is an up-down map on X.

Proof. Suppose at first that $\xi = P$. In this case, see (12), we have $\{\alpha_1, \cdots, \alpha_m\} = \{u_1, \cdots, u_p\}$, where $\{u_1, \cdots, u_p\} = \uparrow \{w\}$, and hence

$$A_w(u) = \begin{cases} A(u) \text{ if } u \in \Omega_A \\ P \quad\ \ \text{ if } u \in \{u_1, \cdots, u_p\}. \end{cases} \tag{15}$$

Observe that A_w is well-defined. Indeed, if $u_i \in \Omega_A$, for some $i = 1, \cdots, p$, then $A(u_i) = P$ (by contradiction, if $A(u_i) = N$ from the hypothesis $w \sqsubseteq u_i$ and from the fact that A is down-negative, it follows that $w \in \Omega_A$ and that $A(w) = N$, against the hypothesis that $w \notin \Omega_A$).

Now we verify that A_w is up-positive. Let $w_1, w_2 \in X$ such that $w_1 \in \Omega_A[w]$, $w_1 \sqsubseteq w_2$ and suppose that $A_w(w_1) = P$. Then, if $w_1 \in \Omega_A$, we have that $A(w_1) = A_w(w_1) = P$, and since A is up-positive we have that $w_2 \in \Omega_A$ and $A_w(w_2) = A(w_2) = P$. Thus, we can assume that $w_1 \in \{u_1, \cdots u_p\} \setminus \Omega_A$.

Since $\{u_1, \cdots u_p\} = \uparrow w$, and $w_1 \in \{u_1, \cdots u_p\}$ and $w_1 \sqsubseteq w_2$, it follows that $w_2 \in \{u_1, \cdots u_p\}$ and hence $w_2 \in \Omega_A[w]$ and $A_w(w_2) = P$ thanks to (15). This proves that A_w is up-positive.

Let $w_1, w_2 \in X$ be such that $w_2 \in \Omega_A[w]$, $w_1 \sqsubseteq w_2$ and $A_w(w_2) = N$. From (15), it follows that $w_2 \in \Omega_A$ and that $A(w_2) = A_w(w_2) = N$, and, since A is down-negative, it follows that $w_1 \in \Omega_A$ and $A_w(w_1) = A(w_1) = N$. This proves that A_w is down-negative.

The case $\xi = N$ is similar. $\square$

Hence, as proved, the rules s_1), s_2) and s_3) define a transition function

$$\delta : \mathscr{ST}_2(X) \times \mathscr{TR}_2(X) \to \mathscr{ST}_2(X).$$

In view of that, the automaton $op - \mathscr{AUT}(X, \mathbf{2})$ is formally given by

$$\langle \mathscr{ST}_2(X), \mathscr{TR}_2(X), \mathfrak{I}_{2,X}, \delta, \mathscr{OP}(X, \mathbf{2}) \rangle.$$

The following proposition shows the connection between δ and $\trianglelefteq$.

Proposition 4.1. Let (A, Ω_A) and (B, Ω_B) two non-exit states of $op - \mathscr{AUT}(X, \mathbf{2})$ (i.e. two up-down maps on X) and let $w\xi$ be a transition symbol such that $\delta(A, w\xi) = B$. Then $(A, \Omega_A) \trianglelefteq (B, \Omega_B)$.

Proof. Since B is a non-exit state, it can coincide with A or it can be of the form (14). In every case, it follows that $\Omega_A \subseteq \Omega_B$ and that $B_{|\Omega_A} = A$, from which the assertion follows. $\square$

The preceding proposition asserts that the transition function can move a state A to a *exit-state* or to a state B that in the Hasse diagram of $\mathscr{UD}(X, \mathbf{2})$ is necessarily above A.

5 The DFA $wt - \mathscr{AUT}(X, \mathbf{2})$

In this section, X will denote an arbitrary finite IP. Consider $\mathscr{WP}(X, \mathbf{2})$ as a sub-poset of $\mathscr{UD}(X, \mathbf{2})$. The subset of all the total maps of $\mathscr{WP}(X, \mathbf{2})$ is exactly $\mathscr{W}(X, \mathbf{2})$. We define now a structure of deterministic finite automaton which will be denoted by $wt - \mathscr{AUT}(X, \mathbf{2})$.

The set $w\mathscr{ST}_2(X)$ of the states of $wt - \mathscr{AUT}(X, \mathbf{2})$ is exactly $\mathscr{WP}(X, \mathbf{2})$ by adding a new state EXIT. Formally, as in the previous section, $w\mathscr{ST}_2(X)$ is the linear sum of the poset $\mathscr{WP}(X,\mathbf{2})$ and of $\mathbf{1}$(=EXIT). However, also in this case, the partial order on $w\mathscr{ST}_2(X)$ will be denoted again by $\trianglelefteq$. The set of the final states of $wt-\mathscr{AUT}(X,\mathbf{2})$ is $\mathscr{W}(X,\mathbf{2})$.

If $X=\{x_1,\cdots,x_m\}$, the set $w\mathscr{TR}_2(X)$ of the transitions symbols of $wt-\mathscr{AUT}(X,\mathbf{2})$ is $\{x_1P,x_1N\cdots,x_mP,x_mN\}$.

The initial state $w\mathfrak{I}_{2,X}$ is the empty partial map of $\mathscr{WP}(X,\mathbf{2})$, i.e., the Boolean map having empty domain, which is also the minimum of the poset $((X\rightsquigarrow\mathbf{2}),\trianglelefteq)$.

As in the previous case, we need to fix some preliminary concepts. Let $w\in X$ be fixed, and let $u_1,\cdots,u_p,v_1,\cdots,v_q,z_1,\cdots,z_t$ be the elements of X such that $\{u_1,\cdots,u_p\}=\uparrow w$, $\{v_1,\cdots v_q\}=\downarrow w$, $\{z_1,\cdots,z_t\}=\uparrow\{v_1^c,\cdots,v_q^c\}$.

Let ξ be an arbitrary Boolean value in $\mathbf{2}$. The following sequence of transition symbols

$$w\xi[\cdots]=\begin{cases}u_1P\cdots u_pP & \text{if } \xi=P\\ v_1N\cdots v_qNz_1P\cdots z_tP & \text{if } \xi=N\end{cases}\qquad(16)$$

is called *production of the transition symbol* $w\xi$, and it is denoted by $w\xi[\cdots]$.

Let $(A,\Omega_A)\in\mathscr{WP}(X,\mathbf{2})$ as in (13), where $\Omega_A=\{w_1,\cdots,w_s\}$. We define now how the transition symbols act on the states. Given a non-exit state $A\equiv w_1\xi_1\cdots w_s\xi_s$ and a transition symbol $w\xi$, then:

wt_1) If $w\notin\{w_1,\cdots,w_s\}$ and w is not-complemented , then $w\xi$ transforms the state A into the state

$$A'\equiv w_1\xi_1\cdots w_s\xi_s w\xi[\cdots]\qquad(17)$$

wt_2) If $w\notin\{w_1,\cdots,w_s\}$ and w is complemented, then $w\xi$ transforms the state A into the state

$$A'\equiv w_1\xi_1\cdots w_s\xi_s wP[\cdots]\qquad(18)$$

if $\xi=P$, or into a exit-state if $\xi=N$.

wt_3) If $w\in\{w_1,\cdots,w_s\}$ and $w=w_i$, for some $i=1,\cdots,s$, then:
- if $\xi\neq\xi_i$, $w\xi$ sends the state A into a exit-state;
- if $\xi=\xi_i$, $w\xi$ sends the state A into itself.

wt_4) Each transition symbol $w\xi$ sends an exit-state into itself.

In order to prove that this automaton $wt-\mathscr{AUT}(X,\mathbf{2})$ is well-defined, we need to prove that the sequences of transition symbols in (17) define effectively a WBPM on X, i.e., an element of $\mathscr{WP}(X,\mathbf{2})$. This is established in the following theorem.

Theorem 5.1. Let (A,Ω_A) be a WBPM on X and let $w\in X\setminus\Omega_A$. Let $\xi\in\mathbf{2}$ and suppose that $w\xi[\cdots]=\alpha_1\xi_1\cdots\alpha_m\xi_m$ is the production of $w\xi$. Let $\Omega_A[w]=\Omega_A\cup\{\alpha_1,\cdots,\alpha_m\}$ and $A_w:\Omega_A[w]\to\mathbf{2}$ defined by

$$A_w(u)=\begin{cases}A(u) & \text{if } u\in\Omega_A\\ \xi_i & \text{if } u=\alpha_i\ (i=1,\cdots,m).\end{cases}$$

Then, if $\xi=P$ or $\xi=N$ and w is not-complemented, the couple $(A_w,\Omega_A[w])$ is an WBPM on X.

Proof. First, suppose that $\xi=P$. In this case, see (16), we have $\{\alpha_1,\cdots,\alpha_m\}=\{u_1,\cdots,u_p\}$, where $\{u_1,\cdots,u_p\}=\uparrow\{w\}$, and hence

$$A_w(u)=\begin{cases}A(u) & \text{if } u\in\Omega_A\\ P & \text{if } u\in\{u_1,\cdots,u_p\}.\end{cases}\qquad(19)$$

Observe that A_w is well-defined, up-positive and down-negative by using the same argument of Theorem 4.1. Let $u\in\Omega_A[w]$ be such that $A_w(u)=N$. From (19), it follows that $u\in\Omega_A$ and that $A(u)=A_w(u)=N$. Since A is a WBPM on X, it is also complemented positive. Therefore, $u^c\in\Omega_A$ and $A_w(u^c)=A(u^c)=P$. This proves that A_w is complemented positive.

Hence, if $\xi=P$, the couple $(A_w,\Omega_A[w])$ is a WBPM on X.

Now suppose that w is not-complemented and $\xi=N$. In this case, from (16), it can be deduced that $\{\alpha_1,\cdots,\alpha_m\}=\{v_1,\cdots v_q,z_1,\cdots,z_t\}$, where $\{v_1,\cdots,v_q\}=\downarrow w$, and $\{z_1,\cdots,z_t\}=\uparrow\{v_1^c,\cdots,v_q^c\}$,. Consequently,

$$A_w(u)=\begin{cases}A(u) & \text{if } u\in\Omega_A\\ N & \text{if } u\in\{v_1,\cdots v_q\}\\ P & \text{if } u\in\{z_1,\cdots z_t\}.\end{cases}\qquad(20)$$

At this point, we need to prove that A_w is well defined. Observe that $\{z_1\cdots z_t\}\cap\{v_1,\cdots v_q\}=\emptyset$. Indeed, suppose, by contradiction, that for some $j\in\{1,\cdots,t\}$ and for some $i\in\{1,\cdots q\}$ we have that $z_j=v_i$. Since $\{z_1,\cdots z_t\}=\uparrow\{v_1^c\cdots,v_q^c\}$, there will exist a $k\in\{1,\cdots,q\}$ such that $v_k^c\sqsubseteq z_j$. Since $\{v_1,\cdots,v_q\}=\downarrow w$, then $v_k\sqsubseteq w$; since X is an IP, by $I2$) we have

$$w^c\sqsubseteq v_k^c\sqsubseteq z_j=v_i\sqsubseteq w.$$

Hence w is complemented and this is a contradiction.

We need to prove now that if $v_i\in\Omega_A$, for some $i=1,\cdots,q$, then it will be $A(v_i)=N$. Suppose by contradiction that $v_i\in\Omega_A$ and that $A(v_i)=P$. Since $v_i\sqsubseteq w$ and A is up-positive, it follows that $w\in\Omega_A$ and that $A(w)=P$, in contradiction with the hypothesis $w\notin\Omega_A$.

Finally, we prove that if $z_j\in\Omega_A$, for some $j=1,\cdots t$, then it will be $A(z_j)=P$. If, by contradiction, $z_j\in\Omega_A$ and $A(z_j)=N$, as by the hypothesis A is complemented positive, it follows that $z_j^c\in\Omega_A$ and $A(z_j^c)=P$. Now let $k\in\{1,\cdots q\}$ be such that $v_k^c\sqsubseteq z_j$. Since $v_k\sqsubseteq w$ and X is an IP, it holds that $z_j^c\sqsubseteq v_k\sqsubseteq w$. Then, we will have that $z_j^c\in\Omega_A$, $A(z_j^c)=P$ and $z_j^c\sqsubseteq w$; since A is up-positive, it follows that $w\in\Omega_A$ and $A(w)=P$, against the hypothesis $w\notin\Omega_A$. Hence A_w is well defined.

From this point on, we will show that A_w is a WBPM on X, i.e., that it is up-positive, down-negative and complemented positive.

Let $w_1, w_2 \in X$ be such that $w_1 \in \Omega_A[w]$, $w_1 \sqsubseteq w_2$ and suppose that $A_w(w_1) = P$. If $w_1 \in \Omega_A$, we can proceed as in the case $\xi = P$. Then, suppose that $w_1 \in \{z_1 \cdots z_t\}$ and $w_1 \notin \Omega_A$. Since $\{z_1, \cdots, z_t\} =\uparrow \{v_1^c, \cdots, v_q^c\}$ and $w_1 \in \{z_1, \cdots, z_t\}$, by the hypothesis $w_1 \sqsubseteq w_2$, it follows that $w_2 \in\uparrow \{v_1^c, \cdots, v_q^c\}$, and hence $w_2 \in \Omega_A[w]$ and $A_w(w_2) = P$ thanks to (20). This proves that A_w is up-positive.

Let $w_1, w_2 \in X$ be such that $w_2 \in \Omega_A[w]$, $w_1 \sqsubseteq w_2$ and $A_w(w_2) = N$. If $w_2 \in \Omega_A$, then we can use the same argument as in the case $\xi = P$. If $w_2 \in \{v_1, \cdots, v_q\}$ and $w_2 \notin \Omega_A$, thanks to the equality $\{v_1, \cdots, v_q\} =\downarrow w$, and to the hypothesis $w_1 \sqsubseteq w_2$, it follows that $w_1 \in \{v_1, \cdots v_q\}$, and hence $w_1 \in \Omega_A[w]$ and $A_w(w_1) = N$ thanks to (20). This proves that A_w is down-negative

Finally, suppose that $u \in \Omega_A[w]$ is such that $A_w(u) = N$. If $u \in \Omega_A$, we can use the same argument as in the case $\xi = P$. If $u \in \{v_1, \cdots, v_q\}$, then $u = v_i$ for some $i \in \{1, \cdots q\}$, and hence $u^c = v_i^c \in \{z_1, \cdots, z_t\} =\uparrow \{v_1^c, \cdots, v_q^c\}$. Therefore, $u^c \in \Omega_A[w]$ and $A_w(u^c) = P$. This proves that A_w is also complemented positive and the thesis follows. □

Hence, as proved, the rules $wt_1) - wt_4)$ define a transition function

$$\delta : w\mathscr{S}\mathscr{T}_2(X) \times w\mathscr{T}\mathscr{R}_2(X) \to w\mathscr{S}\mathscr{T}_2(X).$$

Therefore, the automaton $w - \mathscr{A}\mathscr{U}\mathscr{T}(X, \mathbf{2})$ is formally given by

$$\langle w\mathscr{S}\mathscr{T}_2(X), w\mathscr{T}\mathscr{R}_2(X), w\mathfrak{I}_{2,X}, \delta, \mathscr{W}(X, \mathbf{2})\rangle.$$

Proposition 5.1. Let (A, Ω_A), and (B, Ω_B) two non-exit states of $wt - \mathscr{A}\mathscr{U}\mathscr{T}(X, \mathbf{2})$ (i.e. two WBPM maps on X) and let $w\xi$ be a transition symbol such that $\delta(A, w\xi) = B$. Then $(A, \Omega_A) \trianglelefteq (B, \Omega_B)$.

Proof. Since B is a non-exit state, it can coincide with A or it can be of the form (17) or of the form (18). In all cases, $\Omega_A \subseteq \Omega_B$ and that $B_{|\Omega_A} = A$, from which the assertion follows. □

As in the previous section, the preceding proposition asserts that the transition function can move a state A to an *exit-state* or to a state B which in the Hasse diagram of $\mathscr{W}\mathscr{P}(X, \mathbf{2})$ is necessarily above A.

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