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Analytical Solution of ψ Fractional Initial Value Problems

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Abstract: This paper gives the solution of the generalized fractional Initial Value Problems (IVP) by using Sumudu transform. Using this we obtained compact solution of Cauchy initial value problem.

Keywords: Ψ-Fractional derivative, Sumudu transform, initial value problem.

1 Introduction

The fractional differential equations plays very important role in various field of science and technology. Since last three decades the development in this field is very vast. Because of this it became one of the important branch of the research in the field of the Mathematics. In this journey of the fractional calculus several fractional operators are developed which is started from Riemann-Liouville(R-L), Caputo, Hadamard, Ritz, Hilfer, Katugampola, Ψ-Hilfer etc. Now days the more general fractional operator called Ψ-Hilfer which is very useful and applied in the various fields. Which is appearing in many natural phenomenon. Therefore, these phenomenons forms initial value problems (IVP) or initial boundary value problems (IBVP). For solving these problems several methods are existed in the literature viz. analytical methods, approximation methods and numerical methods. But the integral transform methods are the most powerful methods to solve such problem. The number of researcher made an attempt to solve the fractional IVP and fractional IBVP by any one of the above methods. The popular among them are integral transform method [1,2,3,4,5,6,7], iterative method [8], Adomian decomposition method [9,8,10,11,12] and finite difference mehtod[13,14]. As the generalization of fractional operators carried during the last decade the researcher called Almeida et al. [15] studied fractional differential equations with dependence on a Caputo fractional derivative of real order and experiment on fractional differential equations which model more efficiently certain problems than ordinary differential equations. Yang [16] introduce new integral transform method which is different from the Laplace transform, Sumudu transform, and Elazaki transform operators. Katugampola generalized well-known standard fractional Riemann-Liouville integrals and obtained unification to Riemann-Liouville(R-L), Hadamard and other fractional derivatives in [17]. Recently, Abdo et al. [18] concerned with boundary value problem for a nonlinear fractional differential equation involving a general form of Caputo fractional derivative operator with respect to new function Ψ . Sausa and Oliviera in [19] introduced new fractional derivative setting of Ψ-fractional operator called Ψ-Hilfer fractional derivative. Recently, Fahad and Rehman introduced new integral transform called Ψ-Laplace transform and obtained on fractional derivatives in the settings of Ψ-fractional calculus. Moreover found the analytic solutions to some classes of fractional differential equations involving Ψ-RL, Ψ-C and Ψ-Hilfer fractional derivatives in [20] .

In the present paper, we made an attempt to obtain the analytical solution of IVP of generalized fractional order operator by using Sumudu transform. We are presenting the results in the setting of generalized operators Ψ- Caputo and Ψ- Hilfer fractional derivatives.

The plan of the paper is as follow. The basic definitions and terminologies are given in section II. The section III, is devoted for the development of Sumudu transform for generalized fractional operators. In the section IV the result are

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obtain for Sumudu transform of generalised fractional derivatives and integration. Section V is devoted for the application of Sumudu transform for solving Cauchy's IVP.

2 Preliminaries

In this section, we have given basic definitions, formulas and other important results.

Definition 1.[21] *Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$,$m = \mu+1$, f be an integrable function defined on $[a,b]$ and $g \in C^1([a,b])$ be increasing function such that $g^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the left Riemann-Liouville fractional integral and derivative of f with respect to function g of order μ is defined as*

$$({}_aI_g^{\mu} f)(t) = \frac{1}{\Gamma(\mu)} \int_a^t (g(t)-g(u))^{\mu-1} g^{'}(u) f(u) du.$$

$$\begin{aligned}({}_aD_g^{\mu} f)(t) &= \left(\frac{1}{g^{'}(t)}\frac{d}{dt}\right)^n ({}_aI_g^{n-\mu} f)(t) \\ &= \frac{\left(\frac{1}{g^{'}(t)}\frac{d}{dt}\right)^n}{\Gamma(n-\mu)} \int_a^t (g(t)-g(u))^{n-\mu-1} g^{'}(u) f(u) du.\end{aligned}$$

It is to be noted that for $g(t) \to t$, ${}_aI_g^{\mu} f(t) \to {}_a I_g^{\mu} f(t)$ which is the standard Riemann- Liouville integral. Moreover for $g(t) \to ln(t)$ the integral defined in (1) approaches to the Hadamard fractional integral.

Inspired by Caputo's concept [22] of fractional derivative, Almeidea [23] presents the following Caputo version of (2) and studies some important properties of fractional calculus.

Definition 2.[23] *Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu+1$, f, $g(t) \in C^m$ ($[a, b]$) be the functions such that $g(t)$ is increasing and $g(t) \neq 0$ for all $t \in [a, b]$. Then, the left Caputo fractional derivative of a function f of order μ is defined as*

$${}_a{}^C D_g^{\mu} = {}_a I_g^{n-\mu} f(t) \left(\frac{1}{g^{'}(t)}\frac{d}{dt}\right)^n$$

Taking $g \to ln(t)$ and $g \to t$, we get the Caputo-type Hadamard fractional derivative [?] Caputo fractional derivative [21] respectively.

Motivated by the definitions of Riemann-Liouville and Hilfer fractional derivatives, Sousa and Oliveria [19] introduced the $\Psi-$ Hilfer fractional derivative which we recall in the following definition replacing Ψ by g and defined as

Definition 3.*Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu+1$, f, $g \in C^m$ ($[a, b]$) be the functions such that g is increasing and $g \neq 0$ for all $t \in [a, b]$. Then, the generalized Hilfer fractional derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by*

$${}_aD_g^{\mu,\nu} = {}_a I_g^{\nu(m-\mu)} \left(\frac{1}{g^{'}(t)}\frac{d}{dt}\right)^m {}_a I_g^{(1-\nu)(n-\mu)} f(t).$$

Definition 4.*Let $g \in C^n[a,b]$ such that $g^{'}(t) > 0$ on $[a,b]$. Then*

$$AC_g^n[a,b] = \{f : [a,b] \to \mathbb{C} \quad and \quad f^{[n-1]} \in AC[a,b], f^{[n-1]} = (\frac{1}{g^{'}(t)}\frac{d}{dt})^{n-1} f\}$$

.

and

$$C_{\varepsilon,g}[a,b] = \{f : (a,b] \to \mathbb{R} \quad such\ that \quad (g(t)-g(a))^{\varepsilon} f(t) \in C[a,b]\}$$

,

where $C_{0,g}[a,b] = C[a,b]$.

Definition 5.*The entire function*

$$W(z,\mu,\nu) := \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\mu i+\nu)}, \qquad \mu > -1, \nu \in \mathbb{C}$$

which is valid in the whole complex plane, is known as the Wright function. In [21] Wright investigation in the asymptotic theory of partitions. In [21], Mittag-leffler introduced the well-known Mittag-Leffler function $E_{\mu}(z)$, given by

$$E_{\mu}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(\mu i+1)},\qquad \mu\in\mathbb{C},\quad Re(\mu)>0.$$

In [21] Wiman discussed a natural generalization of $E_{\mu}(z)$ and two parameter Mittag-Leffler function $E_{\mu}(z)$ as

$$E_{\mu,\nu}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(\mu i+\nu)},\qquad \mu,\nu\in\mathbb{C},\quad Re(\mu)>0.$$

Some important properties of this function can be seen in [24].

Definition 6.[25] *Let f and h be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. We define the generalized convolution of f and h by*

$$((f*_{g}h)(t))=\int_{a}^{t}f(\tau)h\Big(g^{-1}(g(t)-g(a)-g(\tau))\Big)g^{'}(\tau)d\tau$$

The generalized convolution of two functions is commutative.

Lemma 1.[25] *Let f and h be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. Then*

$$f*_{g}h=h*_{g}f.$$

Proof. The proof can be easily stated once the change of variable $u=g^{-1}(g(t)+g(a)-g(\tau))$ is utilized.

Lemma 2.[21] *Let $\mathbb{R}(\mu)>0$ and $\mathbb{R}(\nu)>0$, then*

$$_{a}I_{g}^{\mu}(g(x)-g(a))^{\nu-1}(t)=\frac{\Gamma(\nu)}{\Gamma(\nu+\mu)}(g(t)-g(a))^{\nu+\mu-1}\tag{1}$$

Lemma 3.[21] *Let $\mathbb{R}(\mu)\geq 0$ and $\mathbb{R}(\nu)>0$, then*

$$_{a}D_{g}^{\mu}(g(x)-g(a))^{\nu-1}(t)=\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}(g(t)-g(a))^{\nu-\mu-1}\tag{2}$$

Theorem 1.[25] *Let $\mu>m, m\in\mathscr{N}$. Then,*

$$(\frac{1}{g^{'}(t)}\frac{d}{dt})^{m}_{a}I_{g}^{\mu}f(t)=_{a}I_{g}^{\mu-m}f(t).\tag{3}$$

Proof.

$$(\frac{1}{g^{'}(t)}\frac{d}{dt})^{m}_{a}I_{g}^{\mu}f(t)=\frac{(\frac{1}{g^{'}(t)}\frac{d}{dt})^{m}}{\Gamma(\mu)}\int_{a}^{t}(g(t)-g(u))^{\mu-1}f(u)g^{'}(u)du.\tag{4}$$

The result is then obtained by applying the operator $(\frac{1}{g^{'}(t)}\frac{d}{dt})$ m-times to the integral on the right hand side.

3 Main Results

In this section we are obtaining the main results by using Sumudu transform.

3.1 Generalized Sumudu Transform

We consider functions in the set A is defined by,

$$A = \{f(t)/\exists M, \tau_1, \tau_2 |f(t)| \le M e^{|t|/T_j}, if \quad t\varepsilon(-1^j) \times [0,\infty)\}$$

Definition 7. *Let $f,g : [0,\infty) \to \mathbb{R}$ be a real valued functions such that $g(t)$ is continuous and $g(t)' > 0$ on $[0,\infty)$. The generalized Sumudu transform of f is defined by*

$$T(u) := S_g\{f(t)\}(u) := \frac{1}{u}\int_a^\infty e^{-\frac{(g(t)-g(a))}{u}} f(t) g'(t) dt \tag{5}$$

for all u.

Theorem 2. *Let $f,g : [a,\infty) \to \mathbb{R}$ be a real valued functions such that $g(t)$ is continuous and $g(t)' > 0$ on $[0,\infty)$ and such that the generalized Sumudu transform of f exists. Then*

$$S_g\{f(t)\}(u) = S\left\{f\left(g^{-1}(t+g(a))\right)\right\}(u), \tag{6}$$

where $S\{f\}$ is the usual Sumudu transform of f.

Proof. The proof is straight forward if one use the change of variable $u = g(t) - g(a)$ in equation (5)

Definition 8. *A function $f : [0,\infty) \to \mathbb{R}$ is of $g(t)-$exponential order $c > 0$ if there exist non-negative constants M,c,T such that for all $t > T$*

$$|f(t)| \le M e^{cg(t)}.$$

Theorem 3. *If the generalized Sumudu transform of $f_1 : [a,\infty) \to \mathbb{R}$ exists for $u > c_1$ and the generalized Sumudu transform of $f_1 : [a,\infty) \to \mathbb{R}$ exists for $u > c_2$. Then, for any constants a_1 and a_2, the generalized Sumudu transform of $a_1 f_1 + a_2 f_2$, where a_1 and a_2 are constants, exists and*

$$S_g\{a_1 f_1 + a_2 f_2\}(u) = a_1 S_g\{f_1(t)\}(u) + a_2 S_g\{f_2(t)\}(u), \quad for \quad u > max\{c_1, c_2\}.$$

Lemma 4. *The generalized Sumudu transform of some elementary functions were given in the following lemma.*
(a) $S_g\{1\}(u) = 1, \quad u > 0.$
(b) $S_g\{(g(t)-g(a))^\mu\}(u) = u^\mu \Gamma(\mu+1), \quad for \quad u > 0.$
(c) $S_g\{e^{\lambda g(t)}\}(u) = \frac{e^{\lambda g(a)}}{1-u\lambda}, \quad for \quad u > \lambda.$

Lemma 5. *Let $\mathbb{R} > 0$ and $|\lambda u^\mu| < 1$. Then*

$$S_g\left\{E_\mu(\lambda((g(t)-g(a)))^\mu)\right\}(u) = \frac{1}{1-\lambda u^\mu}, \tag{7}$$

and

$$S_g\left\{(g(t)-g(a))^{\nu-1} E_{\mu,\nu}(\lambda(g(t)-g(a))^\mu)\right\}(u) = \frac{u^{\nu-1}}{1-\lambda u^\mu} \tag{8}$$

Proof. First we will prove (7)

$$\begin{aligned} S_g\left\{E_\mu(\lambda((g(t)-g(a)))^\mu)\right\}(u) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\mu+1)} S_g\{((g(t)-g(a)))^{k\mu}\} \\ &= \sum_{k=0}^{\infty} \lambda^k u^{\mu k} \\ &= \frac{1}{1-\lambda u^\mu}. \end{aligned}$$

Hence the equation (7) is proved.

$$S_g(g(t)-g(a))^{\nu-1}E_{\mu,\nu}(\lambda(g(t)-g(a)))^{\mu}(u) = S_g\left\{\sum_{i=0}^{\infty}\frac{\lambda^i((g(t)-g(a)))^{\mu i+\nu-1}}{\Gamma(\mu i+\nu)}\right\}(u)$$
$$= \sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\mu i+\nu)}S_g\{((g(t)-g(a)))^{\mu i+\nu-1}\}$$
$$= \frac{u^{\nu-1}}{1-\lambda u^{\mu}}$$

Thus the proof (8)

Theorem 4. *Let the function $f(t)\in C_g[a,T]$ and of $g(t)$-exponential such that $f^{[1]}(t)$ is piecewise continuous over every finite interval $[a,T]$. Then generalized Sumudu transform of $f^{[1]}(t)$ exists,*

$$S_g\{f^{[1]}(t)\}(u) = \frac{1}{u}S_g\{f(t)\}(u) - \frac{1}{u}f(a). \tag{9}$$

Proof. Let $a < t_1, t_2, ..., t_n < T$ be the points in the interval [a, T] where $f^{[1]}$ is discontinuous. Then we have

$$\frac{1}{u}\int_a^T e^{-\frac{(g(t)-g(a))}{u}}f^{[1]}(t)g'(t)dt = \frac{1}{u}\int_a^T e^{-\frac{g(t)-g(a)}{u}}f'(t)dt$$
$$= \frac{1}{u}\left\{\int_a^{t_1} e^{-\frac{g(t)-g(a)}{u}}f'(t)dt + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} e^{-\frac{g(t)-g(a)}{u}}f'(t)dt\right\}$$
$$+ \int_{t_n}^T e^{-\frac{g(t)-g(a)}{u}}f'(t)dt.$$

Using integrating by parts, we have

$$\frac{1}{u}\int_a^T e^{-\frac{(g(t)-g(a))}{u}}f^{[1]}(t)g'(t)dt = \frac{1}{u}e^{-\frac{g(T)-g(a)}{u}}f(T) - \frac{1}{u}f(0) + \frac{1}{u}\int_{t_0}^T e^{-\frac{g(t)-g(a)}{u}}f(t)g'(t)dt.$$

The result is obtained by taking the limit as $T\to\infty$ of both sides.
The above theorem can be generalized as follows.

Corollary 1. *Let $f\in C_g^{n-1}[a,t)$ such that f^i, $i=0,1,2,...,n-1$ are g-exponential order. Let $f^{[n]}$ be a piecewise continuous function on the interval $[a,T]$. Then, the generalized Sumudu transform of $f^{[n]}(t)$ exists and*

$$S_g\{f^{[n]}(t)\}(u) = u^{-n}S_g\{f(t)\}(u) - \sum_{k=0}^{n-1}u^{-n+k}(f^k)(a). \tag{10}$$

Proof. We can prove this by using mathematical induction.
To be able to find the generalized Sumudu transform of the generalized fractional operators, we need to define the generalized convolution integral.
Below we present the ρ-Sumudu transform of the ρ-convolution integral.

Theorem 5. *Let f and h be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. Then*

$$S_g\{f *_g h\}(u) = uS_g\{f\}S_g\{h\}(u).$$

Proof.

$$S_g\{f\}S_g\{h\}(u) = \frac{1}{u}\int_a^\infty e^{-\frac{(g(t)-g(a))}{u}} f(t)g'(t)dt \frac{1}{u}\int_a^\infty e^{-\frac{(g(t)-g(a))}{u}} h(u)g'(u)du$$
$$= \frac{1}{u}\frac{1}{u}\int_a^\infty \int_a^\infty e^{-\frac{(g(t)-g(u)-2g(a))}{u}} f(t)h(u)g'(t)g'(u)dtdu$$

Now, setting choosing τ satisfying $g(\tau) = g(t) + g(u) - g(a)$, we get

$$S_g\{f\}S_g\{h\}(u) = \frac{1}{u}\frac{1}{u}\int_0^\infty \int_u^\infty e^{-\frac{(g(\tau)-g(a))}{u}} f\big(g^{-1}(g(\tau)-g(u)+g(a))\big)$$
$$\times h(u)g'(\tau)g'(u)d\tau du$$
$$= \frac{1}{u}\frac{1}{u}\int_a^\infty e^{-\frac{(g(\tau)-g(a))}{u}} \left[\int_a^\tau f\big(g^{-1}(g(\tau)-g(u)+g(a))\big)\right.$$
$$\left.\times h(u)g'(u)du\right] g'(\tau)d\tau$$
$$uS_g\{f\}S_g\{h\}(u) = \frac{1}{u}S_g\{f *_g h\}(u).$$

Now, we can find the generalized Sumudu transform of the generalized fractional operators

4 The Generalized Sumudu Transform of the Generalized Fractional Integrals and Derivatives.

In the following theorem, we present the generalized Sumudu transform of the left generalized fractional integral starting at a.

Theorem 6. *Let $\mu > 0$ and f be a piecewise continuous function on each interval $[a,t]$ and of $g(t)$-exponential order. Then*

$$S_g\{({}_aI^{\mu,\rho} f)(t)\} = u^\mu S_g\{f(t)\}. \tag{11}$$

Proof. The proof can be done using the definition of the generalized fractional integral (1). Theorem 5 and Lemma 4 we get

$$S_g\{({}_aI_g^{\mu,\rho} f)(t)\}(u) = \frac{1}{\Gamma(\mu)} S_g\{(g(t)-g(a))^{\mu-1} *_g f(t)\}(u)$$
$$= u\frac{1}{\Gamma(\mu)} u^{\mu-1}\Gamma(\mu)S_g\{f(t)\}$$
$$= u^\mu S_g\{f(t)\}$$

Now, we can find the generalized Sumudu transform of the left generalized fractional derivative.

Corollary 2. *Let $\mu > 0$ and $f \in AC_g^n[a,b]$ for any $b > a, g \in C^m[a,b]$ such that $g'(t) > 0$ and ${}_aI_g^{n-k-\mu,\rho} f, k = 0,1,2,...,n-1$ be of $g(t)$-exponential order. Then*

$$S_g\left\{{}_aD_g^{\mu,} f(t)\right\}(u) = u^{-\mu}S_g\{f(t)\} - \sum_{k=0}^{n-1} u^{-n+k}({}_aI_g^{n-k-\mu})f(a^+). \tag{12}$$

Proof.

$$\begin{aligned} S_g\{({}_aD_g^{\mu})(t)\}(u) &= S_g\{({}_aI^{n-\mu}f)^{[n]}(t)\} \\ &= u^{-n}S_g\{({}_aI_g^{n-\mu}f)(t)\} - \sum_{k=o}^{n-1} u^{-n+k}(({}_aI_g^{n-\mu})^{[k]}f)(a^+) \\ &= u^{-n}u^{n-\mu}S_g\{f(t)\} - \sum_{k=0}^{n-1} u^{-n+k}(({}_aI_g^{n-\mu})^{[k]}f)(a^+) \\ &= u^{-\mu}S_g\{f(t)\} - \sum_{k=0}^{n-1} u^{-n+k}(({}_aI^{n-\mu})^{[k]}f)(a^+). \end{aligned}$$

Now, the proof is completed by using Theorem 6.

Corollary 3. *Let $\mu > 0$ and $f \in AC_{\gamma}^{n}[a,b]$ for any $b > a$ and $f^{[k]}, k = 0,1,2,...,n$ be of $g(t)$-exponential order. Then*

$$S_g\left\{{}_a{}^CD_g^{\mu,}f(t)\right\}(u) = u^{-\mu}\left[S_g\{f(t)\} - \sum_{k=0}^{n-1} u^k(f^k)(a^+)\right]. \tag{13}$$

Proof.

$$\begin{aligned} S_g\left\{{}_a{}^CD_g^{\mu,}f(t)\right\}(u) &= S_g\{({}_aI_g^{n-\mu}f^{[n]})(t)\} \\ &= u^{-\mu+n}S_g\{f^{[n]}(t)\} \\ &= u^{-\mu+n}[u^{-n}S_gf(t)(u) - \sum_{k=0}^{n-1} u^{-n+k}(f^{[k]})(a^+)] \\ &= u^{-\mu}[S_gf(t) - \sum_{k=0}^{n-1} u^{k-\mu}(f^{[k]})(a^+)]. \end{aligned}$$

Theorem 7. *If $\mu > 0$, $m = [\mu]+1$, $0 \le \nu \le 1$, and f(t), ${}_aD_g^k{}_aI_g^{(1-\nu)(m-\mu)}f(t) \in C[0,\infty)$ and of $g(t)$- exponential order for $k = 0,1,2,...,m-1$, while ${}_aD_g^{\mu,\nu}f(t)$ is piecewise continuous on [0, ∞). Then*

$$S_g\left\{{}_aD_g^{\mu,\nu}f(t)\right\} = u^{-\mu}S_g\{f(t)\} - \sum_{k=0}^{m-1} u^{m(\nu-1)-\nu\mu+k-\mu}({}_aI_g^{(1-\nu)(m-\mu)-k}f)(0).$$

Proof. From Definition 3 and 7, we have

$$S_g\left\{{}_aD_g^{\mu,\nu}f(t)\right\} = S_g\left\{{}_aI_g^{\nu(m-\mu)}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^m {}_aI_0^{(1-\nu)(m-\mu)}f(t)\right\}. \tag{14}$$

Using Theorem 3 and 4, we get

$$\begin{aligned} S_g\left\{{}_aD_g^{\mu,\nu}f(t)\right\} &= u^{\nu(m-\mu)}S_g\left\{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^m {}_aI_g^{(1-\nu)(m-\mu)}f(t)\right\} \\ &= u^{\nu(m-\mu)}\left[u^{-m}u^{(1-\nu)(m-\mu)}S_g\left\{{}_aI_g^{(1-\nu)(m-\mu)}f(t)\right\}\right. \\ &\left. - \sum_{k=0}^{m-1} u^{-m+k}({}_aD_g^k{}_aI_g^{(1-\nu)(m-\mu)}f)(0)\right] \\ &= u^{-\mu}S_g\left\{f(t)\right\} - \sum_{k=0}^{m-1} u^{m(\nu-1)-\nu\mu+k}({}_aI_g^{(1-\nu)(m-\mu)-k}f)(0) \end{aligned}$$

5 Applications

In this section, the applications of the generalized Sumudu transform is given below.

Theorem 8. *The Cauchy problem*

$$ {}_aD_g^{\mu} y(t) - \lambda y(t) = f(t), \quad t > a, \quad 0 < \mu \leq 1, \quad \lambda \in \mathbb{R}, \qquad (15) $$
$$ ({}_aI_g^{1-\mu})(a^+) = c, \quad c \in \mathbb{R}, $$

has the solution

$$ y(t) = c(g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) $$
$$ + \int_a^t (g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) f(\tau) g^{'}(\tau) d\tau \qquad (16) $$

Proof. Applying the Generalized Sumudu transform to both sides of the equation (15) and then using Corollary 2 with $n = 1$, one gets

$$ \begin{aligned} S_g\{y(t)\} &= c\frac{1}{u(u^{-\mu} - \lambda)} + \frac{1}{u^{-\mu} - \lambda} S_g\{f(t)\} \\ &= \frac{u^{\mu-1}}{1 - \lambda u^{\mu}} + \frac{1}{u}\left[\frac{u^{\mu-1}}{1 - \lambda u^{\mu}} S_g\{f(t)\}\right] \\ &= S_g\left\{c(g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) + (g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) *_g f(t)\right\}. \end{aligned} $$

Taking inverse we get,

$$ \begin{aligned} y(t) &= c(g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) \\ &\quad + (g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) *_g f(t) \\ &= c(g(t) - g(a))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) \\ &\quad + \int_a^t (g(t) - g(\tau))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(\tau))^{\mu}\big) f(\tau) g^{'}(\tau) d\tau. \end{aligned} $$

Example 1. In above theorem if we substitute $g(t) = t$; $g(a) = a$, and $f(t) = 1$ then the solution is

$$ \begin{aligned} y(t) =& c(t-a)^{\nu-1} E_{\mu,\nu}(\lambda(t-a)^{\mu}) + t^{\nu} E_{\mu,\nu+1}(\lambda(t-a)^{\mu}) \\ y(t) =& t^{\nu} E_{\mu,\nu+1}(\lambda(t)^{\mu}) \quad (put c = 0 \quad \text{in above equation}) \end{aligned} $$

Theorem 9. *The Cauchy problem*

$$ {}_a^C D_g^{\mu} y(t) - \lambda y(t) = f(t), \quad t > a, \quad 0 < \mu \leq 1, \quad \lambda \in \mathbb{R}, \qquad (17) $$
$$ y(a^+) = c, \quad c \in \mathbb{R}, $$

has the solution

$$ y(t) = cE_{\mu,\nu}\big(\lambda(g(t) - g(a))^{\mu}\big) + \int_a^t (g(t) - g(\tau))^{\nu-1} E_{\mu,\nu}\big(\lambda(g(t) - g(\tau))^{\mu}\big) f(\tau) g^{'}(\tau) d\tau \qquad (18) $$

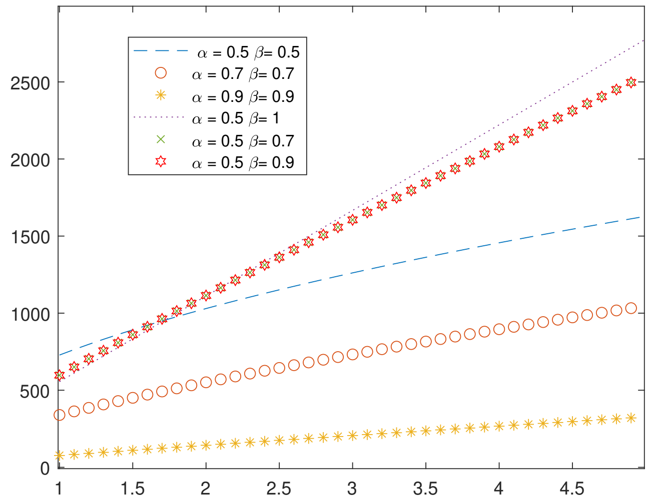

Fig. 1: Graphical representation of the solution for different values of α.

Proof. Applying the generalized Sumudu transform to both sides of the equation (31) and then using Corollary 3 with n=1, we get,

$$\begin{aligned} S_g\{y(t)\} &= c\frac{1}{u^{-\mu}(u^{-\mu}-\lambda)}S_g\{f(t)\} + \frac{1}{u^{-\mu}-\lambda}S\{f(t)\} \\ &= c\frac{1}{1-\lambda u^{\mu}} + \frac{1}{u}\frac{u^{\mu-1}}{1-\lambda u^{\mu}}S_g\{f(t)\} \\ &= cS_g\left\{E_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right)\right\} \\ &+ \frac{1}{u}S_g\left\{(g(t)-g(a))^{\nu-1}E_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right)\right\}S_g\{f(t)\} \\ &= S_g\left\{cE_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right)\right. \\ &\left.+ \frac{1}{u}(g(t)-g(a))^{\nu-1}E_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right) *_g f(t)\right\} \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= cE_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right) + \frac{1}{u}(g(t)-g(a))^{\nu-1}E_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right) *_g f(t) \\ &= cE_{\mu,\nu}\left(\lambda(g(t)-g(a))^{\mu}\right) \\ &+ \int_a^t (g(t)-g(\tau))^{\nu-1}E_{\mu,\nu}\left(\lambda(g(t)-g(\tau))^{\mu}\right)f(\tau)g'(\tau)d\tau. \end{aligned}$$

*Example 2.*In above theorem if we substitute $g(t)=t$; $g(a)=a$, $g'(a)=a$ and $f(t)=1$ then the solution is

$$\begin{aligned} y(t) &= cE_{\mu,\nu}(\lambda(t-a)^{\mu}) + (t-a)^{\nu}E_{\mu,\nu+1}(\lambda(t-a)^{\mu}) \\ &= cE_{\mu,\nu}(\lambda(t)^{\mu}) + (t)^{\nu}E_{\mu,\nu+1}(\lambda(t)^{\mu})\text{for a=0} \\ y(t) &= t^{\nu}E_{\mu,\nu+1}(\lambda(t)^{\mu}) \quad (if\ c=0 \quad \text{in above equation}) \end{aligned}$$

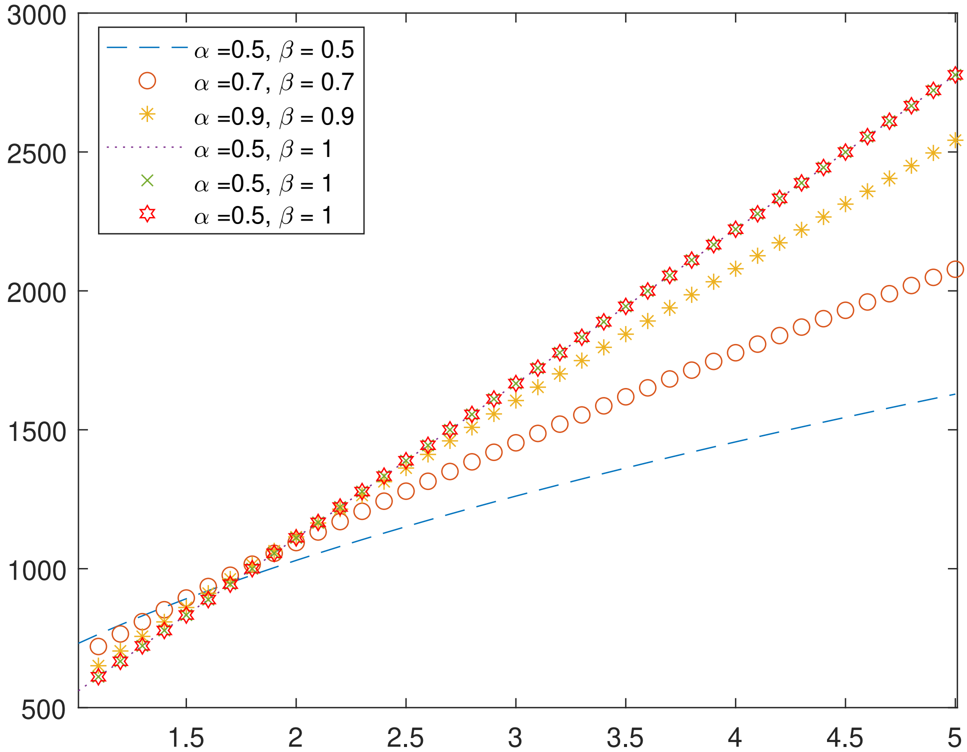

Fig. 2: Graphical representation of the solution for different values of α

6 Conclusion

In the present work, we obtained the solution of generalized fractional initial value problem by using Sumudu transform. Also, the test problem shows the importance of the generalization.

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