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Fractional Riccati Equation Rational Expansion Method For Fractional Differential Equations

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Abstract: In this paper, a new fractional Riccati equation rational expansion method is proposed to establish new exact solutions for fractional differential equations. For illustrating the validity of this method, we apply it to the nonlinear fractional Sharma-Tasso-Olever (STO) equation, the nonlinear time fractional biological population model and the nonlinear fractional foam drainage equation. Compared with the existing results in the literature, more exact solutions are obtained by the proposed method. We also illustrate the application of the established exact solutions.

Keywords: fractional Riccati equation method, fractional differential equations, exact solutions, nonlinear fractional Sharma-Tasso-Olever (STO) equation, nonlinear time fractional biological population model, nonlinear fractional foam drainage equation

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In recent decades, fractional differential equations have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology and other areas. Many articles have investigated some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions [1,2,3,4,5,6,7,8]. Among the investigations for fractional differential equations, research for seeking exact solutions and numerical solutions of fractional differential equations is an important topic, which can also provide valuable reference for other related research. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. For example, these methods include the Adomian decomposition method [9, 10], the variational iterative method [11,12,13], the homotopy perturbation method [14,15], the differential transformation method [16], the finite difference method [17], the finite element method [18] and so on. Based on these methods, a variety of fractional differential equations have been investigated and solved.

Recently, Zhang et al. [19] first proposed a new direct algebraic method named fractional sub-equation method based on the homogeneous balance principle, modified Riemann-liouville derivative by Jumarie [20], and the fractional Riccati equation. The main idea of this method lies in that the solutions of certain fractional differential equations are supposed to have the form $u(\xi)=\sum_{i=0}^{n}a_i\phi^i$, where $\phi=\phi(\xi)$ satisfies the fractional Riccati equation $D_\xi^\alpha\phi=\sigma+\phi^2$. With the aid of mathematical software, the authors established successfully new exact solutions for some fractional differential equations. Then in [21, 22], the authors improved this method to be suitable for more general cases, in which the solutions of certain fractional differential equations are supposed to have the forms $u(\xi)=\sum_{i=-n}^{n}a_i\phi^i$, $u(\xi)=a_0+\sum_{i=1}^{n}a_i(\frac{-\sigma B+D\phi}{D+B\phi})^i$ respectively, where $\phi=\phi(\xi)$ satisfies the fractional Riccati equation $D_\xi^\alpha\phi=\sigma+\phi^2$.

Motivated by the works above, in this paper, by introducing a new ansatz with more general form, we propose a new fractional Riccati equation rational expansion method for solving fractional differential equations, in which the solutions $u(\xi)$ of certain fractional differential equations are supposed to have the

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form

$$u(\xi)=a_0+\sum_{i=1}^{m}\frac{\phi^{i-1}(a_i\phi+b_i\sqrt{D_\xi^\alpha\phi})}{(\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi})^i},$$

where $\phi=\phi(\xi)$ satisfies the fractional Riccati equation $D_\xi^\alpha\phi=\sigma+\phi^2$. We organize the rest of this paper as follows. In Section 2, we give some definitions and properties of Jumarie's modified Riemann-liouville derivative and the description of the fractional Riccati equation rational expansion method. Then in Section 3 we apply the method to solve the nonlinear fractional Sharma-Tasso-Olever (STO) equation, the nonlinear time fractional biological population model, and the nonlinear fractional foam drainage equation. Some conclusions are presented at the end of the paper.

2 Jumarie's modified Riemann-liouville derivative and description of the fractional Riccati equation rational expansion method

The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression [20]:

$$D_t^\alpha f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi,\\ \qquad 0<\alpha<1,\\ (f^{(n)}(t))^{(\alpha-n)},\ n\le\alpha<n+1,\ n\ge1.\end{cases}$$

We list some important properties for the modified Riemann-Liouville derivative as follows (see [20, (3.10)-(3.13)]:

$$D_t^\alpha t^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{1}$$

$$D_t^\alpha(f(t)g(t))=g(t)D_t^\alpha f(t)+f(t)D_t^\alpha g(t), \tag{2}$$

$$D_t^\alpha f[g(t)]=f'_g[g(t)]D_t^\alpha g(t)=D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{3}$$

Suppose that a fractional partial differential equation, say in two or three independent variables x,y,t, is given by

$$P(u,u_t,u_x,u_y,D_t^\alpha u,D_x^\alpha u,D_y^\alpha u\ldots)=0, \tag{4}$$

where $u=u(x,y,t)$ is an unknown function, P is a polynomial in $u=u(x,y,t)$ and its various partial derivatives, in which the highest order derivative and nonlinear term are involved.

Step 1. Suppose that

$$u(x,y,t)=U(\xi),\quad \xi=\xi(x,y,t), \tag{5}$$

and then Eq. (4) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\widetilde{P}(U,\ U',U'',D_\xi^\alpha U,\ \ldots)=0. \tag{6}$$

Step 2. Suppose that the solution of (6) can be expressed in ϕ as follows:

$$U(\xi)=a_0+\sum_{i=1}^{m}\frac{\phi^{i-1}(a_i\phi+b_i\sqrt{D_\xi^\alpha\phi})}{(\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi})^i}, \tag{7}$$

where $\phi=\phi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha\phi=\sigma+\phi^2, \tag{8}$$

and $a_0,\ a_i,\ b_i,\ c_i,\ i=1,2,\ldots,m,\ \mu_0,\ \mu_1,\ \mu_2$ are all constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivative and nonlinear term appearing in (6).

In [23], by using the generalized Exp-function method, Zhang et al. first obtained the following solutions of Eq. (8):

$$\phi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi),\ \sigma<0,\\ -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi),\ \sigma<0,\\ \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi),\ \sigma>0,\\ -\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi),\ \sigma>0,\\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega},\ \omega\ is\ a\ constant,\ \sigma=0,\end{cases} \tag{9}$$

where the generalized hyperbolic and trigonometric functions are defined as

$$\sin_\alpha(\xi)=\frac{E_\alpha(i\xi^\alpha)-E_\alpha(-i\xi^\alpha)}{2i},$$

$$\cos_\alpha(\xi)=\frac{E_\alpha(i\xi^\alpha)+E_\alpha(-i\xi^\alpha)}{2},$$

$$\sinh_\alpha(\xi)=\frac{E_\alpha(\xi^\alpha)-E_\alpha(-\xi^\alpha)}{2},$$

$$\cosh_\alpha(\xi)=\frac{E_\alpha(\xi^\alpha)+E_\alpha(-\xi^\alpha)}{2},$$

$$\tan_\alpha(\xi)=\frac{\sin_\alpha(\xi)}{\cos_\alpha(\xi)},\ \cot_\alpha(\xi)=\frac{\cos_\alpha(\xi)}{\sin_\alpha(\xi)},$$

$$\tanh_\alpha(\xi)=\frac{\sinh_\alpha(\xi)}{\cosh_\alpha(\xi)},\ \coth_\alpha(\xi)=\frac{\cosh_\alpha(\xi)}{\sinh_\alpha(\xi)},$$

where $E_\alpha(\xi)=\sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(1+k\alpha)}$, $\alpha>0$ is the Mittag-Leffler function.

Step 3. Substituting (7) into (6) and using (8), the left-hand side of (6) is converted to another polynomial in $\phi^j(\sqrt{\sigma+\phi^2})^i$ after eliminating the denominator. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_0,\ a_i,\ b_i,\ c_i,\ i=1,2,\ldots,m,\ \mu_0,\ \mu_1,\ \mu_2$.

Step 4. Solving the equations system in Step 3, and by using the solutions of Eq. (8), we can construct a variety of exact solutions of Eq. (4).

3 Applications

In this section, we will apply the described method in Section 2 to some fractional differential equations.

3.1 nonlinear fractional Sharma-Tasso-Olever (STO) equation

We consider the nonlinear fractional Sharma-Tasso-Olever (STO) equation with space- and time-fractional derivatives of the form

$$D_t^\alpha u+3a(D_x^\alpha u)^2+3au^2D_x^\alpha u+3auD_x^{2\alpha}u+aD_x^{3\alpha}u=0,$$

$$0<\alpha\le 1, \tag{10}$$

which is the variation of the following nonlinear fractional Sharma-Tasso-Olever (STO) equation [24,25] with time-fractional derivative of the form

$$D_t^\alpha u+3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}=0.$$

To begin with, we suppose $u(x,t)=U(\xi)$, where $\xi=kx+ct+\xi_0$. Then by use of Eq. (3), Eq. (10) can be turned into

$$c^\alpha D_\xi^\alpha U+3ak^{2\alpha}(D_\xi^\alpha U)^2+3aU^2k^\alpha D_\xi^\alpha U$$

$$+3ak^{2\alpha}UD_\xi^{2\alpha}U+ak^{3\alpha}D_\xi^{3\alpha}U=0. \tag{11}$$

Suppose that the solution of Eq. (11) can be expressed by a polynomial in ϕ as follows:

$$U(\xi)=a_0+\sum_{i=1}^{m}\frac{\phi^{i-1}(a_i\phi+b_i\sqrt{D_\xi^\alpha\phi})}{(\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi})^i}, \tag{12}$$

where $\phi=\phi(\xi)$ satisfies Eq. (8).

Balancing the order between the highest order derivative term and nonlinear term in Eq. (11), we can obtain $m=1$. So we have

$$U(\xi)=a_0+\frac{a_1\phi+b_1\sqrt{D_\xi^\alpha\phi}}{\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi}}, \tag{13}$$

Substituting (13) into (11) and collecting all the terms with the same power of $\phi^j(\sqrt{\sigma+\phi^2})^i$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0=0,\ a_1=-\mu_0k^\alpha,\ b_1=0,\ \sigma=\frac{c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 2:

$$a_0=0,\ a_1=-2\mu_0k^\alpha,\ b_1=0,\ \sigma=\frac{4c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 3:

$$a_0=0,\ a_1=-\mu_0k^\alpha,\ b_1=0,\ \sigma=\frac{4c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 4:

$$a_0=0,\ a_1=-2\mu_0k^\alpha,\ b_1=0,\ \sigma=\frac{c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 5:

$$a_0=0,\ a_1=-\mu_0k^\alpha,\ b_1=\pm\frac{\mu_0}{2}k^\alpha,\ \sigma=\frac{c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 6:

$$a_0=0,\ a_1=-\mu_0k^\alpha,\ b_1=\pm\mu_0k^\alpha,\ \sigma=\frac{c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 7:

$$a_0=0,\ a_1=-\frac{\mu_0}{2}k^\alpha,\ b_1=\pm\frac{\mu_0}{2}k^\alpha,\ \sigma=\frac{c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 8:

$$a_0=0,\ a_1=-\frac{\mu_0}{2}k^\alpha,\ b_1=\pm\mu_0k^\alpha,\ \sigma=\frac{c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 9:

$$a_0=0,\ a_1=-\mu_0k^\alpha,\ b_1=\pm\frac{\mu_0}{2}k^\alpha,\ \sigma=\frac{4c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 10:

$$a_0=0,\ a_1=-\mu_0k^\alpha,\ b_1=\pm\mu_0k^\alpha,\ \sigma=\frac{4c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 11:

$$a_0=0,\ a_1=-\frac{\mu_0}{2}k^\alpha,\ b_1=\pm\frac{\mu_0}{2}k^\alpha,\ \sigma=\frac{4c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Case 12:

$$a_0=0,\ a_1=-\frac{\mu_0}{2}k^\alpha,\ b_1=\pm\mu_0k^\alpha,\ \sigma=\frac{4c^\alpha k^{-3\alpha}}{a},$$

$$\mu_1=0,\ \mu_2=0,\ \mu_0=\mu_0,$$

where μ_0 is an arbitrary constant.

Substituting the results above into Eq. (13), and combining with the solutions of Eq. (8) as denoted in (9) we can obtain a rich variety of exact solutions to the nonlinear fractional Sharma-Tasso-Olever (STO) equation with space- and time-fractional derivatives.

From Cases 1-4 we have the following generalized exact solutions for Eq. (10)

$$\begin{cases} u_{1,1}(x,t)=m\sqrt{-\frac{c^\alpha k^{-\alpha}}{a}}\tanh_\alpha[n\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)], \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}<0, \\ u_{1,2}(x,t)=m\sqrt{-\frac{c^\alpha k^{-\alpha}}{a}}\coth_\alpha[n\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)], \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}<0, \\ u_{1,3}(x,t)=-m\sqrt{\frac{c^\alpha k^{-\alpha}}{a}}\tan_\alpha[n\sqrt{\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)], \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}>0, \\ u_{1,4}(x,t)=m\sqrt{\frac{c^\alpha k^{-\alpha}}{a}}\cot_\alpha[n\sqrt{\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)], \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}>0, \end{cases} \tag{14}$$

where $m=1,\ n=1$ or $m=4,\ n=2$ or $m=2,\ n=2$ or $m=2,\ n=1$.

From Cases 5-12 we have the following generalized exact solutions for Eq. (10)

$$\begin{cases} u_{2,1}(x,t)=m\sqrt{-\frac{c^\alpha k^{-\alpha}}{a}}\tanh_\alpha[p\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)] \\ \qquad \pm n\sqrt{\frac{c^\alpha k^{-\alpha}}{a}\{1-\tanh^2_\alpha[p\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)]\}}, \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}<0, \\ u_{2,2}(x,t)=m\sqrt{-\frac{c^\alpha k^{-\alpha}}{a}}\coth_\alpha[p\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)] \\ \qquad \pm n\sqrt{\frac{c^\alpha k^{-\alpha}}{a}\{1-\coth^2_\alpha[p\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)]\}}, \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}<0, \\ u_{2,3}(x,t)=-m\sqrt{\frac{c^\alpha k^{-\alpha}}{a}}\tan_\alpha[p\sqrt{\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)] \\ \qquad \pm n\sqrt{\frac{c^\alpha k^{-\alpha}}{a}\{1+\tan^2_\alpha[p\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)]\}}, \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}>0, \\ u_{2,4}(x,t)=m\sqrt{\frac{c^\alpha k^{-\alpha}}{a}}\cot_\alpha[p\sqrt{\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)] \\ \qquad \pm n\sqrt{\frac{c^\alpha k^{-\alpha}}{a}\{1+\cot^2_\alpha[p\sqrt{-\frac{c^\alpha k^{-3\alpha}}{a}}(kx+ct+\xi_0)]\}}, \\ \qquad\qquad\qquad\qquad \frac{c^\alpha k^{-\alpha}}{a}>0, \end{cases} \tag{15}$$

where $m=1,\ n=\frac{1}{2}$ or $m=1,\ n=1$ or $m=\frac{1}{2},\ n=\frac{1}{2}$ or $m=\frac{1}{2},\ n=1$, and p is determined by $p=1$ or $p=2$.

3.2 nonlinear time fractional biological population model

We consider the nonlinear time fractional biological population model [19,22]:

$$D_t^\alpha u=(u^2)_{xx}+(u^2)_{yy}+h(u^2-r),\ 0<\alpha\le 1, \tag{16}$$

where $h,\ r$ are constants.

Similar as in [19,22], we suppose $u(x,y,t)=U(\xi)$, where $\xi=kx+iky+ct+\xi_0,\ k,\ c,\ \xi_0$ are all constants with $k,\ c\neq 0$, and i is the unit of imaginary numbers. Then by use of Eq. (3), Eq. (16) can be turned into

$$c^\alpha D_\xi^\alpha U=h(U^2-r). \tag{17}$$

Suppose that the solution of Eq. (17) can be expressed by a polynomial in ϕ as follows:

$$U(\xi)=a_0+\sum_{i=1}^{m}\frac{\phi^{i-1}(a_i\phi+b_i\sqrt{D_\xi^\alpha\phi})}{(\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi})^i}, \tag{18}$$

where $\phi=\phi(\xi)$ satisfies Eq. (8).

Balancing the order between the highest order derivative term and nonlinear term in Eq. (17), we can obtain $m=1$. So we have

$$U(\xi)=a_0+\frac{a_1\phi+b_1\sqrt{D_\xi^\alpha\phi}}{\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi}}, \tag{19}$$

Substituting (19) into (17) and collecting all the terms with the same power of $\phi^j(\sqrt{\sigma+\phi^2})^i$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields

Case 1:

$$a_0=\frac{2hr\mu_1c^{-\alpha}}{\mu_0},\ a_1=-\frac{4h^2r\mu_1^2c^{-\alpha}-\mu_0^2c^{\alpha}}{2h\mu_0},$$

$$b_1=0,\ \mu_2=\pm\frac{\mu_0c^{\alpha}}{h\sqrt{-4r}},\ \sigma=-4h^2rc^{-2\alpha},\ \mu_0=\mu_0,\ \mu_1=\mu_1,$$

where $\mu_0,\ \mu_1$ are arbitrary constants.

Case 2:

$$a_0=\frac{2hr\mu_1c^{-\alpha}}{\mu_0},\ a_1=-\frac{4h^2r\mu_1^2c^{-\alpha}-\mu_0^2c^{\alpha}}{2h\mu_0},$$

$$b_1=\frac{-4h^2r\mu_1\mu_2c^{-\alpha}\pm\sqrt{4h^2r\mu_0^2(\mu_2^2-\mu_1^2)+\mu_0^4c^{2\alpha}}}{2h\mu_0},$$

$$\mu_2=\mu_2,\ \sigma=-4h^2rc^{-2\alpha},\ \mu_0=\mu_0,\ \mu_1=\mu_1,$$

where $\mu_0,\ \mu_1,\ \mu_2$ are arbitrary constants.

Case 3:

$$a_0=\frac{hr\mu_1c^{-\alpha}}{\mu_0},\ a_1=-\frac{h^2r\mu_1^2c^{-\alpha}-\mu_0^2c^{\alpha}}{h\mu_0},$$

$$b_1=0,\ \mu_2=0,\ \sigma=-h^2rc^{-2\alpha},\ \mu_0=\mu_0,\ \mu_1=\mu_1,$$

where $\mu_0,\ \mu_1$ are arbitrary constants.

Substituting the results in the three cases above into Eq. (19), and combining with (9) we can obtain the following exact solutions to Eq. (16).

Family 1:

$$\begin{cases} u_{1,1}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{r}\tanh_\alpha(2hc^{-\alpha}\sqrt{r}\xi)}{p_{1,1}},\ r>0,\\ u_{12}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{r}\coth_\alpha(2hc^{-\alpha}\sqrt{r}\xi)}{p_{1,2}},\ r>0,\\ u_{13}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{-(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{-r}\tan_\alpha(2hc^{-\alpha}\sqrt{r}\xi)}{p_{1,3}},\ r<0,\\ u_{1,4}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{-r}\cot_\alpha(2hc^{-\alpha}\sqrt{r}\xi)}{p_{1,4}},\ r<0, \end{cases} \quad (20)$$

where $\xi_0,\ k,\ c,\ \mu_0,\ \mu_1$ are all arbitrary constants with $k,\ c,\ \mu_0\neq 0,\ \xi=kx+iky+ct+\xi_0$, and

$$p_{1,1}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{r}\tanh_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$\pm\mu_0^2\sqrt{1-\tanh_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)}$$

$$p_{1,2}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{r}\coth_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$\pm\mu_0^2\sqrt{1-\coth_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)}$$
$$p_{1,3}=\mu_0^2+2\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\tan_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$\pm\mu_0^2\sqrt{1+\tan_\alpha^2(2hc^{-\alpha}\sqrt{-r}\xi)}$$
$$p_{1,4}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\cot_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$\pm\mu_0^2\sqrt{1+\cot_\alpha^2(2hc^{-\alpha}\sqrt{-r}\xi)}$$

Family 2:

$$\begin{cases} u_{2,1}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{r}\tanh_\alpha(2hc^{-\alpha}\sqrt{r}\xi)}{p_{2,1}}+\frac{\varphi_{2,1}}{q_{2,1}},\ r>0,\\ u_{2,2}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{r}\coth_\alpha(2hc^{-\alpha}\sqrt{r}\xi)}{p_{2,2}}+\frac{\varphi_{2,2}}{q_{2,2}},\ r>0,\\ u_{2,3}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{-(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{-r}\tan_\alpha(2hc^{-\alpha}\sqrt{-r}\xi)}{p_{2,3}}+\frac{\varphi_{2,3}}{q_{2,3}},\ r<0,\\ u_{2,4}(x,y,t)=\frac{2hr\mu_1c^{-\alpha}}{\mu_0}+\frac{(4h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{-r}\cot_\alpha(2hc^{-\alpha}\sqrt{-r}\xi)}{p_{2,4}}+\frac{\varphi_{2,4}}{q_{2,4}},\ r<0,\\ u_{2,5}(x,y,t)=\frac{-\mu_0c^{\alpha}\Gamma(1+\alpha)}{h\{\mu_0(\xi^{\alpha}+\omega)-\mu\Gamma(1+\alpha)\}},\ r=0, \end{cases} \quad (21)$$

where $\xi_0,\ \omega,\ \mu_0,\ k,\ c,\ \mu=\mu_1\pm\mu_2$ are all arbitrary constants with $k,\ c,\ \mu_0\neq 0,\ \xi=kx+iky+ct+\xi_0$, and

$$p_{2,1}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{r}\tanh_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$+2i\mu_0\mu_2hc^{-\alpha}\sqrt{r}\sqrt{1-\tanh_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$p_{2,2}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{r}\coth_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$+2i\mu_0\mu_2hc^{-\alpha}\sqrt{r}\sqrt{1-\coth_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$p_{2,3}=\mu_0^2+2\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\tan_\alpha(2hc^{-\alpha}\sqrt{-r}\xi)$$
$$+2\mu_0\mu_2hc^{-\alpha}\sqrt{-r}\sqrt{1+\tan_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$p_{2,4}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\cot_\alpha(2hc^{-\alpha}\sqrt{-r}\xi)$$
$$+2\mu_0\mu_2hc^{-\alpha}\sqrt{-r}\sqrt{1+\cot_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$q_{2,1}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{r}\tanh_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$+2i\mu_0\mu_2hc^{-\alpha}\sqrt{r}\sqrt{1-\tanh_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$q_{2,2}=\mu_0^2-2\mu_0\mu_1hc^{-\alpha}\sqrt{r}\coth_\alpha(2hc^{-\alpha}\sqrt{r}\xi)$$
$$+2i\mu_0\mu_2hc^{-\alpha}\sqrt{r}\sqrt{1-\coth_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$q_{2,3}=\mu_0^2+2\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\tan_\alpha(2hc^{-\alpha}\sqrt{-r}\xi)$$

$$+2\mu_0\mu_2 hc^{-\alpha}\sqrt{-r}\sqrt{1+\cot_\alpha^2(2hc^{-\alpha}\sqrt{-r}\xi)},$$
$$q_{2,4}=\mu_0^2-2\mu_0\mu_1 hc^{-\alpha}\sqrt{-r}\cot_\alpha(2hc^{-\alpha}\sqrt{-r}\xi)$$
$$+2\mu_0\mu_2 hc^{-\alpha}\sqrt{-r}\sqrt{1+\cot_\alpha^2(2hc^{-\alpha}\sqrt{-r}\xi)}$$
$$\varphi_{2,1}=ic^{-\alpha}\sqrt{r}(-4h^2r\mu_1\mu_2c^{-\alpha}\pm\sqrt{4h^2r\mu_0^2(\mu_2^2-\mu_1^2)+\mu_0^4c^{2\alpha}})$$
$$\sqrt{1-\tanh_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$\varphi_{2,2}=ic^{-\alpha}\sqrt{r}(-4h^2r\mu_1\mu_2c^{-\alpha}\pm\sqrt{4h^2r\mu_0^2(\mu_2^2-\mu_1^2)+\mu_0^4c^{2\alpha}})$$
$$\sqrt{1-\coth_\alpha^2(2hc^{-\alpha}\sqrt{r}\xi)},$$
$$\varphi_{2,3}=c^{-\alpha}\sqrt{-r}(-4h^2r\mu_1\mu_2c^{-\alpha}\pm\sqrt{4h^2r\mu_0^2(\mu_2^2-\mu_1^2)+\mu_0^4c^{2\alpha}})$$
$$\sqrt{1+\tan_\alpha^2(2hc^{-\alpha}\sqrt{-r}\xi)},$$
$$\varphi_{2,4}=c^{-\alpha}\sqrt{-r}(-4h^2r\mu_1\mu_2c^{-\alpha}\pm\sqrt{4h^2r\mu_0^2(\mu_2^2-\mu_1^2)+\mu_0^4c^{2\alpha}})$$
$$\sqrt{1+\cot_\alpha^2(2hc^{-\alpha}\sqrt{-r}\xi)}.$$

Family 3:

$$\begin{cases} u_{3,1}(x,y,t)=\dfrac{hr\mu_1c^{-\alpha}}{\mu_0} \\ \quad +\dfrac{(h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{r}\tanh_\alpha(hc^{-\alpha}\sqrt{r}\xi)}{\mu_0^2-\mu_0\mu_1hc^{-\alpha}\sqrt{r}\tanh_\alpha(hc^{-\alpha}\sqrt{r}\xi)},\ r>0, \\ u_{3,2}(x,y,t)=\dfrac{hr\mu_1c^{-\alpha}}{\mu_0} \\ \quad +\dfrac{(h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{r}\coth_\alpha(hc^{-\alpha}\sqrt{r}\xi)}{\mu_0^2-\mu_0\mu_1hc^{-\alpha}\sqrt{r}\coth_\alpha(hc^{-\alpha}\sqrt{r}\xi)},\ r>0, \\ u_{3,3}(x,y,t)=\dfrac{hr\mu_1c^{-\alpha}}{\mu_0} \\ \quad -\dfrac{(h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{-r}\tan_\alpha(hc^{-\alpha}\sqrt{-r}\xi)}{\mu_0^2+\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\tan_\alpha(hc^{-\alpha}\sqrt{-r}\xi)},\ r<0, \\ u_{3,4}(x,y,t)=\dfrac{hr\mu_1c^{-\alpha}}{\mu_0} \\ \quad +\dfrac{(h^2r\mu_1^2c^{-2\alpha}-\mu_0^2)\sqrt{-r}\cot_\alpha(hc^{-\alpha}\sqrt{-r}\xi)}{\mu_0^2-\mu_0\mu_1hc^{-\alpha}\sqrt{-r}\cot_\alpha(hc^{-\alpha}\sqrt{-r}\xi)},\ r<0, \\ u_{3,5}(x,y,t)=\dfrac{-\mu_0c^\alpha\Gamma(1+\alpha)}{h\{\mu_0(\xi^\alpha+\omega)-\mu_1\Gamma(1+\alpha)\}},\ r=0, \end{cases} \tag{22}$$

where $\xi_0,\ k,\ c,\ \mu_0,\ \mu_1$ are all arbitrary constants with $k,\ c,\ \mu_0\neq 0$, and $\xi=kx+iky+ct+\xi_0$.

Remark 1. If we set $\mu_0=D,\ \mu_1=B,\ k=1,\ c=\lambda$, then the solutions in (22) reduce to the solutions established in [22, Eqs. (36)-(40)]. If we set $\mu_1=0$, then the solutions in (22) reduce to the solutions established in [19, Eqs. (16)-(20)].

3.3 *nonlinear fractional foam drainage equation*

We consider the nonlinear foam drainage equation with time and space-fractional derivatives [22,26,27]:

$$D_t^\alpha u=\frac{u}{2}D_x^\alpha D_x^\alpha u-2u^2D_x^\alpha u+(D_x^\alpha u)^2,\ 0<\alpha\leq 1. \tag{23}$$

The foam drainage equation is a model of the flow of liquid through channels and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [28].

First we suppose $u(x,y,t)=U(\xi)$, where $\xi=kx+ct+\xi_0$. Then by use of Eq. (3), Eq. (23) can be turned into

$$c^\alpha D_\xi^\alpha U=\frac{U}{2}k^{2\alpha}D_\xi^\alpha D_\xi^\alpha U-2U^2k^\alpha D_\xi^\alpha U+k^{2\alpha}(D_\xi^\alpha U)^2. \tag{24}$$

Suppose that the solution of Eq. (24) can be expressed by a polynomial in ϕ as follows:

$$U(\xi)=a_0+\sum_{i=1}^{m}\frac{\phi^{i-1}(a_i\phi+b_i\sqrt{D_\xi^\alpha\phi})}{(\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi})^i}, \tag{25}$$

where $\phi=\phi(\xi)$ satisfies Eq. (8).

Balancing the order between the highest order derivative term and nonlinear term in Eq. (24), we can obtain $m=1$. So we have

$$U(\xi)=a_0+\frac{a_1\phi+b_1\sqrt{D_\xi^\alpha\phi}}{\mu_0+\mu_1\phi+\mu_2\sqrt{D_\xi^\alpha\phi}}, \tag{26}$$

Substituting (26) into (24) and collecting all the terms with the same power of $\phi^j(\sqrt{\sigma+\phi^2})^i$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields

Case 1:

$$a_0=-\frac{2\mu_1c^\alpha k^{-2\alpha}}{\mu_0},\ a_1=\frac{4k^{-2\alpha}\mu_1^2c^\alpha+\mu_0^2k^\alpha}{2\mu_0},\ b_1=0$$

$$\mu_2=\pm\frac{\mu_0}{2}k^{\frac{3}{2}\alpha}c^{-\frac{\alpha}{2}},\ \sigma=4c^\alpha k^{-3\alpha},\ \mu_0=\mu_0,\ \mu_1=\mu_1,$$

where $\mu_0,\ \mu_1$ are arbitrary constants.

Case 2:

$$a_0=0,\ a_1=\frac{\mu_0k^\alpha}{2},$$

$$b_1=\pm\frac{\mu_0k^\alpha}{2},\ \mu_2=0,\ \sigma=4c^\alpha k^{-3\alpha},\ \mu_0=\mu_0,\ \mu_1=0,$$

where μ_0 is an arbitrary constant.

Case 3:

$$a_0=-\frac{\mu_1c^\alpha k^{-2\alpha}}{\mu_0},\ a_1=\frac{k^{-2\alpha}\mu_1^2c^\alpha+\mu_0^2k^\alpha}{\mu_0},$$

$$b_1=0,\ \mu_2=0,\ \sigma=c^\alpha k^{-3\alpha},\ \mu_0=\mu_0,\ \mu_1=\mu_1,$$

where $\mu_0,\ \mu_1$ are arbitrary constants.

Substituting the results in the three cases above into Eq. (26), and combining with (9) we can obtain the following exact solutions to Eq. (23).

Family 1:

$$\begin{cases} u_{1,1}(x,t) = \dfrac{2\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad - \dfrac{(4k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{-c^\alpha k^{-\alpha}}\tanh_\alpha(2\sqrt{-c^\alpha k^{-3\alpha}}\xi)}{l_{1,1}}, \\ \qquad c^\alpha k^{-\alpha} < 0, \\ u_{1,2}(x,t) = \dfrac{2\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad - \dfrac{(4k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{-c^\alpha k^{-\alpha}}\coth_\alpha(2\sqrt{-c^\alpha k^{-3\alpha}}\xi)}{l_{1,2}}, \\ \qquad c^\alpha k^{-\alpha} < 0, \\ u_{1,3}(x,t) = \dfrac{2\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad + \dfrac{(4k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{c^\alpha k^{-\alpha}}\tan_\alpha(2\sqrt{c^\alpha k^{-3\alpha}}\xi)}{l_{1,3}}, \\ \qquad c^\alpha k^{\alpha} > 0, \\ u_{1,4}(x,t) = \dfrac{2\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad - \dfrac{(4k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{c^\alpha k^{-\alpha}}\cot_\alpha(2\sqrt{c^\alpha k^{-3\alpha}}\xi)}{l_{1,4}}, \\ \qquad c^\alpha k^{\alpha} > 0, \end{cases} \tag{27}$$

where $\xi_0,\ k,\ c,\ \mu_0,\ \mu_1$ are all arbitrary constants with $k,\ c,\ \mu_0 \neq 0,\ \xi = kx + ct + \xi_0$, and

$$l_{1,1} = \mu_0^2 - 2\mu_0\mu_1\sqrt{-c^\alpha k^{-3\alpha}}\tanh_\alpha(2\sqrt{-c^\alpha k^{-3\alpha}}\xi) \pm \mu_0^2 c^\alpha k^{-3\alpha}\sqrt{1-\tanh_\alpha^2(2\sqrt{-c^\alpha k^{-3\alpha}}\xi)},$$

$$l_{1,2} = \mu_0^2 - 2\mu_0\mu_1\sqrt{-c^\alpha k^{-3\alpha}}\coth_\alpha(2\sqrt{-c^\alpha k^{-3\alpha}}\xi) \pm \mu_0^2 c^\alpha k^{-3\alpha}\sqrt{1-\coth_\alpha^2(2\sqrt{-c^\alpha k^{-3\alpha}}\xi)},$$

$$l_{1,3} = \mu_0^2 + 2\mu_0\mu_1\sqrt{c^\alpha k^{-3\alpha}}\tan_\alpha(2\sqrt{c^\alpha k^{-3\alpha}}\xi) \pm \mu_0^2 c^\alpha k^{-3\alpha}\sqrt{1+\tan_\alpha^2(2\sqrt{c^\alpha k^{-3\alpha}}\xi)},$$

$$l_{1,4} = \mu_0^2 - 2\mu_0\mu_1\sqrt{c^\alpha k^{-3\alpha}}\cot_\alpha(2\sqrt{c^\alpha k^{-3\alpha}}\xi) \pm \mu_0^2 c^\alpha k^{-3\alpha}\sqrt{1+\cot_\alpha^2(2\sqrt{c^\alpha k^{-3\alpha}}\xi)},$$

Family 2:

$$\begin{cases} u_{2,1}(x,t) = -\sqrt{-c^\alpha k^{-\alpha}}\tanh_\alpha(2\sqrt{-c^\alpha k^{-3\alpha}}\xi) \\ \quad \pm i\sqrt{-c^\alpha k^{-\alpha}\{1-\tanh_\alpha^2(2\sqrt{-c^\alpha k^{-3\alpha}}\xi)\}},\ c^\alpha k^{-\alpha} < 0, \\ u_{2,2}(x,t) = -\sqrt{-c^\alpha k^{-\alpha}}\coth_\alpha(2\sqrt{-c^\alpha k^{-3\alpha}}\xi) \\ \quad \pm i\sqrt{-c^\alpha k^{-\alpha}\{1-\coth_\alpha^2(2\sqrt{-c^\alpha k^{-3\alpha}}\xi)\}},\ c^\alpha k^{-\alpha} < 0, \\ u_{2,3}(x,t) = \sqrt{c^\alpha k^{-\alpha}}\tan_\alpha(2\sqrt{c^\alpha k^{-3\alpha}}\xi) \\ \quad \pm i\sqrt{c^\alpha k^{-\alpha}\{1+\tan_\alpha^2(2\sqrt{c^\alpha k^{-3\alpha}}\xi)\}},\ c^\alpha k^{-\alpha} > 0, \\ u_{2,4}(x,t) = -\sqrt{c^\alpha k^{-\alpha}}\cot_\alpha(2\sqrt{c^\alpha k^{-3\alpha}}\xi) \\ \quad \pm i\sqrt{c^\alpha k^{-\alpha}\{1+\cot_\alpha^2(2\sqrt{c^\alpha k^{-3\alpha}}\xi)\}},\ c^\alpha k^{-\alpha} > 0, \end{cases} \tag{28}$$

where $\xi_0,\ k,\ c,\ \mu_0$ are all arbitrary constants with $k,\ c,\ \mu_0 \neq 0$, and $\xi = kx + ct + \xi_0$.

Family 3:

$$\begin{cases} u_{3,1}(x,t) = -\dfrac{\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad - \dfrac{(k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{-c^\alpha k^{-\alpha}}\tanh_\alpha(\sqrt{-c^\alpha k^{-3\alpha}}\xi)}{\mu_0^2 - \mu_0\mu_1\sqrt{-c^\alpha k^{-3\alpha}}\tanh_\alpha(\sqrt{-c^\alpha k^{-3\alpha}}\xi)}, \\ \qquad c^\alpha k^{-\alpha} < 0, \\ u_{3,2}(x,t) = -\dfrac{\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad - \dfrac{(k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{-c^\alpha k^{-\alpha}}\coth_\alpha(\sqrt{-c^\alpha k^{-3\alpha}}\xi)}{\mu_0^2 - \mu_0\mu_1\sqrt{-c^\alpha k^{-3\alpha}}\coth_\alpha(\sqrt{-c^\alpha k^{-3\alpha}}\xi)}, \\ \qquad c^\alpha k^{-\alpha} < 0, \\ u_{3,3}(x,t) = -\dfrac{\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad + \dfrac{(k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{c^\alpha k^{-\alpha}}\tan_\alpha(\sqrt{c^\alpha k^{-3\alpha}}\xi)}{\mu_0^2 + \mu_0\mu_1\sqrt{c^\alpha k^{-3\alpha}}\tan_\alpha(\sqrt{c^\alpha k^{-3\alpha}}\xi)}, \\ \qquad c^\alpha k^{-\alpha} > 0, \\ u_{3,4}(x,t) = -\dfrac{\mu_1 c^\alpha k^{-2\alpha}}{\mu_0} \\ \quad - \dfrac{(k^{-3\alpha}\mu_1^2 c^\alpha + \mu_0^2)\sqrt{c^\alpha k^{-\alpha}}\cot_\alpha(\sqrt{c^\alpha k^{-3\alpha}}\xi)}{\mu_0^2 - \mu_0\mu_1\sqrt{c^\alpha k^{-3\alpha}}\cot_\alpha(\sqrt{c^\alpha k^{-3\alpha}}\xi)}, \\ \qquad c^\alpha k^{-\alpha} > 0, \end{cases} \tag{29}$$

where $\xi_0,\ k,\ c,\ \mu_0,\ \mu_1$ are all arbitrary constants with $k,\ c,\ \mu_0 \neq 0$, and $\xi = kx + ct + \xi_0$.

Remark 2. If we set $\mu_0 = D,\ \mu_1 = B,\ k = 1,\ c = \lambda$, then the solutions $u_{33}(x,t),\ u_{34}(x,t)$ reduce to the solutions established in [22, Eq. (31)].

4 Illustration of the presented results

In this section, we will illustrate the application of the results established above.

The nonlinear fractional Sharma-Tasso-Olever (STO) equation (10) is a KdV-like equation, and play an important role in describing the nonlinear wave phenomena. Exact solutions for it with different forms can describe different nonlinear waves. For the established exact solutions $u_{11}(x,t),\ u_{12}(x,t),\ u_{21}(x,t),\ u_{22}(x,t)$ with hyperbolic functions forms in (14)-(15), solitary wave phenomenon can be demonstrated by them, while for the established exact solutions $u_{13}(x,t),\ u_{14}(x,t),\ u_{23}(x,t),\ u_{24}(x,t)$ with periodic functions forms in (14)-(15), periodic wave phenomenon can be demonstrated. For the better understanding the solitary wave phenomenon and periodic wave phenomenon, we take $u_{11}(x,t),\ u_{13}(x,t)$ for example, and show them in Figs. 1-4 with some given parameters, in which the value of the variable c represents wave velocity.

From Figs. 1-2 one can see, with the increasing of the order α of the fractional derivative, the time and space coordinates at which the solitary wave appears get smaller, which implies the existing period of the solitary wave gets shorter. From Figs. 3-4 one can see, with the

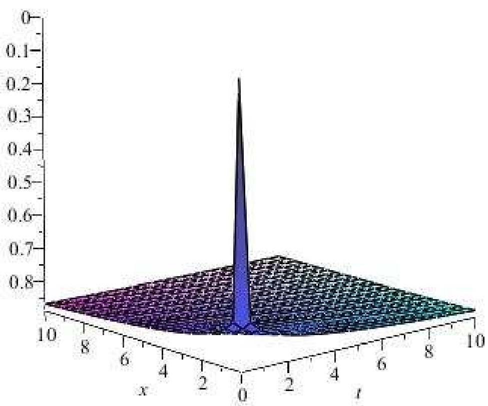

Fig. 1 The solution $u_{11}(x,t)$ in (14) with $\alpha=1/10$, $a=-1$, $m=n=c=k=1$, $\xi_0=0$,

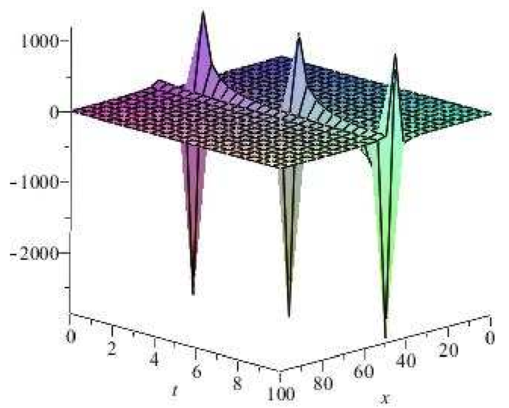

Fig. 3 The solution $u_{13}(x,t)$ in (14) with $\alpha=1/10$, $a=1$, $m=n=c=k=1$, $\xi_0=0$,

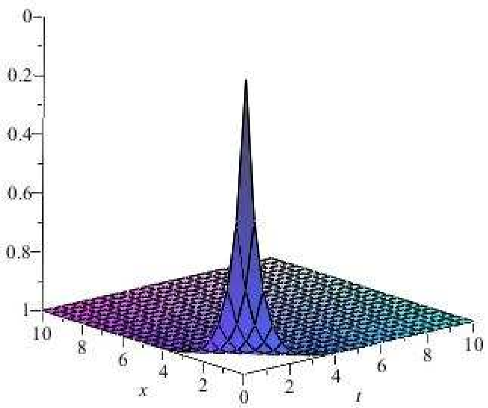

Fig. 2 The solution $u_{11}(x,t)$ in (14) with $\alpha=4/5$, $a=-1$, $m=n=c=k=1$, $\xi_0=0$,

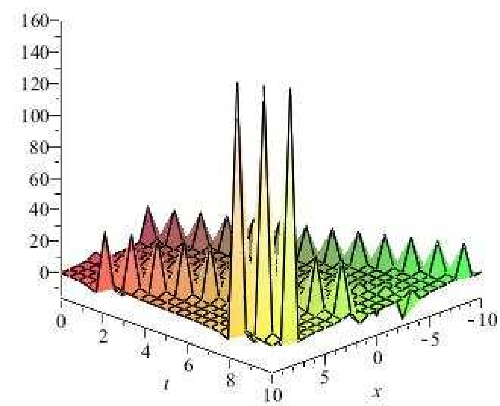

Fig. 4 The solution $u_{13}(x,t)$ in (14) with $\alpha=4/5$, $a=1$, $m=n=c=k=1$, $\xi_0=0$,

increasing of the order α of the fractional derivative, it gets more frequent that the periodic wave reaches its local climax.

In the nonlinear time fractional biological population model (16), the function u denotes the population density and $h(u^2-r)$ represents the population supply due to births and deaths. The reason of using fractional differential equations to modeling biological population is that fractional differential equations are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The resulting solutions spread faster than the classical solutions and may exhibit asymmetry. For the sake of illustrating the variation trend of the population density, we take the solution $u_{33}(x,y,t)$ in (22) for example, and show it in Fig. 5.

The nonlinear fractional foam drainage equation (23) and the established solutions (27)-(29) for it play a fundamental role in describing the foam drainage process, where the variables x and t represent scaled position and time coordinates respectively. Foaming occurs in many distillation and absorption processes. Recent research in foams has centered on three topics which are often treated

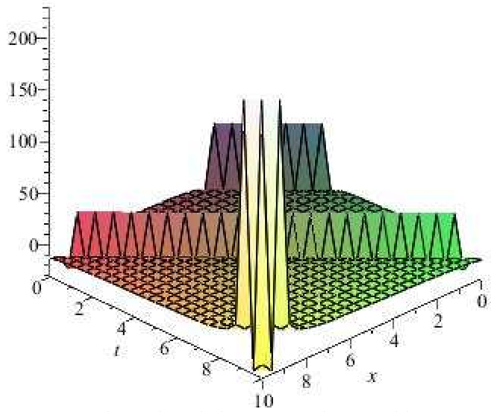

Fig. 5 The solution $u_{33}(x, t)$ in (22) with $\alpha = 1/2, h = \mu_0 = \mu_1 = c = k = 1, \xi_0 = 0, r = -4$

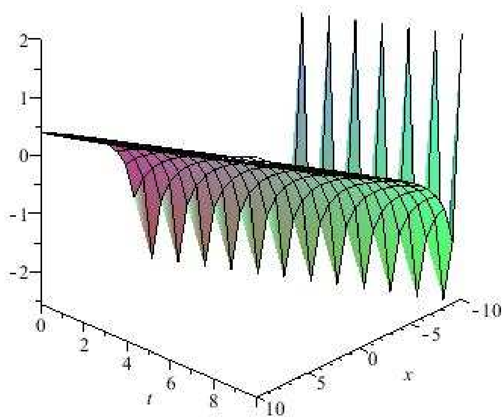

Fig. 6 The solution $u_{34}(x, t)$ in (29) with $\alpha = 1/3, \mu_0 = 2, \mu_1 = c = k = 1, \xi_0 = 0$

separately, but are in fact interdependent: drainage, coarsening, and rheology. Drainage plays an important role in foam stability: indeed, when a foam dries, its structure becomes more fragile; the liquid films between adjacent bubbles being thinner, then can break, leading to foam collapse. The drainage of liquid foams involves the interplay of gravity, surface tension, and viscous forces. Forced foam drainage may well be the best prototype for certain general phenomena described by nonlinear differential equations, particularly the type of solitary wave which is most familiar in tidal bores. For better understanding the function of the solutions (27)-(29) in describing the foam drainage process, we take the solitary wave solution $u_{34}(x,t)$ in (29) for example, and show it in Fig. 6, in which the value of the variable c represents wave velocity.

5 Conclusions

We have proposed a new fractional Riccati equation rational expansion method for solving fractional differential equations, and applied it to find exact solutions of the nonlinear fractional Sharma-Tasso-Olever (STO) equation, the nonlinear time fractional biological population model and the nonlinear fractional foam drainage equation. As a result, some generalized and new exact solutions for them have been successfully found. Being concise and powerful, this method can also be applied to solve other fractional differential equations as long as the homogeneous balance principle is satisfied.

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