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Volume Time Functions and K-Causality

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Abstract: Recent researches show that there are more relations rather than causal and chronological relations which are important in general relativity. One of these relations is K^+, the smallest closed, transitive relation which contains I^+. In This paper an equivalent condition for inner continuity of $int(K^+(.))$, by using of admissible measure is given.

Keywords: General relativity, Spacetime, Causally continuous, K- causal, Admissible measure.

1 Introduction

The causal relations are usually presented through their point based counterparts, namely the sets $I^{\pm}(x)$, $J^{\pm}(x)$ [1, 4]. However it is shown in [9,5,6] that more natural and effective approach is to regard them as subsets of $M\times M$.
The relation $R^+ \subseteq M\times M$ is transitive if for all p,q and $z\in M$, $(p,q)\in R^+$ and $(q,z)\in R^+$ implies that $(p,z)\in R^+$. It is reflexive if for all $x\in M$, $(x,x)\in R^+$.
The relation R^+ is antisymmetric if for all $p,q\in M$, $(p,q)\in R^+$ and $(q,p)\in R^+$ implies that $p=q$. R^+ is a causal relation if $I^+\subseteq R^+$.
R^+ is a reflexive partial order if it is reflexive, transitive and antisymmetric.
Given a relation R^+ one can define two operations:

– closure: $R^+ \to \overline{R^+}$.
– transitivization: $R^+ \to R^{+\infty} = \cup_{i=1}^{\infty}(R^+)^i$, that $(R^+)^i = \{(p,q): \exists\, p_1,..,p_{i-1}\in M : (p,p_1)\in R^+, (p_1,p_2)\in R^+,..,(p_{i-1},q)\in R^+\}$, for $i\geq 1$.

These operators are useful for the definition of a new causal relation. Sorkin and Woolgar [9] have defined $K^+\subseteq M\times M$ as the smallest transitive closed relation which contains I^+. This definition arose from the fact that J^+ is transitive but not necessarily closed and $\overline{J^+}$ is closed but not necessarily transitive. The spacetime (M,g) is $K-$causal if K^+ is antisymmetric. It is proved that K-causality is equivalent to the stable causality [7].
Definition 1.1. $R^{\pm}$, is inner(resp. outer) continuous at some $p\in M$ if, for any compact subset $K\subseteq R^{\pm}(p)$(resp. $K\subseteq M-\overline{R^{\pm}(p)}$), there exists an open neighborhood $U\ni p$ such that $K\subseteq R^{\pm}(q)$(resp. $K\subseteq M-\overline{R^{\pm}(q)}$) for all $q\in U$.
For example $I^{\pm}$ are always inner continuous but they are not necessarily outer continuous [1,4,8]. The spacetime (M,g) is called causally continuous if $I^{\pm}$ are outer continuous.

2 Admissible measure

An equivalent relation for causal continuity is given by using of admissible measure.
Geroch used volumes of $I^{\pm}(.)$, in [3]. But such volumes must be finite. So he used Admissible measure. Let us recall the construction of a Borel measure on M, that is a measure on the σ- algebra generated by the open subsets of M. This measure is called Admissible measure [1,8,4].
Let ω be an oriented volume element associated to the metric g. Choose a countable atlas on M, with ω- measure smaller than one and a partition of unity $\{\rho_n\}$ subordinated to this covering. Let m be the associated measure to the volume element

$$\omega^* = \sum 2^{-n}\rho_n\omega.$$

If we choose any auxiliary Riemannian metric g_R with associated oriented volume element ω_R then for some smooth function, f, we have:

$$\omega^* = e^f\omega_R.$$

Thus ω^* is also the volume element associated to the Riemannian metric $g_R^* = e^{2f/n_0}g_R$, where n_0 is the

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dimension of M. We can assume that m is completed in the standard way by adding to the Borel sigma algebra all subsets of any subset of measure 0. The relevant properties of this measure is as follows:

– Finiteness: $m(M) < \infty$.
– For any nonempty open subset U, $m(U) > 0$.
– The boundaries $\partial I \pm (p)$ have measure 0, for any $p \in M$.
– For any measurable subset $A \subset M$ there exists a sequence $\{G_n\}$ of open subsets which contains A, and a sequence $\{K_n\}$ of compact subsets contained in A such that $G_n \supset G_{n+1}$, $K_n \subset K_{n+1}$ for all n and:

$$m(A) = lim\, m(G_n) = lim\, m(K_n).$$

Theorem 2.1.[4] If S is a future set then the boundary of S is a closed, imbedded, achronal submanifold.
We recall that a set S is a future set if $I^+(S) \subset S$ and a set R is a achronal set if $I^+(R) \cap R = \emptyset$.
The third property of admissible measure is satisfied because the boundary of $I^+(.)$, which is a future set, by using of the above theorem is closed, imbedded, achronal hypersurface and hence can be written as Lipschizian graphs, which have measure 0.
Definition 2.1. Let (M,g) be a spacetime with an admissible measure m. The future t^- and past t^+ volume functions associated to m are defined as:

$$t^-(p) = m(I^-(p)), \quad t^+(p) = m(I^+(p)), \quad \forall p \in M.$$

Theorem 2.2.[8] The following properties are equivalent for a spacetime:

– The set volume map I^- (resp. I^+) is outer continuous.
– Volume function t^- (resp. t^+) is continuous.

$int(K^+(.))$ is a future set too. Hence by using of theorem 2.1 its boundary is a closed, imbedded, achronal submanifold and consequently $m(\partial(K^{\pm}(p))) = 0$, for any $p \in M$.
We define the future and past volume K- functions respectively by:

$$k^-(p) = m(int(K^-(p))), \quad k^+(p) = m(int(K^+(p))), \quad \forall p \in M.$$

Lemma 2.1.[2] $int(K^+(.))$ and $int(K^-(.))$ are outer continuous.

Proof. Given a point x and a compact set $C \subseteq M$ with $C \subseteq M - \overline{int(K^+(x))} = M - K^+(x)$, $x \notin K^-(C)$ and therefore, by closure of $K^-(C)$, there must be a neighbourhood U_x with $U_x \cap K^-(C) = \emptyset$.

Lemma 2.2.[2] In a K- causal spacetime (M,g), $int(K^+(.))$ and $int(K^-(.))$ are inner continuous if and only if for every $x, y \in M$, $x \in int(K^-(y)) \Leftrightarrow y \in int(K^+(x))$.

Proof. Suppose $int(K^+(.))$ and $int(K^-(.))$ are inner continuous. If $x \in int(K^-(y))$, there must be a neighbourhood U_y of y such that $x \in int(K^-(y_0))$, for every $y_0 \in U_y$. therefore $U_y \subseteq K^+(x)$ and $y \in int(K^+(x))$.
Conversely, suppose that for every $x,y \in M$, $x \in int(K^-(y)) \Leftrightarrow y \in int(K^+(x))$. Consider any $y \in M$ and any compact $C \subseteq int(K^-(y))$. For every $x \in C$, the condition implies that we can find points $z \gg x$ and $w \ll y$ such that $z \in int(K^-(w))$ and therefore neighbourhoods $U_x \subseteq I^-(z)$ of x and $U_y^x \subseteq I^+(w)$ of y so that $U_x \subseteq int(K^-(y_0))$, for every $y_0 \in U_y^x$. The cover $\{U_x, \ x \in C\}$ of C must have a finite subcover, so $C \subseteq \cup_{j=1}^n U_{x_j}$, then $C \subseteq int(K^-(y_0))$, for every $y_0 \in U_y$, so that $int(K^-(.))$ is inner continuous.

Theorem 2.3. The outer continuity of $int(K^-(.))$ (resp. $int(K^+(.))$) is equivalent to the upper (resp. lower) semi continuity of k^- (resp. k^+).

Proof. As $int(K^-(.))$ is outer continuous only the implication to the right must be proved. Fix ε. Let K be a compact subset of $M - K^-(p)$ with $m(K) > m(M - K^-(p)) - \varepsilon$. If $\{p_n\} \to p$ then for large n, $k^-(p_n) \leq m(M) - m(K) < k^-(p) + \varepsilon$.

Theorem 2.4. The inner continuity of $int(K^-(.))$ (resp. $int(K^+(.))$) is equivalent to the lower (resp. upper) semi continuity of k^- (resp. k^+).

Proof. Let $\{p_n\} \to p$, fix $\varepsilon > 0$. Let K be the compact subset of $int(K^-(p))$ such that $m(K) > m(K^-(p)) - \varepsilon = k^-(p) - \varepsilon$. $K \subset int(K^-(p_n))$, for large n. Thus $k^-(p_n) \geq m(K) > k^-(p) - \varepsilon$.
conversely suppose that $int(K^-(.))$ is not inner continuous. There is a compact set $K \subset int(K^-(p))$ and a sequence $\{p_n\}$, $p_n \to p$, such that $r_n \in K \cap (M - int(K^-(p_n)))$. Since K is compact, $r_n \to r \in K$. We choose $s \in I^+(r)$ with $s \in int(K^-(p))$. There are neighborhoods $U \subseteq int(K^-(p))$ and $V \subseteq int(K^-(p))$ of r and s, respectively such that $(U,V) \subseteq I^+$. $V \subseteq M - K^-(p_n)$, for sufficiently large n, since if there is $v \in V$ such that $v \in K^-(p_n)$ then $r_n \in int(K^-(p_n))$ which is a contradiction. We choose the sequence $q_j \to p$, with $q_j \ll q_{j+1} \ll p$. Let $\varepsilon = m(V)$. $V \cap int(K^-(q_j)) = \emptyset$ since if $v \in V \cap int(K^-(q_j))$ then $r_n \in int(K^-(p_n))$ which is a contradiction. Hence $k^-(q_j) \leq k^-(p) - \varepsilon$.

Corollary 2.1. The following properties are equivalent for a spacetime.

– $int(K^-(.))$ (resp. $int(K^+(.))$) is inner continuous
– Volume k- function k^- (resp. k^+) is continuous.

Theorem 2.5. If (M,g) is a K- causal spacetime, then k^- and k^+ are generalized time functions.

Proof. Suppose that $(p,q) \in K^+$, $p \neq q$ and $k^-(p) = k^-(q)$. $K^-(p) \subseteq K^-(q)$ and since $m(K^-(q)) = m(K^-(p))$, almost all the points in $K^-(q)$ belongs to $K^-(p)$. hence there is a sequence q_n in $K^-(p)$ that converges to q. Since $K^-(p)$ is closed $q \in K^-(p)$, which is a contradiction to the K- causality of space time.

3 Conclusion

Since the relation K^+ plays an important role in causality theory, investigating about its inner and outer continuity is valuable. In this paper it is shown that inner continuity (outer continuity) of $K^\pm$ is equivalent to lower (upper) continuity of functions, $k^\pm$. It seems that these results leads us to add a new type of spacetime in the causal ladder of spacetime, between causal continuous spacetime and stably causal spacetime.

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