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q-Analogue of Aleph-Function and Its Transformation Formulae with q-Derivative

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Abstract: In the present paper, the authors have derived the alternative definition of q-analogue of Aleph-Function, introduced by Dutta et. al.[13], by using q-Gamma function, which is an q-extension of the generalized H-function and I-function earlier defined by Saxena [4] and some transformation formulae are also derived. The basic analogue for this function provides elegant generalizations of the various results given by Saxena in connection with q- calculus. Some special cases have also been discussed.

Keywords: Aleph Function, q-analogue of Aleph Function, q-analogue of I- Function, q-analogue of H- Function, q-analogue of G-Function, q-analogue of E-Function, q-Gamma function, q-Calculus, q-derivative operator.

1 Introduction

The q-calculus is the extension of the ordinary calculus. The subject deals with the investigations of q-integrals and q-derivatives of arbitrary order, and has gained importance due to its various applications in the areas like ordinary calculus, solution of the q-differential and q-integral equations, q-transform analysis [3,16,17,18]. Motivated by these avenues of applications, a number of workers have made use of these operators to evaluate q-calculus, basic analogue of H-function, basic analogue of I-function, general class of q-polynomials etc. Here in the present paper we too make use of these operators on new basic hypergeometric function (Aleph-Function) which is an q-extension of the generalized H-function and I-function earlier defined by Saxena [2,4].
We present some usual notions and notations used in the q-calculus see [8]. Throughout this paper, we assume q to be a fixed number satisfying $0 < q < 1$. The q-calculus begins with the definition of the q-analogue $d_q f(x)$ of the differential of functions,

$$d_q f(x) = f(qx) - f(x).$$

Having said this, we immediately get the q-analogue of the derivative of $f(x)$, called its q-derivative and is given by [15] as:

$$(D_q f)(x) = \frac{(d_q f(x))}{(d_q x)} = \frac{f(x) - f(qx)}{(1-q)x}, if x \neq 0, \tag{1}$$

$(D_q f)(0) = f^{'}(0)$, provided $f^{'}(0)$ exists. If f is differentiable, then $(D_q f)(x)$ tends to $f^{'}(0)$ as q tends to 1.
We have

$$D^n_{x,q} x^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-n+1)} x^{\mu-n}, Re(\mu)+1 > 0. \tag{2}$$

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The q-analogue of x and ∞ is defined by

$$[x]=\frac{1-q^{x}}{1-q}, and[\infty]=\frac{1}{1-q}. \tag{3}$$

Sdland et. al. [14] studied the generalized fractional drift-less Fokker-Planck equation with power law coefficient. As a result, a special function was found, which is a particular case of the Aleph-function. The Aleph function is defined by means of Mellin-Barnes type integral (Mathai and Saxena, 1978) in the following manner [1,9]:

$$\aleph(z)=\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[\left(z\left|\begin{matrix}(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]=\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)z^{-s}ds \tag{4}$$

where $z\neq 0, \omega=\sqrt{-1}$ and

$$\Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)]}$$

The integration path $L=L_{\omega\gamma\infty}, \gamma\in R$, extends from $\gamma-\omega\infty$ to $\gamma+\omega\infty$, and is such that the poles, assumed to be simple, of $\Gamma(1-a_j-A_js)$, j = 1,...,n do not coincide with the poles of $\Gamma(b_j+B_js)$, j= 1,...,m. The parameters p_i, q_i are non-negative integers satisfying $0\leq n\leq p_i$, $1\leq m\leq q_i$, $\tau_i>0$ for i =1,2,3,...,r. The parameters A_j, B_j , A_{ji}, $B_{ji}>0$ and a_j, b_j, a_{ji}, $b_{ji}\in C$. The empty product is interpreted as unity. The existence conditions for the defining integral (4) are given below:

$$\varphi_l>0, |arg(z)|<\frac{\pi}{2}\varphi_l \text{ and } R(\zeta_l)+1<0, l=1,2,3,...,r$$

where

$$\varphi_l=\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j+\tau_i(\sum_{j=m+1}^{q_i}b_{ji}-\sum_{j=n+1}^{p_i}a_{ji})$$

$$\zeta_l=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\tau_i(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}B_{ji})+\frac{1}{2}(p_i-q_i), l=1,2,3,...,r.$$

Saxena et. al.[12] introduced the following basic analogue of I-Function in terms of the Mellin-Barnes type basic contour integral as:

$$I(z)=I_{A_i,B_i;r}^{m,n}\left[\left(z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}G(q^{(b_j-\beta_js)})\prod_{j=1}^{n}G(q^{(1-a_j+\alpha_js)})}{\sum_{i=1}^{r}[\prod_{j=m+1}^{B_i}G(q^{(1-b_{ji}+\beta_{ji}s)})\prod_{j=n+1}^{A_i}G(q^{(a_{ji}-\alpha_{ji}s)})G(q^s)G(q^{1-s})sin\pi s]}\pi z^s ds \tag{5}$$

where $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive, a_j,b_j,a_{ji},b_{ji} are complex numbers and

$$G(q^{\alpha})=\prod_{n=0}^{\infty}(1-q^{\alpha+n})^{-1}=\frac{1}{(q^{\alpha};q)_{\infty}}$$

where $z\neq 0, 0<|q|<1 and \omega=\sqrt{-1}$

where L is contour of integration running from $-\omega\infty$ to$\omega\infty$ in such a manner so that all poles of $G(q^{(b_j-\beta_js)}); 1\leq j\leq m$ are to right of the path and those of $G(q^{(1-a_j+\alpha_js)}); 1\leq j\leq n$,are to left. The integral converges if $Re[slog(x)-logsin\pi s]<0$, for large values of $|s|$ on the contour L. Setting $r=1, A_i=A, B_i=B$, in equation (5) we get q-analogue of H-Function defined by Saxena et.al.[12] as follows:

$$H_{A,B}^{m,n}\left[\left(z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,A}\\ (b_j,\beta_j)_{1,m}\end{matrix}\right.\right)\right]=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}G(q^{(b_j-\beta_js)})\prod_{j=1}^{n}G(q^{(1-a_j+\alpha_js)})}{\prod_{j=m+1}^{B}G(q^{(1-b_j+\beta_js)})\prod_{j=n+1}^{A}G(q^{(a_j-\alpha_js)})G(q^s)G(q^{1-s})sin\pi s}\pi z^s ds \tag{6}$$

Further if we put $\alpha_j = \beta_j = 1$, equation (6) reduces to the basic analogue of Meijer's G-Function given by Saxena et. al.[12].

$$G_{A,B}^{m,n}\left[\left(z;q\middle|\begin{matrix}(a_1,a_2,...,a_A)\\(b_1,b_2,...,b_B)\end{matrix}\right)\right] = \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m} G(q^{(b_j-s)})\prod_{j=1}^{n} G(q^{(1-a_j+s)})}{\prod_{j=m+1}^{B} G(q^{(1-b_j+s)})\prod_{j=n+1}^{A} G(q^{(a_j-s)})G(q^s)G(q^{1-s})sin\pi s}\pi z^s ds \quad (7)$$

Dutta et. al.[13] defined the q-analogue of Aleph-Function in term of Mellin-Barnes type contour integral in the following manner:

$$\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[\left(z;q\middle|\begin{matrix}(a_j,A_j)_{1,n} \ldots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} \ldots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right)\right]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m} G(q^{(b_j-B_js)})\prod_{j=1}^{n} G(q^{(1-a_j-A_js)})}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i} G(q^{(1-b_{ji}+B_{ji}s)})\prod_{j=n+1}^{p_i} G(q^{(a_{ji}-A_{ji}s)})G(q^s)G(q^{1-s})sin\pi s]}\pi z^s ds \quad (8)$$

where $z \neq 0, 0 < |q| < 1 and \omega = \sqrt{-1}$
The parameters p_i, q_i are non-negative integers satisfying the inequality $0 \leq n \leq p_i, 0 \leq m \leq q_i$ and $\tau_i > 0; i = 1,2,3,,r$ is finite and A_j, B_j, A_{ji}, B_{ji} are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers. The $C = C_{\omega\gamma\infty}$ is a suitable contour of Mellin-Barnes type in the complex s-plane, which runs from $\gamma - \omega\infty$ to $\gamma + \omega\infty$ with $\gamma \in C$, in such a manner so that all poles of $G(q^{(b_j-B_js)}); 1 \leq j \leq m$, separating from those of $G(q^{(1-a_j+A_js)})$; $1 \leq j \leq n$. All the poles of the integrand (8) are assumed to be simple and empty products are interpreted as unity. The integral converges if $Re[slog(z) - logsin\pi s] < 0$, for large values of $|s|$ on the contour L, that is if $|(arg(z) - w_2 w_1^{-1} log|z|)| < \pi$, where $0 < |q| < 1, logq = -w = -(w_1 + iw_2), w, w_1, w_2$ are definite quantities, w_1, w_2 being real. If we take $\tau_i = 1$ in (8), then (5) is recovered and if we set r=1 in (5), then we get (6). If we set $A_i = B_j = 1$ for all i and j in (6), then it reduces to (7).
If we set $n = 0, m = B$ in (7), then it reduces to the basic analogue of MacRobert's E-function given below:

$$G_{A,B}^{B,0}\left[\left(z;q\middle|\begin{matrix}(a_1,a_2,...,a_A)\\(b_1,b_2,...,b_B)\end{matrix}\right)\right] = E_q[B; b_j : A; a_j : z]$$

2 Main Results

In this section, the authors have defined the alternative definition of q-analogue of Aleph-Function by using q-Gamma function and have derived some of its transformation formulae in connection with q-calculus.

2.1. q-analogue of Aleph function:

We shall make use of $\aleph(z;q)$ notation for q-analogue of Aleph-Function and the same is defined as:
Theorem 1: Let the parameters p_i, q_i are non-negative integers satisfying the inequality $0 \leq n \leq p_i, 0 \leq m \leq q_i and \tau_i > 0; i = 1,2,3,,r$ is finite and A_j, B_j, A_{ji}, B_{ji} are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers, then

$$[(1-q)^{\sum_{t=1}^{n} a_t - \sum_{t=1}^{m} b_t + m + n - 1 + \sum_{i=1}^{r}\tau_i[\sum_{t=n+1}^{p_i} a_{ti} - \sum_{t=m+1}^{q_i} b_{ti} - A_i]} G(q)^{\sum_{i=1}^{r} p_i + q_i - 2(m+n-1)}] \times$$

$$\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[\left(z(1-q)^{\sum_{t=1}^{m} B_t - \sum_{t=1}^{n} A_t + \sum_{i=1}^{r}\tau_i[\sum_{t=m+1}^{q_i} B_{ti} - \sum_{t=n+1}^{p_i} A_{ti}]}; q\middle|\begin{matrix}(a_j,A_j)_{1,n} \ldots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} \ldots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}(1,1)\end{matrix}\right)\right]$$

$$= \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[\left(z;q\middle|\begin{matrix}(a_j,A_j)_{1,n} \ldots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} \ldots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right)\right] \quad (9)$$

where $z \neq 0, 0 < |q| < 1 and \omega = \sqrt{-1}$

Proof: To prove (9) we consider the expression

$$\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z(1-q)^{\sum_{t=1}^{m}B_t-\sum_{t=1}^{n}A_t+\sum_{i=1}^{r}\tau_i[\sum_{t=m+1}^{q_i}B_{ti}-\sum_{t=n+1}^{p_i}A_{ti}]};q\left|\begin{matrix}(a_j,A_j)_{1,n}\ \dots\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\ \dots\ [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}(1,1)\end{matrix}\right.\right)\right]$$

$$=\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}G(q^{(b_j-B_js)})\prod_{j=1}^{n}G(q^{(1-a_j-A_js)})\pi z^s(1-q)^{s[\sum_{t=1}^{m}B_t-\sum_{t=1}^{n}A_t+\sum_{i=1}^{r}\tau_i[\sum_{t=m+1}^{q_i}B_{ti}-\sum_{t=n+1}^{p_i}A_{ti}]]}}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}G(q^{(1-b_{ji}+B_{ji}s)})\prod_{j=n+1}^{p_i}G(q^{(a_{ji}-A_{ji}s)})G(q^s)G(q^{1-s})sin\pi s]}ds \tag{10}$$

On multiplying (10) by

$$[(1-q)^{\sum_{t=1}^{n}a_t-\sum_{t=1}^{m}b_t+m+n-1+\sum_{i=1}^{r}\tau_i[\sum_{t=n+1}^{p_i}a_{ti}-\sum_{t=m+1}^{q_i}b_{ti}-A_i]}G(q)^{\sum_{i=1}^{r}p_i+q_i-2(m+n-1)}]$$

And making use of the identity given by Askey [5]

$$\Gamma_q(x)=\frac{G(q^x)}{(1-q)^{x-1}G(q)};|q|<1,$$

we get (9) as follows:

$$=\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j-A_js)\pi z^s}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$=\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n}\ \dots\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\ \dots\ [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

where L is contour of integration running from $-\omega\infty$ to $+\omega\infty$ in such a manner so that all poles of$\Gamma_q(b_j+B_js); 1 \leq j \leq m$ are to right of the path and those of $\Gamma_q(1-a_j-A_js); 1 \leq j \leq n$,are to left. The integral converges if $Re[slog(z)-logsin\pi s] < 0$, for large values of $|s|$ on the contour L, that is if $|(arg(z)-w_2w_1^{-1}log|z|)| < \pi$, where $0 < |q| < 1, logq = -w = -(w_1+iw_2), w, w_1, w_2$ are definite quantities, w_1, w_2being real.

Remark: By setting $\tau_i = 1$ in (9), we get well known result for basic analogue of I-function as reported in [4] which is as follows:

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,A_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)}{\sum_{i=1}^{r}[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}\pi z^s ds \tag{11}$$

The existence conditions for the integral in (11) are the same as for q-analogue of Aleph-function with $\tau_i = 1, i = 1, 2, , r$. Moreover taking r=1 in (11) we get well known result as in [4] as:

$$H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right]=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)}{\prod_{j=m+1}^{Q}\Gamma_q(1-b_j+B_js)\prod_{j=n+1}^{P}\Gamma_q(a_j-A_js)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}\pi z^s ds \tag{12}$$

The existence conditions for the integral in (12) are the same as for q-analogue of I-Function with r = 1.

2.2. Some transformation formulae of $\aleph(z;q)$ Function

(I) Let the parameters p_i, q_i are non-negative integers satisfying the inequality $0 \le n \le p_i, 0 \le m \le q_i and \tau_i > 0; i = 1,2,3,,r$ is finite and A_j, B_j, A_{ji}, B_{ji} are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers, then

$$\aleph(z;q) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a,0)(a_j,A_j)_{2,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \Gamma_q(1-a)\,\aleph^{m,n-1}_{p_i-1,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{2,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{13}$$

where $z \neq 0, 0 < |q| < 1 and \omega = \sqrt{-1}$.
Proof: By definition of $\aleph$(z;q)-function, we get L.H.S.

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j+B_js)\Gamma_q(1-a-0.s)\prod_{j=2}^{n}\Gamma_q(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}+A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$= \Gamma_q(1-a)\times\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=2}^{n}\Gamma_q(1-a_j+A_js)\pi z^{s}}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$= \Gamma_q(1-a)\,\aleph^{m,n-1}_{p_i-1,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{2,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

= R.H.S. This proves the theorem.
In the same manner we can prove the following results.
(II)

$$\aleph(z;q) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i-1(a,0)}\\ (b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \frac{1}{\Gamma_q(a)}\aleph^{m,n}_{p_i-1,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{2,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i-1}\\ (b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{14}$$

(III)

$$\aleph(z;q) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b,0)(b_j,B_j)_{2,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \Gamma_q(b)\,\aleph^{m-1,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{2,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{15}$$

(IV)

$$\aleph(z;q) = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b,0)(b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i-1(b,0)}\end{matrix}\right.\right)\right]$$

$$= \frac{1}{\Gamma_q(1-b)}\aleph^{m,n}_{p_i,q_i-1,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i-1}\end{matrix}\right.\right)\right] \tag{16}$$

Special Cases:

(i) By setting τ_i= 1 in (13), we get well known result for basic analogue of I-function as reported in [3,4] which is as follows:

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a,0)(a_j,A_j)_{2,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]=\Gamma_q(1-a)I^{m,n-1}_{p_i-1,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{2,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{17}$$

Moreover taking r = 1 in (17) we get well known result as in [4]as:

$$H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a,0)(a_j,A_j)_{2,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right]=\Gamma_q(1-a)H^{m,n-1}_{P-1,Q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{2,P-1}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right] \tag{18}$$

(ii) Taking τ_i= 1 in (14), we get well known formula for basic analogue of I-function as reported in [3,4] as:

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i-1(a,0)}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]=\frac{1}{\Gamma_q(a)}I^{m,n}_{p_i-1,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i-1}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{19}$$

Again, taking r = 1 in (19) we get well known formula of Fox's basic analogue of H-function as:

$$H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P-1}(a,0)\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right]=\frac{1}{\Gamma_q(a)}H^{m,n}_{P-1,Q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P-1}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right] \tag{20}$$

(iii) Taking τ_i= 1 in (15), we get well known formula for basic analogue of I-function as reported in [3,4] as:

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b,0)(b_j,B_j)_{2,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]=\Gamma_q(b)I^{m-1,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i-1}\\(b_j,B_j)_{2,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{21}$$

Again, taking r = 1 in (21) we get well known formula of Fox's basic analogue of H-function as:

$$H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b,0)(b_j,B_j)_{2,Q}\end{matrix}\right.\right)\right]=\Gamma_q(b)H^{m,n}_{P,Q-1}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{2,Q}\end{matrix}\right.\right)\right] \tag{22}$$

(iv) Taking τ_i= 1 in (16), we get well known formula for basic analogue of I-function as reported in [3,4] as:

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i-1}(b,0)\end{matrix}\right.\right)\right]=\frac{1}{\Gamma_q(1-b)}I^{m,n}_{p_i,q_i-1;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i-1}\end{matrix}\right.\right)\right] \tag{23}$$

Again, taking r = 1 in (23) we get well known formula of Fox's basic analogue of H-function as:

$$H^{m,n}_{P,Q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q-1}(b,0)\end{matrix}\right.\right)\right]=\frac{1}{\Gamma_q(1-b)}H^{m,n}_{P,Q-1}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q-1}\end{matrix}\right.\right)\right] \tag{24}$$

(2.3) In this section, we will evaluate the q-derivative operator involving q-analogue of Aleph-Function.

Theorem 2: Let the parameters p_i, q_i are non-negative integers satisfying the inequality $0 \le n \le p_i, 0 \le m \le q_i$ and $\tau_i > 0; i = 1,2,3,,r$ is finite and A_j, B_j, A_{ji}, B_{ji} are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers, then

$$zD_{z,q}[z^{1-a_1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,1)_{1,n} & \dots & [\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m} & \dots & [\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right.\right)\right]]$$

$$=z^{1-a_1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n} & \dots & [\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m} & \dots & [\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{25}$$

where $z \neq 0, 0 < |q| < 1 and \omega = \sqrt{-1}$.

Proof: To prove theorem (25) when $a_1 \geq 0$, we apply equation (2)

$$L.H.S. = zD_{z,q}[z^{1-a_1}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi z^s}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds]$$

$$= z\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi D_{z,q}[z^{1-a_1+s}]}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi[1-a_1+s]_q[z^{1-a_1+s}]}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

Since,

$$\Gamma_q(1+a) = \frac{1-q^a}{1-q}\Gamma_q(a) = [a]_q\Gamma_q(a)$$

$$[a]_q\Gamma_q(a) = \Gamma_q(1+a)$$

Therefore $[1-a_1+s]_q\Gamma_q(1-a_1+s) = \Gamma_q(1-(a_1-1)+s)$
Thus

$$L.H.S. = \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\Gamma_q(1-(a_1-1)+s)\prod_{j=2}^{n}\Gamma_q(1-a_j+s)\pi z^s z^{1-a_1}}{\sum_{i=1}^r \tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

Which implies,

$$zD_{z,q}[z^{1-a_1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,1)_{1,n} \dots [\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m} \dots [\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right.\right)\right]] = z^{1-a_1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n} \dots [\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m} \dots [\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

Hence the result.
Theorem 3: Let the parameters p_i, q_i are non-negative integers satisfying the inequality $0 \leq n \leq p_i, 0 \leq m \leq q_i$ and $\tau_i > 0; i = 1,2,3,,r$ is finite and A_j, B_j, A_{ji}, B_{ji} are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers, then

$$D^{\mu}_{z,q}[\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z^{\lambda};q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]]$$

$$= z^{-\mu}\aleph^{m,n+1}_{p_i,q_i+1,\tau_i;r}\left[\left(z^{\lambda};q\left|\begin{matrix}(0,\lambda)(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}(\mu,\lambda)\end{matrix}\right.\right)\right] \quad (26)$$

where $z \neq 0, 0 < |q| < 1 and \omega = \sqrt{-1}$.
Proof: To prove theorem (26) when $\lambda \geq 0$, we apply equation (2)

$$D^{\mu}_{z,q}[\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[\left(z^{\lambda};q\left|\begin{matrix}(a_j,A_j)_{1,n} \dots [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m} \dots [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j - B_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_j s)\pi D_{z,q}^{\mu}[z^{\lambda s}]}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j - B_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_j s)\pi\Gamma_q(\lambda s+1)[z^{\lambda s-\mu}]}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-A_{ji}s)\Gamma_q(\lambda s-\mu+1)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$= \frac{z^{-\mu}}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j - B_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_j s)\Gamma_q(1-0+\lambda s)\pi[z^{\lambda s}]}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\Gamma_q(1-\mu+\lambda s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}-A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

$$= z^{-\mu}\aleph_{p_i,q_i+1,\tau_i;r}^{m,n+1}\left[\left(z^{\lambda};q\left|\begin{matrix}(0,\lambda)(a_j,A_j)_{1,n} & \dots & [\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (b_j,B_j)_{1,m} & \dots & [\tau_i(b_{ji},B_{ji})]_{m+1,q_i}(\mu,\lambda)\end{matrix}\right.\right)\right] \tag{27}$$

Hence the result.

Conclusion

The results proved in this paper give some contributions to the theory of the basic hypergeometric functions and are believed to be a new to the theory of q- calculus and are likely to find certain applications to the solution of the q-integral equations involving various q-hypergeometric functions.

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