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Nonlocal Initial Value Problems For Nonlinear Neutral Pantograph Equations With Hilfer-Hadamard Fractional Derivative

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Abstract: In this article, we study the existence and Ulam stability of solutions for Hilfer-Hadamard fractional type nonlinear neutral pantograph equations with nonlocal conditions. The Scheafer's fixed point theorem and Banach contraction principle are used to obtain the desired results.

Keywords: Hilfer-Hadamard fractional derivative; Neutral pantograph equations; fixed point; Generalized Ulam-Hyers stability.

1 Introduction

The fractional differential equations have received increasing attention, because the behavior of many physical systems can be properly described by using the fractional order system theory; one can see the monographs of [1, 3–5] and the references therein. Recently, some mathematicians have considered fractional differential equations depending on the Hadamard fractional derivative [6–8, 10]. Nowadays, Hilfer-Hadamard fractional derivative (for short H-H fractional derivative) have attracted much attention to researchers. A detailed description of H-H fractional derivative can be found in [2, 9, 11–14].

In [15], K. Balachandran et al. studied the fractional nonlinear neutral pantograph equations. Due to its importance in many fields, it is interesting to study the fractional model of the pantograph equations. Such model can be suitable to be applied when the corresponding process occurs through strongly anomalous media.

In this paper, we discuss mainly the existence and Ulam stability of solution for Hilfer-Hadamard fractional neutral pantograph equation with nonlocal condition of the following type

$$\begin{cases} {}_H D^{\alpha,\beta}_{1^+} x(t) = f(t, x(t), x(\lambda t), {}_H D^{\alpha,\beta}_{1^+} x(\lambda t)), \\ t \in J := [1, b], \\ {}_H I^{1-\gamma}_{1^+} x(1) = \sum_{i=1}^m c_i x(\tau_i), \\ \alpha \le \gamma = \alpha + \beta - \alpha\beta < 1, \quad \tau_i \in [1, b], \end{cases} \tag{1}$$

where $\alpha, \lambda \in (0, 1)$, $\beta \in [0, 1]$, ${}_H D^{\alpha,\beta}_{1^+}$ is the H-H fractional derivative of order α and type β, introduced by Hilfer in [1]. Let X be a Banach space, $f : J \times X^3 \to X$ is given continuous function and ${}_H I^{1-\gamma}_{1^+}$ is the left-sided mixed Hadamard integral of order $1 - \gamma$. For brevity of notation, we shall take ${}_H I^{1-\gamma}_{1^+}$ as $I^{1-\gamma}_{1^+}$.

In passing, we remark that the application of nonlocal condition $I^{1-\gamma}_{1^+} x(1) = \sum_{i=1}^m c_i x(\tau_i)$ in physical problems yields better effect than the initial condition $I^{1-\gamma}_{1^+} x(1) = x_0$.

We adopt some ideas from [16, 17]. For sake of brevity, let us take

$${}_H D^{\alpha,\beta}_{1^+} x(t) := g_x(t) = f(t, x(t), x(\lambda t), g_x(t)).$$

A new and important equivalent mixed type integral equation for our system (1) can be established. We adopt some ideas in [13] to estabilish an equivalent mixed type

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integral equation:

$$x(t) = \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_x(s) \frac{ds}{s}, \quad (2)$$

where

$$Z := \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{m} c_i (\log \tau_i)^{\gamma-1}}, \quad (3)$$

$$\text{if} \quad \Gamma(\gamma) \neq \sum_{i=1}^{m} c_i (\log \tau_i)^{\gamma-1}.$$

In this regard, in Section 2, we recall some basic definition, lemmas and results concerning with H-H fractional derivative. In section 3, we study the existence results for the problem (1). In Section 4, we prove the stability of solution of the problem (1).

2 Elementary definitions and lemmas

For convenience, this section summarizes some concepts, definitions and basic results from fractional calculus, which are useful for the further developments in this paper.

Let $C[J, X]$ be the Banach space of all continuous functions with the norm

$$\|x\|_c = \max \{|x(t)| : t \in [0, b]\}.$$

For $0 \leq \gamma < 1$, we denote the space $C_{\gamma,\log}[J, X]$ as

$$C_{\gamma,\log}[J, X] := \{f(t) : [0, b] \to X | (\log t)^{\gamma} f(t) \in C[J, X]\},$$

where $C_{\gamma,\log}[J, X]$ is the weighted space of the continuous functions f on the finite interval $[0, b]$.

Obviously, $C_{\gamma,\log}[J, X]$ is the Banach space with the norm

$$\|f\|_{C_{\gamma,\log}} = \|(\log t)^{\gamma} f(t)\|_C.$$

Meanwhile, $C^n_{\gamma,\log}[J, X] := \left\{f \in C^{n-1}[J, X] : f^{(n)} \in C_{\gamma,\log}[J, X]\right\}$ is the Banach space with the norm

$$\|f\|_{C^n_{\gamma,\log}} = \sum_{i=0}^{n-1} \|f^k\|_C + \left\|f^{(n)}\right\|_{C_{\gamma,\log}}, \; n \in \mathbb{N}.$$

Moreover, $C^0_{\gamma,\log}[J, X] := C_{\gamma,\log}[J, X]$.

Definition 1. *The Hadamard fractional integral of order α for a continuous function h is defined as*

$$I^{\alpha}_{1+} h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s}, \quad \alpha > 0,$$

provided the integral exists.

Definition 2. *The Hadamard derivative of fractional order α for a continuous function $h : [1, \infty) \to X$ is defined as*

$${}_H D^{\alpha}_{1+} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} h(s) \frac{ds}{s},$$

$$n - 1 < \alpha < n, \quad n = \lceil \alpha \rceil + 1,$$

where $\lceil \alpha \rceil$ denotes the integer part of real number α and $\log(\cdot) = \log_e(\cdot)$.

We review few basic properties of H-H fractional derivative that are needed for this work. For details, see [1, 9, 10, 12, 13] and references therein.

Definition 3. *The H-H fractional derivative of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function $h(t)$ is defined by*

$${}_H D^{\alpha,\beta}_{1+} h(t) = \left(I^{\beta(1-\alpha)}_{1+} D \left(I^{(1-\beta)(1-\alpha)}_{1+} h\right)\right)(t),$$

where $D := \frac{d}{dt}$.

Remark. 1. The operator ${}_H D^{\alpha,\beta}_{1+}$ also can be written as

$${}_H D^{\alpha,\beta}_{1+} = I^{\beta(1-\alpha)}_{1+} D I^{(1-\beta)(1-\alpha)}_{1+} = I^{\beta(1-\alpha)}_{1+} \; {}_H D^{\gamma}_{1+},$$

$$\gamma = \alpha + \beta - \alpha\beta.$$

2. Let $\beta = 0$, the Hadamard Riemann-Liouville fractional derivative can be presented as ${}_H D^{\alpha}_{1+} := {}_H D^{\alpha,0}_{1+}$.
3. Let $\beta = 1$, the Hadamard Caputo fractional derivative can be presented as ${}^c_H D^{\alpha}_{1+} := I^{1-\alpha}_{1+} D$.

Lemma 1. *If $\alpha > 0$ and $\beta > 0$ and $0 < \alpha < b < \infty$, there exist*

$$\left[I^{\alpha}_{1+} (\log s)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log t)^{\beta+\alpha-1}$$

and

$$\left[{}_H D^{\alpha}_{1+} (\log s)^{\alpha-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log t)^{\beta-\alpha-1},.$$

In particular, if $\beta = 1$ and $\alpha \geq 0$, then the Hadamard fractional derivative of a constant is not equal to zero:

$$\left({}_H D^{\alpha}_{1+} 1\right)(t) = \frac{1}{\Gamma(1-\alpha)} (\log t)^{-\alpha}, \quad 0 < \alpha < 1.$$

Lemma 2. *If $\alpha > 0$, $\beta > 0$, and $h \in L^1\{R^+\}$, for $t \in [0, T]$ there exist the following properties*

$$\left(I^{\alpha}_{1+} I^{\beta}_{1+} h\right)(t) = \left(I^{\alpha+\beta}_{1+} h\right)(t),$$

and

$$\left({}_H D^{\alpha}_{1+} I^{\alpha}_{1+} h\right)(t) = h(t).$$

In particular, if $h \in C_{\gamma,\log}[J, X]$ or $h \in C[J, X]$, then these equalities hold at $t \in [0, b]$.

Lemma 3. *Let $0<\alpha<1$, $0\le\gamma<1$. If $h\in C_{\gamma,\log}[J,X]$ and $I_{1^+}^{1-\alpha}h\in C^1_{\gamma,\log}[J,X]$, then*

$$I_{1^+}^{\alpha}\ {}_HD_{1^+}^{\alpha}h(t)=h(t)-\frac{\left(I_{1^+}^{1-\alpha}h\right)(1)}{\Gamma(\alpha)}(\log t)^{\alpha-1},\quad \forall\, t\in[0,b].$$

Lemma 4. *For $0\le\gamma<1$ and $h\in C_{\gamma,\log}[J,X]$, then*

$$\left(I_{1^+}^{\alpha}h\right)(1):=\lim_{t\to1^+}I_{1^+}^{\alpha}h(t)=0,\quad 0\le\gamma<\alpha.$$

Lemma 5. *Let $\alpha>0$, $\beta>0$, and $\gamma=\alpha+\beta-\alpha\beta$. If $h\in C^{\gamma}_{1-\gamma,\log}[J,X]$, then*

$$I_{1^+}^{\gamma}\ {}_HD_{1^+}^{\gamma}h=I_{1^+}^{\alpha}\ {}_HD_{1^+}^{\alpha,\beta}h,\ \ {}_HD_{1^+}^{\gamma}I_{1^+}^{\alpha}h=\ {}_HD_{1^+}^{\beta(1-\alpha)}h(t).$$

Lemma 6. *Let $h\in L^1\{R_+\}$ and ${}_HD_{1^+}^{\beta(1-\alpha)}h\in L^1\{R_+\}$ existed, then*

$${}_HD_{1^+}^{\alpha,\beta}I_{1^+}^{\alpha}h=I_{1^+}^{\beta(1-\alpha)}\ {}_HD_{1^+}^{\beta(1-\alpha)}h.$$

3 Existence results

In this section, we introduce spaces that helps us to solve and reduce the problem (1) to an equivalent integral equation (2).

$$C^{\alpha,\beta}_{1-\gamma,\log}=\left\{f\in C_{1-\gamma,\log}[J,X],{}_H\,D_{1^+}^{\alpha,\beta}f\in C_{1-\gamma,\log}[J,X]\right\},$$

and

$$C^{\gamma}_{1-\gamma,\log}=\left\{f\in C_{1-\gamma,\log}[J,X],{}_H\,D_{1^+}^{\gamma}f\in C_{1-\gamma,\log}[J,X]\right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma,\log}[J,X]\subset C^{\alpha,\beta}_{1-\gamma,\log}[J,X].$$

Lemma 7. *[24] Let $f:J\times X\to X$ be a function such that $f(\cdot,x(\cdot))\in C_{1-\gamma,\log}[J,X]$ for any $x\in C_{1-\gamma,\log}[J,X]$. A function $x\in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of fractional initial value problem:*

$$\begin{cases}{}_HD_{1^+}^{\alpha,\beta}x(t)=f(t,x(t)),\ 0<\alpha<1,\ 0\le\beta\le1, t\in J,\\ I_{1^+}^{1-\gamma}x(1)=x_0,\quad \gamma=\alpha+\beta-\alpha\beta,\end{cases}$$

if and only if x satisfies the following Volterra integral equation:

$$x(t)=\frac{x_0(\log t)^{\gamma-1}}{\Gamma(\gamma)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}.$$

Further details can be found in [24]. From Lemma 7 we have the following result.

Lemma 8. *Let $f:J\times X^3\to X$ be a function such that $f\in C_{1-\gamma,\log}[J,X]$ for any $x\in C_{1-\gamma,\log}[J,X]$. A function $x\in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of the system (1) if and only if x satisfies the mixed type integral (2).*

Proof. According to Lemma 7, a solution of system (1) can be expressed by

$$\begin{aligned}x(t)=&\ \frac{I_{1^+}^{1-\gamma}x(1)}{\Gamma(\gamma)}(\log t)^{\gamma-1}\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s}.\end{aligned}\tag{4}$$

Next, we substitute $t=\tau_i$ into the above equation,

$$\begin{aligned}x(\tau_i)=&\ \frac{I_{1^+}^{1-\gamma}x(1)}{\Gamma(\gamma)}(\log\tau_i)^{\gamma-1}\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s},\end{aligned}\tag{5}$$

by multiplying c_i to both sides of (5), we can write

$$c_ix(\tau_i)=\frac{I_{1^+}^{1-\gamma}x(1)}{\Gamma(\gamma)}c_i(\log\tau_i)^{\gamma-1}$$

$$+\frac{1}{\Gamma(\alpha)}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s}.$$

Thus, we have

$$\begin{aligned}I_{1^+}^{1-\gamma}x(1)&=\sum_{i=1}^m c_ix(\tau_i)\\&=\frac{I_{1^+}^{1-\gamma}x(1)}{\Gamma(\gamma)}\sum_{i=1}^m c_i(\log\tau_i)^{\gamma-1}\\&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s},\end{aligned}$$

which implies

$$I_{1^+}^{1-\gamma}x(1)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}Z\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s}\tag{6}$$

Submitting (6) to (4), we derive that (2). It is probative that x is also a solution of the integral equation (2), when x is a solution of (1).

The necessity has been already proved. Next, we are ready to prove its sufficiency. Applying $I_{1^+}^{1-\gamma}$ to both sides of (2), we have

$$\begin{aligned}I_{1^+}^{1-\gamma}x(t)=&I_{1^+}^{1-\gamma}(\log t)^{\gamma-1}\frac{Z}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\\&g_x(s)\frac{ds}{s}+I_{1^+}^{1-\gamma}I_{1^+}^{\alpha}g_x(t),\end{aligned}$$

using the Lemmas 1 and 2,

$$\begin{aligned}I_{1^+}^{1-\gamma}x(t)=&\frac{\Gamma(\gamma)}{\Gamma(\alpha)}Z\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s}\\&+I_{1^+}^{1-\beta(1-\alpha)}g_x(t).\end{aligned}$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 4 can be used when taking the limits as $t \to 1$,

$$I_{1^+}^{1-\gamma}x(1) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} \quad (7)$$

Substituting $t = \tau_i$ into (2), we have

$$x(\tau_i) = \frac{Z}{\Gamma(\alpha)}(\log \tau_i)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s}.$$

Then, we derive

$$\begin{aligned}\sum_{i=1}^{m} c_i x(\tau_i) &= \frac{Z}{\Gamma(\alpha)} \times \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} \sum_{i=1}^{m} c_i (\log \tau_i)^{\gamma-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} \times \left(1 + Z \sum_{i=1}^{m} c_i (\log \tau_i)^{\gamma-1}\right) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s},\end{aligned}$$

that is

$$\sum_{i=1}^{m} c_i x(\tau_i) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} \quad (8)$$

It follows (7) and (8) that

$$I_{1^+}^{1-\gamma}x(1) = \sum_{i=1}^{m} c_i x(\tau_i).$$

Now by applying ${}_H D_{1^+}^{\gamma}$ to both sides of (2), it follows from Lemma 1 and 5 that

$$\begin{aligned}{}_H D_{1^+}^{\gamma} x(t) &= {}_H D_{1^+}^{\beta(1-\alpha)} g_x(t) \quad (9) \\ &= {}_H D_{1^+}^{\beta(1-\alpha)} f(t, x(t), x(\lambda t), {}_H D_{1^+}^{\alpha,\beta} x(\lambda t)).\end{aligned}$$

Since $x \in C_{1-\gamma,\log}^{\gamma}[J,X]$ and by the definition of $C_{1-\gamma,\log}^{\gamma}[J,X]$, we have ${}_H D_{1^+}^{\gamma} x \in C_{1-\gamma,\log}[J,X]$, then, ${}_H D_{1^+}^{\beta(1-\alpha)} f = D I_{1^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\log}[J,X]$. For $f \in C_{1-\gamma,\log}[J,X]$, it is obivious that $I_{1^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\log}[J,X]$, then $I_{1^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\log}^{1}[J,X]$. Thus f and $I_{1^+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 3.

Next, by applying $I_{1^+}^{\beta(1-\alpha)}$ to both sides of (9) and using Lemma 3, we can obtain

$${}_H D_{1^+}^{\alpha,\beta} x(t) = g_x(t) - \frac{\left(I_{1^+}^{1-\beta(1-\alpha)} g_x\right)(1)}{\Gamma(\beta(1-\alpha))} (\log t)^{\beta(1-\alpha)-1},$$

where $\left(I_{1^+}^{\beta(1-\alpha)} g_x\right)(1) = 0$ is implied by Lemma 4.
Hence, it reduces to ${}_H D_{1^+}^{\alpha,\beta} x(t) = g_x(t) = f(t, x(t), x(\lambda t), {}_H D_{1^+}^{\alpha,\beta} x(\lambda t))$.
The results are proved completely.

To study the existence and uniqueness of solutions to (1) we require the following assumptions:

(A1)The function $f : J \times X^3 \to X$ is continuous.
(A2)There exist $l, p, q, r \in C_{1-\gamma,\log}[J,X]$ with $l^* = \sup_{t\in J} l(t) < 1$ such that

$$|f(t,u,v,w)| \le l(t) + p(t)\,|u| + q(t)\,|v| + r(t)\,|w|$$

for $t \in J$, $u, v, w \in X$.
(A3)There exist positive constants $K > 0$ and $L > 0$ such that

$$|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})| \le K\left(|u-\overline{u}| + |v-\overline{v}|\right) + L\,|w-\overline{w}|$$

for any $u, v, w, \overline{u}, \overline{v}, \overline{w} \in X$ and $t \in J$.

Our first theorem is based on the Banach contraction principle.

Theorem 1. *Assume that (A1) and (A3) hold. If*

$$\left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(\alpha+1)} \left(|Z| \sum_{i=1}^{m} c_i (\log \tau_i)^{\alpha} + (\log b)^{1-\gamma+\alpha}\right) < 1 \quad (10)$$

then the problem (1) has a unique solution.

Proof. Let the operator $N : C_{1-\gamma,\log}[J,X] \to C_{1-\gamma,\log}[J,X]$.

$$(Nx)(t) = \frac{Z}{\Gamma(\alpha)}(\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s}.$$

By Lemma 8, it is clear that the fixed points of N are solutions of system (1).
Let $x_1, x_2 \in C_{1-\gamma,\log}[J,X]$ and $t \in J$, then we have

$$\begin{aligned}&\left|\left((Nx_1)(t) - (Nx_2)(t)\right)(\log t)^{1-\gamma}\right| \\ &\le \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} |g_{x_1}(s) - g_{x_2}(s)| \frac{ds}{s} \\ &\quad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |g_{x_1}(s) - g_{x_2}(s)| \frac{ds}{s} \quad (11)\end{aligned}$$

and

$$\begin{aligned}&|g_{x_1}(t) - g_{x_2}(t)| \\ &= |f(t, x_1(t), x_1(\lambda t), g_{x_1}(t)) - f(t, x_2(t), x_2(\lambda t), g_{x_2}(t))| \\ &\le K\left(|x_1(t) - x_2(t)| + |x_1(\lambda t) - x_2(\lambda t)|\right) + L\,|g_{x_1}(t) - g_{x_2}(t)| \\ &\le \left(\frac{2K}{1-L}\right) |x_1(t) - x_2(t)|. \quad (12)\end{aligned}$$

By replacing (12) in the inequality (11), we get

$$\begin{aligned}&\left|((Nx_1)(t)-(Nx_2)(t))(\log t)^{1-\gamma}\right|\\&\le \frac{|Z|}{\Gamma(\alpha+1)}\sum_{i=1}^{m}c_i\left(\frac{2K}{1-L}\right)(\log\tau_i)^{\alpha}\|x_1-x_2\|_{C_{1-\gamma,\log}}\\&\quad+\left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha+1)}(\log b)^{1-\gamma+\alpha}\|x_1-x_2\|_{C_{1-\gamma,\log}}\\&\le\left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha+1)}\\&\times\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha}+(\log b)^{1-\gamma+\alpha}\right)\|x_1-x_2\|_{C_{1-\gamma,\log}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\|Nx_1-Nx_2\|_{C_{1-\gamma,\log}}\\&\le\left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha+1)}\\&\times\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha}+(\log b)^{1-\gamma+\alpha}\right)\|x_1-x_2\|_{C_{1-\gamma,\log}}.\end{aligned}$$

From (10), it follows that N has a unique fixed point which is solution of system (1). The proof of the Theorem 1 is complete.

The existence result for the problem (1) will be proved by using the Schaefer's fixed point theorem.

Theorem 2. *Assume that (A1),(A2) hold. Then, the problem (1) has at least one solution in $C^{\gamma}_{1-\gamma,\log}[J,X]\subset C^{\alpha,\beta}_{1-\gamma,\log}[J,X]$.*

Proof. For sake of clarity, we split the proof into a sequence of steps.

Consider the operator $N: C_{1-\gamma,\log}[J,X]\to C_{1-\gamma,\log}[J,X]$.

$$\begin{aligned}(Nx)(t)&=\frac{Z}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s}\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s}\end{aligned}\tag{13}$$

It is obvious that the operator N is well defined.

Step 1. N is continuous.

Let x_n be a sequence such that $x_n\to x$ in $C_{1-\gamma,\log}[J,X]$. Then for each $t\in J$,

$$\begin{aligned}&\left|(\log t)^{1-\gamma}((Nx_n)(t)-(Nx)(t))\right|\\&\le\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|g_{x_n}(s)-g_x(s)|\frac{d}{ds}\\&+\frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|g_{x_n}(s)-g_x(s)|\frac{d}{ds}\\&\le\frac{|Z|}{\Gamma(\alpha+1)}\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha}\|g_{x_n}(\cdot)-g_x(\cdot)\|_{C_{1-\gamma,\log}}\\&+\frac{(\log t)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\|g_{x_n}(\cdot)-g_x(\cdot)\|_{C_{1-\gamma,\log}}\\&\le\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha}+(\log b)^{1-\gamma+\alpha}\right)\|g_{x_n}(\cdot)-g_x(\cdot)\|_{C_{1-\gamma,\log}}.\end{aligned}$$

Since g_x is continuous (i.e., f is continuous), then we have

$$\|Nx_n-Nx\|_{C_{1-\gamma,\log}}\to 0\quad\text{as}\quad n\to\infty.$$

Step 2. N maps bounded sets into bounded sets in $C_{1-\gamma,\log}[J,X]$.

Indeed, it is enough to show that $\eta>0$, there exists a positive constant l such that $x\in B_\eta=\{x\in C_{1-\gamma,\log}[J,X]:\|x\|\le\eta\}$, we have $\|Nx\|_{C_{1-\gamma,\log}}\le l$.

$$\begin{aligned}&\left|(Nx)(t)(\log t)^{1-\gamma}\right|\\&\le\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|g_x(s)|\frac{d}{ds}\\&+\frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|g_x(s)|\frac{d}{ds}\\&:=A_1+A_2.\end{aligned}$$

and

$$\begin{aligned}|g_x(t)|&=|f(t,x(t),x(\lambda t),g_x(t))|\\&\le l(t)+p(t)|x(t)|+q(t)|x(\lambda t)|+r(t)|g_x(t)|\\&\le l^*+p^*|x(t)|+q^*|x(\lambda t)|+r^*|K_x(t)|\\&\le\frac{l^*+p^*|x(t)|+q^*|x(\lambda t)|}{1-r^*},\end{aligned}$$

where

$$\begin{aligned}A_1&=\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|g_x(s)|\frac{ds}{s}\\&=\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\\&\times\left(\frac{l^*+p^*|x(s)|+q^*|x(\lambda s)|}{1-r^*}\right)\frac{ds}{s}\\&=\frac{|Z|}{(1-r^*)}\\&\times\sum_{i=1}^{m}c_i\left(\frac{l^*(\log\tau_i)^{\alpha}}{\Gamma(\alpha+1)}+(p^*+q^*)\frac{(\log\tau_i)^{\alpha}}{\Gamma(\alpha+1)}\|x\|_{C_{1-\gamma,\log}}\right).\\A_2&=\frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|g_x(s)|\frac{ds}{s}\\&=\frac{1}{1-r^*}\\&\times\left(\frac{l^*(\log b)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}+(p^*+q^*)\frac{(\log b)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\|x\|_{C_{1-\gamma,\log}}\right).\end{aligned}$$

From A_1 and A_2, we have

$$\begin{aligned}&\left|(Nx)(t)(\log t)^{1-\gamma}\right|\\&\le\frac{l^*}{(1-r^*)\Gamma(\alpha+1)}\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha}+(\log b)^{1-\gamma+\alpha}\right)\\&+\frac{(p^*+q^*)}{(1-r^*)\Gamma(\alpha+1)}\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha}+(\log b)^{1-\gamma+\alpha}\right)\|x\|_{C_{1-\gamma,\log}}\\&:=l.\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous set of $C_{1-\gamma,\log}[J,X]$.

Let $t_1, t_2 \in J$, $t_2 \leq t_1$ and $x \in B_\eta$. Using the fact f is bounded on the compact set $J \times B_\eta$ (thus $\sup_{(t,x)\in J\times B_\eta} \|K_x(t)\| := C_0 < \infty$), we will get

$$\begin{aligned}
&|(Nx)(t_1) - (Nx)(t_2)| \\
&\leq \frac{|Z|}{\Gamma(\alpha)} C_0 \sum_{i=1}^{m} \frac{(\log \tau_i)^\alpha}{\alpha} \left((\log t_1)^{\gamma-1} - (\log t_2)^{\gamma-1}\right) \\
&+ \frac{C_0}{\Gamma(\alpha)} \left| \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} \frac{ds}{s} - \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \right| \\
&\leq \frac{|Z|\, C_0 \sum_{i=1}^{m} c_i (\log \tau_i)^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{C_0}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{ds}{s} \\
&\quad + \frac{C_0}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Step 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $N : C_{1-\gamma,\log}[J,X] \to C_{1-\gamma,\log}[J,X]$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$\omega = \{x \in C_{1-\gamma,\log}[J,X] : x = \delta(Nx), \quad 0 < \delta < 1\}$$

is bounded set.

Let $x \in \omega$, $x = \delta(Nx)$ for some $0 < \delta < 1$. Thus for each $t \in J$. We have

$$\begin{aligned}
&x(t) = \\
&\delta \left[\frac{|Z|}{\Gamma(\alpha)} (\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right) g_x(s) \frac{ds}{s} \right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_x(s) \frac{ds}{s} \right].
\end{aligned}$$

This implies by (H2) that for each $t \in J$, we have

$$\begin{aligned}
&\left|x(t)(\log t)^{1-\gamma}\right| \\
&\leq \left|(Nx)(t)(\log t)^{1-\gamma}\right| \\
&\leq \frac{l^*}{(1-r^*)\Gamma(\alpha+1)} \left(|Z| \sum_{i=1}^{m} c_i (\log \tau_i)^\alpha + (\log b)^{1-\gamma+\alpha} \right) \\
&+ \frac{(p^*+q^*)}{(1-r^*)\Gamma(\alpha+1)} \left(|Z| \sum_{i=1}^{m} c_i (\log \tau_i)^\alpha + (\log b)^{1-\gamma+\alpha} \right) \\
&\|x\|_{C_{1-\gamma,\log}} := R.
\end{aligned}$$

that $\|\mu\|_{C_{1-\gamma}} \leq R$.

This shows that the set ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of problem (1). The proof is complete.

4 Ulam-Hyers-Rassias stability

In the theory of functional equations there is some special kind of data dependence [20, 22, 25, 26]. For the advanced contribution on Ulam stability for fractional differential equations, we refer the reader to [21, 28–30]. In this paper, we pose different types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the implicit differential equations with H-H fractional derivative. Moreover the Ulam-Hyers stability for fractional differential equations with Hilfer fractional derivative was imposed in [18, 27].

In this section, we employ the well-known definitions of four kinds of Ulam stability. For more details, one can refer to [17, 26].

Definition 4. *The equation (1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^\gamma_{1-\gamma,\log}[J,X]$ of the inequality*

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t))\right| \leq \epsilon, \quad t \in J, \quad (14)$$

there exists a solution $x \in C^\gamma_{1-\gamma,\log}[J,X]$ of equation (1) with

$$|z(t) - x(t)| \leq C_f \epsilon, \quad t \in J.$$

Definition 5. *The equation (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C_{1-\gamma}([0,\infty),[0,\infty))$, $\psi_f(0) = 0$ such that for each solution $z \in C^\gamma_{1-\gamma,\log}[J,X]$ of the inequality*

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)\right| \leq \epsilon, \quad t \in J, \quad (15)$$

there exists a solution $x \in C^\gamma_{1-\gamma,\log}[J,X]$ of equation (1) with

$$|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 6. *The equation (1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\log}[J,X]$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^\gamma_{1-\gamma,\log}[J,X]$ of the inequality*

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)\right| \leq \epsilon\varphi(t), \quad t \in J, \quad (16)$$

there exists a solution $x \in C^\gamma_{1-\gamma,\log}[J,X]$ of equation (1) with

$$|z(t) - x(t)| \leq C_f \epsilon \varphi(t), \quad t \in J.$$

Definition 7. *The equation (1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\log}[J,X]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C^\gamma_{1-\gamma,\log}[J,X]$ of the inequality*

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)\right| \leq \varphi(t), \quad t \in J, \quad (17)$$

there exists a solution $x \in C^\gamma_{1-\gamma,\log}[J,X]$ of equation (1) with

$$|z(t) - x(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.$$

*Remark.*A function $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of the inequality (14) if and only if there exist a function $g \in C^{\gamma}_{1-\gamma,\log}[J,X]$ such that
(i) $\mid g(t) \mid \leq \epsilon, \quad \forall\, t \in J.$
(ii)
${}_HD^{\alpha,\beta}_{1^+}z(t) = f(t, z(t), z(\lambda t), {}_HD^{\alpha,\beta}_{1^+}z(\lambda t) + g(t), t \in J,$

Lemma 9.*If a function $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of the inequality (14), then with*

$$A_z = \frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} g_z(s)\frac{ds}{s}.$$

From this it follows that

$$\left|z(t) - A_z - \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g_z(s)\frac{ds}{s}\right| \leq \frac{\epsilon(\log b)^{\alpha}}{\Gamma(\alpha+1)}. \quad (18)$$

*Proof.*The proof directly follows from Remark 4 and Lemma 8

One can have similar remarks for the inequalities (16) and (17).

*Remark.*It is clear that:

1.Definition 4$\Rightarrow$ Definition 5.
2.Definition 6$\Rightarrow$ Definition 7.

The following generalized Gronwall inequalities with Caputo singular kernel will be widely used to deal with our problems in the sequence.

Lemma 10. *[30] Suppose $1 > \alpha > 0$, $\overline{a} > 0$ and $\overline{b} > 0$ and suppose $u(t)$ is nonnegative and locally integral on $[1,+\infty)$ with*

$$u(t) \leq \overline{a} + \overline{b}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} u(s)\frac{ds}{s}, \quad t \in [1,+\infty).$$

Then

$$u(t) \leq \overline{a} + \int_1^t\left[\sum_{n=1}^{\infty}\frac{(\overline{b}\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}\overline{a}\right]\frac{ds}{s}, \quad t \in [1,+\infty).$$

*Remark.*Under the assumptions of Lemma 10, let $u(t)$ be a nondecreasing function on $[1,\infty)$. Then we have

$$u(t) \leq \overline{a}E_{\alpha,1}\left(\overline{b}\Gamma(\alpha)(\log t)^{\alpha}\right),$$

where $E_{\alpha,1}$ is the Mittag-leffler function defined by

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \quad z \in \mathbb{C}.$$

Now, we are ready to state and prove the stability results.

Theorem 3.*Assume that (A3) and (10) hold. Then, the problem (1) is Ulam-Hyers stable.*

*Proof.*Let $\epsilon > 0$ and let $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ be a function which satisfies the inequality (14) and let $x \in C^{\gamma}_{1-\gamma,\log}[J,X]$ the unique solution of the following problem

$${}_HD^{\alpha,\beta}_{1^+}x(t) = f(t, x(t), x(\lambda t), {}_HD^{\alpha,\beta}_{1^+}x(\lambda t)), \quad t \in J := [0,b],$$
$$I^{1-\gamma}_{1^+}z(1) = I^{1-\gamma}_{1^+}x(1) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in [0,b],\ \gamma = \alpha+\beta-\alpha\beta,$$

where $0 < \alpha < 1, \quad 0 \leq \beta \leq 1$ and $\lambda \in (0,1)$.
Using Lemma 8, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s},$$

where

$$A_x = \frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s}.$$

On the other hand, if $I^{1-\gamma}_{1^+}z(1) = I^{1-\gamma}_{1^+}x(1)$, and $x(\tau_i) = z(\tau_i)$ then $A_x = A_z$.
Indeed,

$$\begin{aligned}
&|A_x - A_z| \\
&\leq \frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|g_x(s) - g_z(s)|\frac{ds}{s} \\
&\leq \frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} \\
&\times\left(\frac{2K}{1-L}\right)|x(s) - z(s)|\frac{ds}{s} \\
&\leq \left(\frac{2K}{1-L}\right)|Z|\,(\log t)^{\gamma-1}\sum_{i=1}^{m} c_i I^{\alpha}_{1^+}\,|x(\tau_i) - z(\tau_i)| \\
&= 0.
\end{aligned}$$

Thus,

$$A_x = A_z.$$

Then, we have

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g_x(s)\frac{ds}{s}.$$

By integration of the inequality (14) and applying Lemma 9, we obtain

$$\left|z(t) - A_z - \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g_z(s)\frac{ds}{s}\right| \leq \frac{\epsilon(\log b)^{\alpha}}{\Gamma(\alpha+1)}. \quad (19)$$

We have for any $t \in J$,

$$\begin{aligned}
&|z(t) - x(t)| \\
&\leq \left|z(t) - A_z - \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g_z(s)\frac{ds}{s}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|g_z(s) - g_x(s)|\frac{ds}{s} \\
&\leq \left|z(t) - A_z - \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g_z(s)\frac{ds}{s}\right| \\
&+ \left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|z(s) - x(s)|\frac{ds}{s}.
\end{aligned}$$

By using (19), we have,

$$|z(t)-x(t)| \le \frac{\epsilon(\log b)^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}|z(s)-x(s)|\frac{ds}{s},$$

and to apply Lemma 10 and Remark 4, we obtain

$$|z(t)-x(t)| \le \frac{1}{\Gamma(\alpha+1)}(\log b)^{\alpha}E_{\alpha,1}\left(\left(\frac{2K}{1-L}\right)(\log b)^{\alpha}\right)\cdot\epsilon, := C_f\epsilon.$$

Thus, the problem (1) is Ulam-Hyers stable.

Theorem 4. *Assume that (A3) and (10) hold. Suppose that there exists an increasing function $\varphi \in C_{1-\gamma,\log}[J,X]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$*

$$I_{1^+}^{\alpha}\varphi(t) \le \lambda_\varphi\varphi(t).$$

Then problem (1) is generalized Ulam-Hyers-Rassias stable.

Proof. Let $\epsilon > 0$ and let $z \in C_{1-\gamma,\log}^{\gamma}[J,X]$ be a function which satisfies the inequality (16) and let $x \in C_{1-\gamma,\log}^{\gamma}[J,X]$ the unique solution of the following problem

$$_HD_{1^+}^{\alpha,\beta}x(t) = f(t,x(t),x(\lambda t),_H D_{1^+}^{\alpha,\beta}x(\lambda t)), \quad t \in J := [0,b],$$
$$I_{1^+}^{1-\gamma}x(1) = I_{1^+}^{1-\gamma}z(1) = \sum_{i=1}^{m}c_ix(\tau_i), \quad \tau_i \in [0,b],\ \gamma = \alpha+\beta-\alpha\beta,$$

where $0<\alpha<1, \quad 0\le\beta\le 1$ and $\lambda \in (0,1)$.
Using Lemma 8, we obtain

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_x(s)\frac{ds}{s},$$

where

$$A_z = \frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}g_z(s)\frac{ds}{s}.$$

By integration of (16), we obtain

$$\left|z(t)-A_z-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_z(s)\frac{ds}{s}\right| \le \frac{\epsilon}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\varphi(s)\frac{ds}{s} \le \epsilon\lambda_\varphi\varphi(t). \quad (20)$$

On the other hand, we have

$$|z(t)-x(t)| \le \left|z(t)-A_z-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}g_z(s)\frac{ds}{s}\right| + \left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|z(s)-x(s)|\frac{ds}{s}.$$

By using (20), we have

$$|z(t)-x(t)| \le \epsilon\lambda_\varphi\varphi(t) + \left(\frac{2K}{1-L}\right)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|z(s)-x(s)|\frac{ds}{s},$$

and to apply Lemma 10 and Remark 4, we obtain

$$|z(t)-x(t)| \le \epsilon\lambda_\varphi\varphi(t)E_{\alpha,1}\left(\left(\frac{2K}{1-L}\right)(\log b)^{\alpha}\right), \quad t \in [1,b].$$

Thus,the problem (1) is generalized Ulam-Hyers-Rassias stable.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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