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Integrals Involving Normal PDF and CDF and Related Series

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Abstract: This paper expresses integrals of the normal distribution function and its cumulative function as a single series. Basically, to obtain this series, all functions are expanded using Taylor series and binomial expansions resulting in nested multiple series. Then, by applying some transformations and changing the order of the summations, we end up with a single series of special functions. Truncation of the series can be used to approximate the integrals. Besides, the sum of some infinite series involving Hermite polynomials, that correspond to integrals with known closed forms, are now obtained.

Keywords: Normal distribution function, Gamma function, Hypergeometric function, Hermite polynomial, Taylor series

1 Introduction

The integrals of the normal distribution function and its cumulative function appear in many applications such as: the cumulative bivariate normal integral in statistics for biometric and financial data, computation of bit error probabilities in communication [1,2,3], the study of transient heat conduction and diffusion [4] and Gaussian process modeling in machine learning ([5]). Unfortunately there is no closed form for such integrals. However, the cumulative bivariate normal integral has different representations as infinite single series containing special functions such as incomplete Gamma function and/or Hermite polynomials [1,2,3]. In this paper, we derive a generalized form of the scheme proposed in [3] to evaluate some other integrals and their related series. The main advantage of the resultant series is their efficient computation using the recurrence formulas of the special functions.

2 Integrals and the Proposed Scheme

In this section, we describe a methodology that can be used to prove the following formulas (for $d>0$, $a^2<d$ and $c^2<d$):

$$I(a,b,d,k,x)=\int_0^x t^k\exp\left(-dt^2\right)erf\left(at+b\right)dt$$

$$=sign(x)^k\left\{\frac{sign(x)d^{-\frac{k+1}{2}}}{2}\gamma(\frac{k+1}{2},x^2d)erf(b)\right.$$

$$+\frac{\exp(-b^2)}{\sqrt{\pi}}\sum_{u=0}^{\infty}\left[\frac{a^{2u+1}d^{-u-\frac{k}{2}-1}}{(2u+1)!}\gamma(u+\frac{k}{2}+1,x^2d)H_{2u}(b)\right.$$

$$\left.\left.-\frac{sign(x)a^{2u+2}d^{-u-\frac{k}{2}-\frac{3}{2}}}{(2u+2)!}\gamma(u+\frac{k}{2}+\frac{3}{2},x^2d)H_{2u+1}(b)\right]\right\}$$

$$,x\geq 0,k>-1\;or\,x<0,k\in Z^+ \tag{1}$$

$$K(a,b,d,c,x)=\int_0^x\frac{e^{-dt^2}}{t}\left[erf\left(at+b\right)-erf(ct+b)\right]dt=$$

$$\frac{e^{-b^2}}{\sqrt{\pi}}\sum_{u=0}^{\infty}\left\{sign(x)\frac{(\frac{a}{\sqrt{d}})^{2u+1}-(\frac{c}{\sqrt{d}})^{2u+1}}{(2u+1)!}\gamma(u+\frac{1}{2},x^2d)H_{2u}(b)\right.$$

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$$-\frac{(\frac{a}{\sqrt{d}})^{2u+2}-(\frac{c}{\sqrt{d}})^{2u+2}}{(2u+2)!}\gamma(u+1,x^2d)H_{2u+1}(b)\Bigg\} \quad (2)$$

Proof. The integral in (1) can be expressed as an infinite series of the incomplete Gamma function and Hermite polynomial as follows.

Using Taylor series expansions of the exponential and error functions, we get:

$$I(a,b,d,k,x)=$$
$$\frac{2}{\sqrt{\pi}}\int_{t=0}^{x}t^k\sum_{q=0}^{\infty}\frac{(-d)^q}{q!}t^{2q}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}(at+b)^{2n+1}dt$$

Using the binomial expansion of $(at+b)^{2n+1}$ and integrating, we get:

$$I(a,b,d,k,x)=$$
$$\frac{2}{\sqrt{\pi}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{2n+1}\frac{(-1)^{n+q}2n!a^{2n-m+1}d^qx^{2n+k+2q-m+2}b^m}{(2n+k+2q-m+2)q!n!m!2n-m+1!}$$

Let us divide the inner summation into two summations for even and odd values of m as follows:

$$I(a,b,d,k,x)=\frac{2}{\sqrt{\pi}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n+q}d^q\Gamma(2n+1)}{q!\Gamma(n+1)}$$
$$\cdot\Bigg\{\frac{a^{2n-2m+1}x^{2n+k+2q-2m+2}b^{2m}}{(2n+k+2q-2m+2)\Gamma(2m+1)\Gamma(2n-2m+2)}$$
$$+\frac{a^{2n-2m}x^{2n+k+2q-2m+1}b^{2m+1}}{(2n+k+2q-2m+1)\Gamma(2m+2)\Gamma(2n-2m+1)}\Bigg\}$$
$$=\frac{2}{\sqrt{\pi}}(I_1+I_2)$$

I_1 can be written with the interior summation changed to start from $m=-\infty$ (this would not change the sum since the added terms are all zeros), then, using the transformation $u=n-m$ leads to:

$$I_1=\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{n+q}\Gamma(2n+1)a^{2u+1}d^qx^{2u+k+2q+2}b^{2n-2u}}{(2u+k+2q+2)q!n!(2u+1)!\Gamma(2n-2u+1)}$$

By changing the order of summation and using the transformation $v=n-u$, we get:

$$I_1=\sum_{u=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-1)^{u+v+q}(2u+2v)!a^{2u+1}d^qx^{2u+k+2q+2}b^{2v}}{(2u+k+2q+2)q!(2v)!(2u+1)!(u+v)!}$$

where the inner summation counter is changed to start from 0 rather than u since for $v<0$ the terms vanish. Using the duplication formula of Gamma function ([6] p. 256) and the series expansion of the incomplete Gamma function ([6] p. 260, 262):

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\frac{1}{2})$$
$$\gamma(a,x)=\sum_{q=0}^{\infty}\frac{(-1)^qx^{q+a}}{q!(q+a)}$$

we get:

$$I_1=\sum_{u=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-1)^{u+v+q}2^{2u}\Gamma(u+v+\frac{1}{2})}{(2u+k+2q+2)q!v!(2u+1)!\Gamma(v+\frac{1}{2})}\cdot$$
$$a^{2u+1}d^qx^{2u+k+2q+2}b^{2v}$$
$$=\frac{sign(x)^k}{2}\sum_{u=0}^{\infty}\frac{(-1)^ua^{2u+1}d^{-u-\frac{k}{2}-1}}{u!(2u+1)}\gamma(u+\frac{k}{2}+1,x^2d)\cdot$$
$${}_1F_1(u+\tfrac{1}{2},\tfrac{1}{2};-b_1^2)$$

where $(\alpha)_v=\alpha(\alpha+1)\cdots(\alpha+v-1),(\alpha)_0=1$ and ${}_1F_1(\alpha,\beta;x)$ is the confluent hypergeometric function given by:

$${}_1F_1(\alpha,\beta;x)=\sum_{v=0}^{\infty}\frac{(\alpha)_v}{v!(\beta)_v}x^v$$

Following a similar procedure, I_2 can be expressed as:

$$I_2=\sum_{u=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-1)^{u+v+q}\Gamma(2u+2v+1)}{q!(2u+k+2q+1)(u+v)!(2v+1)!(2u)!}$$
$$\cdot a^{2u}d^qx^{2u+k+2q+1}b^{2v+1}$$
$$=\frac{bsign(x)^{k+1}}{2}\sum_{u=0}^{\infty}\frac{(-1)^ua^{2u}d^{-u-\frac{k}{2}-\frac{1}{2}}}{u!}\gamma(u+\tfrac{k}{2}+\tfrac{1}{2},x^2d)\cdot$$
$${}_1F_1(u+\tfrac{1}{2},\tfrac{3}{2};-b^2)$$

Therefore:

$$I(a,b,d,k,x)=\int_{t=0}^{x}t^ke^{-t^2}erf(at+b)dt=\frac{2}{\sqrt{\pi}}(I_1+I_2)$$
$$=\frac{sign(x)^k}{\sqrt{\pi}}\sum_{u=0}^{\infty}\Bigg[\frac{(-1)^ua^{2u+1}d^{-u-\frac{k}{2}-1}}{u!(2u+1)}\gamma(u+\frac{k}{2}+1,x^2d)\cdot$$
$${}_1F_1(u+\tfrac{1}{2},\tfrac{1}{2};-b^2)$$
$$+sign(x)\frac{(-1)^uba^{2u}d^{-u-\frac{k}{2}-\frac{1}{2}}}{u!}\gamma(u+\frac{k}{2}+\tfrac{1}{2},x^2d)\cdot$$
$${}_1F_1\left(u+\tfrac{1}{2},\tfrac{3}{2};-b^2\right)\Bigg]$$

This formula can be expressed in terms of the Hermite polynomial and the incomplete Gamma function using the following relations ([7] p. 309, p.313),

$$\int_0^{\infty}t^{2u}\cos(2xt)\exp(-t^2)\,dt=\frac{1}{2}\Gamma(u+\tfrac{1}{2}){}_1F_1(u+\tfrac{1}{2},\tfrac{1}{2};-x^2)$$
$$=\sqrt{\pi}\frac{(-1)^u}{2^{2u+1}}\exp(-x^2)H_{2u}(x)$$

$$\int_0^{\infty}t^{2u+1}\sin(2xt)\exp(-t^2)\,dt=x\Gamma(u+\tfrac{3}{2}){}_1F_1(u+\tfrac{3}{2},\tfrac{3}{2};-x^2)$$
$$=\sqrt{\pi}\frac{(-1)^u}{2^{2u+2}}\exp(-x^2)H_{2u+1}(x)$$

where $H_j(x)$ is the other standard form of the Hermite polynomial given by:

$$H_j(x) = (-1)^j e^{x^2} D^j e^{-x^2} = j! \sum_{k=0}^{[j/2]} \frac{(-1)^k}{k!(j-2k)!} (2x)^{j-2k}$$

where

$$[j/2] = \begin{cases} j/2 & j \, is \, even \\ (j-1)/2 & j \, is odd \end{cases}$$

$${}_1F_1(\tfrac{1}{2}, \tfrac{3}{2}; -x^2) = \frac{1}{x\sqrt{\pi}} \int_0^\infty \frac{\sin(2xt)}{t} \exp(-t^2)\, dt = \frac{erf(x)}{2x}$$

$$\gamma(\tfrac{1}{2}, x^2) = \sqrt{\pi} erf(x)$$

Therefore,

$$I(a,b,d,k,x) = \int_0^x t^k \exp\left(-dt^2\right) erf\,(at+b)\, dt$$

$$= sign(x)^k \left\{ \frac{sign(x) d^{-\frac{k+1}{2}}}{2} \gamma(\frac{k+1}{2}, x^2 d) erf(b) \right.$$

$$+ \frac{\exp(-b^2)}{\sqrt{\pi}} \sum_{u=0}^{\infty} \left[\frac{a^{2u+1} d^{-u-\frac{k}{2}-1}}{(2u+1)!} \gamma(u+\frac{k}{2}+1, x^2 d) H_{2u}(b) \right.$$

$$\left.\left. - \frac{sign(x) a^{2u+2} d^{-u-\frac{k}{2}-\frac{3}{2}}}{(2u+2)!} \gamma(u+\frac{k}{2}+\frac{3}{2}, x^2 d) H_{2u+1}(b) \right] \right\}$$

Convergence of the above series is studied below.

$$I(a,b,d,k,x) = const. + [S_1 - S_2]$$

$$|S_1| \le \frac{d^{-\frac{k}{2}-1} e^{-b^2}}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{a^{2u+1} d^{-u}}{(2u+1)!} \gamma(u+\frac{k}{2}+1, x^2 d) \, |H_{2u}(b)|$$

$$\le \frac{d^{-\frac{k}{2}-1} \exp(-\frac{b^2}{2})}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{2^{2u+1} u! a^{2u+1} d^{-u}}{(2u+1)!} \gamma(u+\frac{k}{2}+1, x^2 d)$$

$$= d^{-\frac{k}{2}-1} \exp(-\frac{b^2}{2}) \sum_{u=0}^{\infty} \frac{a^{2u+1} d^{-u}}{\Gamma(u+\frac{3}{2})} \gamma(u+\frac{k}{2}+1, x^2 d)$$

$$\le d^{-\frac{k}{2}-1} \exp(-\frac{b^2}{2}) \sum_{u=0}^{\infty} \frac{a^{2u+1} d^{-u}}{u!} \gamma(u+\frac{k}{2}+1, x^2 d)$$

$$= \frac{a d^{-\frac{k}{2}-1} \exp(-\frac{b^2}{2})}{(1-\frac{a^2}{d})^{\frac{k}{2}+1}} \gamma(\frac{k}{2}+1, x^2(d-a^2))$$

where $a^2 < d$ and we have used the following relations ([6] p. 787, [8] p. 646):

$$|H_{2n}(x)| \le \exp\left(\frac{x^2}{2}\right) 2^{2n} n! \left[2 - \frac{1}{2^{2n}} \binom{2n}{n}\right]$$

$$\sum_{u=0}^{\infty} \frac{t^u}{u!} \gamma(u+r, x) = \frac{1}{(1-t)^r} \gamma(r, x-tx)$$

Similarly, using ([6] p. 787):

$$|H_{2n+1}(x)| \le |x| \exp\left(\frac{x^2}{2}\right) \frac{(2n+2)!}{(n+1)!}, x \ge 0$$

we get:

$$|S_2| \le \frac{d^{\frac{-k-3}{2}} e^{-b^2}}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{a^{2u+2} d^{-u}}{(2u+2)!} \gamma(u+\frac{k+3}{2}, x^2 d) \, |H_{2u+1}(b)|$$

$$\le d^{\frac{-k-3}{2}} |b| \frac{\exp(-\frac{b^2}{2})}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{a^{2u+2} d^{-u} (2u+2)!}{(2u+2)!(u+1)!} \gamma(u+\frac{k+3}{2}, x^2 d)$$

$$\le d^{\frac{-k-3}{2}} |b| \frac{\exp(-\frac{b^2}{2})}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{a^{2u+2} d^{-u}}{u!} \gamma(u+\frac{k+3}{2}, x^2 d)$$

$$= \frac{d^{\frac{-k-3}{2}} a^2 |b| \exp(-\frac{b^2}{2})}{\sqrt{\pi} (1-\frac{a^2}{d})^{\frac{k+3}{2}}} \gamma(\frac{k+3}{2}, x^2(d-a^2))$$

Therefore S_1 and S_2 converge.

Following the same procedure and using the following upper bound for the incomplete Gamma function [9]:

$$\gamma(a,x) \le \frac{x^a}{a(a+1)} (1 + a\exp(-x))$$

it is easy to prove the other formula (2).

The above integrals can be computed efficiently using the above series representations and the recurrence formulas of the incomplete Gamma function and the Hermite polynomials ([6] p. 262, 782):

$$\gamma(a+1,x) = a\gamma(a,x) - x^a e^{-x}$$

$$H_{u+1}(x) = 2xH_u(x) - 2uH_{u-1}(x)$$

Some special cases of the above integrals are listed in the appendix.

3 Summation of Some Series

Using the above series expressions of the integrals, the summation of some series could be obtained. For $|a| < 1, |z| < 1$, $s > 0$, we have:

$$\frac{2e^{-b^2}}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{(\frac{a}{2})^{2u+2}}{(u+1)!} H_{2u+1}(b) = erf(b) - erf(\frac{b}{\sqrt{1+a^2}}) \quad (3)$$

$$2\sum_{u=0}^{\infty} \frac{(\frac{a}{2})^{2u+2}}{(u+1)!} H_{2u}(b) = \sqrt{1+a^2} \exp\left(\frac{a^2 b^2}{1+a^2}\right) - 1 - \sqrt{\pi} b \exp(b^2) \left[erf(b) - erf(\frac{b}{\sqrt{1+a^2}})\right] \quad (4)$$

$$\frac{2\exp\left(s-b^2\right)}{\sqrt{\pi}}\sum_{u=0}^{\infty}\frac{(-1)^u z^{u+1}}{u+1!}\gamma\left(\frac{u+3}{2},s\right)H_u(b)=$$
$$erf(b)-erf\left(z\sqrt{s}+b\right)+\frac{z\exp\left(\frac{b^2z^2}{z^2+1}\right)}{\sqrt{z^2+1}}\cdot$$
$$\left[erf\left(\sqrt{s\left(z^2+1\right)}+\frac{zb}{\sqrt{z^2+1}}\right)-erf\left(\frac{zb}{\sqrt{z^2+1}}\right)\right]\Bigg\} \quad (5)$$

$$\sum_{u=0}^{\infty}\frac{(-1)^u\left(\frac{z}{2}\right)^u}{\Gamma\left(\frac{u}{2}+1\right)}H_u(b)=\frac{1}{\sqrt{z^2+1}}\exp\left(\frac{b^2z^2}{z^2+1}\right)erfc\left(\frac{bz}{\sqrt{z^2+1}}\right) \quad (6)$$

Proof. From (1),

$$\int_{-\infty}^{\infty}\exp\left(-t^2\right)erf(at+b)\,dt$$
$$=\sqrt{\pi}erf(b)-\frac{2\exp(-b^2)}{\sqrt{\pi}}\sum_{u=0}^{\infty}\frac{a^{2u+2}}{(2u+2)!}\Gamma(u+\frac{3}{2})H_{2u+1}(b)$$

Using the fact that [10]:

$$\int_{-\infty}^{\infty}\exp\left[-(\alpha t+\beta)^2\right]erf(at+b)\,dt=\frac{\sqrt{\pi}}{\alpha}erf\left[\frac{\alpha b-\beta a}{\sqrt{\alpha^2+a^2}}\right]$$

leads to (3).

Differentiating with respect to b and using the following formula ([8], p. 708):

$$\sum_{u=0}^{\infty}\frac{\left(\frac{a}{2}\right)^{2u}}{u!}H_{2u}(b)=\frac{1}{\sqrt{1+a^2}}\exp\left(\frac{a^2b^2}{1+a^2}\right),|a|<1$$

leads to (4).

From (1) and for $x>0$,

$$I(a,b,d,1,x)=\int_0^x t\exp\left(-dt^2\right)erf(at+b)\,dt$$
$$=\frac{1}{2d}\left(1-\exp\left(-x^2d\right)\right)erf(b)+$$
$$\frac{\exp(-b^2)}{d\sqrt{\pi}}\sum_{u=0}^{\infty}\frac{(-1)^u\left(\frac{a}{\sqrt{d}}\right)^{u+1}}{u+1!}\gamma\left(\frac{u+3}{2},x^2d\right)H_u(b)$$

However, the closed form of this integral is ([8], p. 32):

$$\int_0^x t\exp\left(-dt^2\right)erf(at+b)\,dt=$$
$$=\frac{a\exp\left(\frac{-db^2}{a^2+d}\right)}{2d\sqrt{a^2+d}}\left[erf\left(x\sqrt{a^2+d}+\frac{ab}{\sqrt{a^2+d}}\right)\right.$$
$$\left.-erf\left(\frac{ab}{\sqrt{a^2+d}}\right)\right]-\frac{1}{2d}\left[e^{-x^2d}erf(ax+b)-erf(b)\right]$$

So by comparison and putting $\sqrt{d}z=a, s=x^2d$, we get (5). By taking the limit as $s\to\infty$, we get (6).

4 Conclusions

In this paper, a scheme of transformation of variables and interchanging multiple series is incorporated that successfully leads to expressing some integrals involving the normal distribution function and its cumulative function as a single series of special functions. Truncation of the obtained series can be used efficiently to evaluate the integrals. Moreover, the summations of some infinite series involving Hermite polynomials are obtained.

5 Appendix

In the sequel, we report some special cases for the studied integrals.

$$I(a,b,d,k,\infty)=\int_0^{\infty}t^k\exp\left(-dt^2\right)erf(at+b)\,dt$$
$$=\frac{1}{2}d^{\frac{-k-1}{2}}\Gamma(\frac{k+1}{2})erf(b)$$
$$+\frac{\exp(-b^2)}{\sqrt{\pi}}\sum_{u=0}^{\infty}\frac{(-1)^u a^{u+1}d^{-\frac{u+k+2}{2}}}{(u+1)!}\Gamma(\frac{u+k}{2}+1)H_u(b)$$
$$,k>-1,a^2<d$$

$$I(a,b,1,0,\infty)=\int_0^{\infty}\exp\left(-t^2\right)erf(at+b)\,dt$$
$$=\frac{\sqrt{\pi}}{2}erf(b)+\frac{e^{-b^2}}{\sqrt{\pi}}\sum_{u=0}^{\infty}\frac{(-1)^u a^{u+1}}{(u+1)!}\Gamma(\frac{u+2}{2})H_u(b)$$
$$,|a|<1$$

$$K(a,b,d,c,\infty)=\int_0^{\infty}\frac{e^{-dt^2}}{t}\left[erf(at+b)-erf(ct+b)\right]dt$$
$$=e^{-b^2}\sum_{u=0}^{\infty}(-1)^u\frac{\left(\frac{a}{2\sqrt{d}}\right)^{u+1}-\left(\frac{c}{2\sqrt{d}}\right)^{u+1}}{\left(\frac{u+1}{2}\right)\Gamma\left(\frac{u}{2}+1\right)}H_u(b)$$
$$,a^2<d,c^2<d$$

For $x>0$,

$$K(a,b,d,c,x)=\int_0^x\frac{e^{-dt^2}}{t}\left[erf(at+b)-erf(ct+b)\right]dt$$
$$=\frac{e^{-b^2}}{\sqrt{\pi}}\sum_{u=0}^{\infty}(-1)^u\frac{\left(\frac{a}{\sqrt{d}}\right)^{u+1}-\left(\frac{c}{\sqrt{d}}\right)^{u+1}}{(u+1)!}\gamma(\frac{u+1}{2},x^2d)H_u(b)$$
$$,a^2<d,c^2<d$$

$$K(a,b,0,c,x)=\int_0^x\frac{1}{t}\left[erf(at+b)-erf(ct+b)\right]dt$$
$$=\frac{2\exp(-b^2)}{\sqrt{\pi}}\sum_{u=0}^{\infty}(-1)^u\frac{\left[(ax)^{u+1}-(cx)^{u+1}\right]}{(u+1)(u+1)!}H_u(b)$$

$$K(a,b,1,0,x) = \int_0^x \frac{e^{-t^2}}{t}\left[erf(at+b) - erf(b)\right]dt$$

$$= \frac{e^{-b^2}}{\sqrt{\pi}} \sum_{u=0}^{\infty} \frac{(-1)^u a^{u+1}}{(u+1)!} \gamma\left(\frac{u+1}{2}, x^2\right) H_u(b)\ , |a| < 1$$

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