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A New Distribution to Analyze a Practical Problem with Applications

Mahmoud M. Mansour * *and Enayat M. Abd Elrazik*

Dept. of Statistics, Mathematics and Insurance, Benha University, Egypt.

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Abstract: This paper introduces a new four-parameter distribution motivated mainly by dealing with series-parallel or parallel-series systems is introduced. Moments, conditional moments, mean deviations, moment generating function, probability weighted moments, quantile, Lorenz and Bonferroni curves of the new distribution including are presented a. Entropy measures are given and estimation of its parameters is studied. A real data application is described to show its superior performance versus some known lifetime models.

Keywords: Weibull distribution; Geometric distribution; Maximum likelihood estimation; truncated Poisson distribution

1 Introduction

This research aims at introducing a new four-parameter lifetime distribution with physical motivations. The proposed distribution gives preferred fits over a large number of the known lifetime distributions, including those with four parameters.
Our second motivation is based on a practical situation, where a company has two factories. The productions of the two factories are assembled together before going to the market. However, the production manager will interested in analyzing the characteristics of each production line separately. The problem here is that one factory consists of machines in sequence (connected in series) with a replicated of each individual machine, while the other factory is a collection of machines connected in tandem as subsystems and the subsystems are connected in parallel. So, this system that can handle a mix of series-parallel or parallel-series configuration, i.e., Min Max X_i or Max Min X_i. Then, the cumulative distribution function of X, say $G(x)$, can be derived as follows:

1.1 Distribution of series-parallel system (Mini Max)

Nadarajah et al. [8] introduced a two-parameter distribution which represent a general model by taking the probability density function of the cumulative distribution function of failure times to be given by $f()$ and $F()$, respectively. Its cdf is given by

$$G(x) = \frac{exp\left(-\lambda+\lambda F(x)\right)-e^{-\lambda}}{1-e^{-\lambda}-\pi+\pi exp\left(-\lambda+\lambda F(x)\right)}, \tag{1}$$

for $x>0, \lambda>0$ and $0<\pi<1$. the corresponding probability density function is,

$$g(x) = \frac{\lambda(1-\pi)(1-e^{-\lambda})f(x)exp\left(-\lambda+\lambda F(x)\right)}{\left(1-e^{-\lambda}-\pi+\pi exp\left(-\lambda+\lambda F(x)\right)\right)^2}, \tag{2}$$

* Corresponding author e-mail: mahmoud.mansour@fcom.bu.edu.eg

The Weibull distribution is a widely used statistical model for studying fatigue and endurance life in engineering devices and materials. So we chose it as a base distribution. A random variable X is said to have a Weibull distribution with parameterα,β.if its probability density is defined as,

$$f(x)=\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}},x>0,\alpha>0,\beta>0. \tag{3}$$

The corresponding cumulative distribution function (cdf) is:

$$F(x)=1-e^{-(\beta x)^{\alpha}},x>0,\alpha>0,\beta>0. \tag{4}$$

Assume that the failure times of the units for the i th system, say $Z_{(i,1)},Z_{(i,2)},\ldots,Z_{(i,M)}$, are independent and identical weibull random variables with the scale and shape Parameter β,α. Let Y_i denote the failure time of the ith system. Let X denote the time to failure of the first of the N functioning systems. We can write $X=min(Y_1,Y_2,,Y_n)$. The conditional cumulative distribution function of X given N is,

$$G(x)=\frac{exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)-e^{-\lambda}}{1-e^{-\lambda}-\pi+\pi exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)}, \tag{5}$$

for $x>0,\alpha>0,\beta>0,\lambda>0 and 0<\pi<1$. The corresponding probability density function is,

$$g(x)=\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})x^{(\alpha-1)}exp\left(-(\beta x)^{\alpha}-\lambda e^{-(\beta x)^{\alpha}}\right)}{\left(1-e^{-\lambda}-\pi[1-exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)]\right)^{2}}, \tag{6}$$

for $x>0,\alpha>0,\beta>0,\lambda>0 and 0<\pi<1$. we shall refer to the distribution given by (5) and (6) as the geometric weibull Poisson (GWP).

1.2 Distribution of Parallel-Series System (Max Mini)

Suppose the machine is made of M series units, so that the machine will fail if one of the units fail. Assume that $M$is a truncated Poisson random variable independent of$N$.in addition, the failure times of the units for the i th system, say $Z_{(i,1)},Z_{(i,2)},\ldots,Z_{(i,M)}$ are independent and identical weibull random variables with the scale and shape Parameter β,α. Let Y_i denote the failure time of the ith system. Let Xdenote the time to failure of the last of the$N$ functioning systems. We can write $X=max?(Y_1,Y_2,,Y_n)$. Then, the cumulative distribution function of X, say $G(x)$, can be derived as follows: the conditional cumulative distribution function of X given N is,

$$G(x)=\frac{(1-\pi)Pr(Y\leq x)}{1-\pi Pr(Y\leq x)}, \tag{7}$$

and

$$Pr(Y\leq x)=Pr\left[(Z_{(i,1)},Z_{(i,2)},\ldots,Z_{(i,M)})\leq x\right]$$

$$=\frac{exp(\lambda)-exp\left(\lambda e^{-(\beta x)^{\alpha}}\right)}{e^{\lambda}-1},$$

so

$$G(x)=\frac{(1-\pi)[1-exp\left(-\lambda+\lambda e^{-(\beta x)^{\alpha}}\right)-e^{-\lambda}]}{1-e^{-\lambda}-\pi+\pi exp\left(-\lambda+\lambda e^{-(\beta x)^{\alpha}}\right)} \tag{8}$$

for $x>0,\alpha>0,\beta>0,\lambda>0$ and $0<\pi<1$. the corresponding probability density function is,

$$g(x)=\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})x^{(\alpha-1)}exp\left(-(\beta x)^{\alpha}-\lambda+\lambda e^{-(\beta x)^{\alpha}}\right)}{\left(1-e^{-\lambda}-\pi[1-exp\left(-\lambda+\lambda e^{-(\beta x)^{\alpha}}\right)]\right)^{2}}, \tag{9}$$

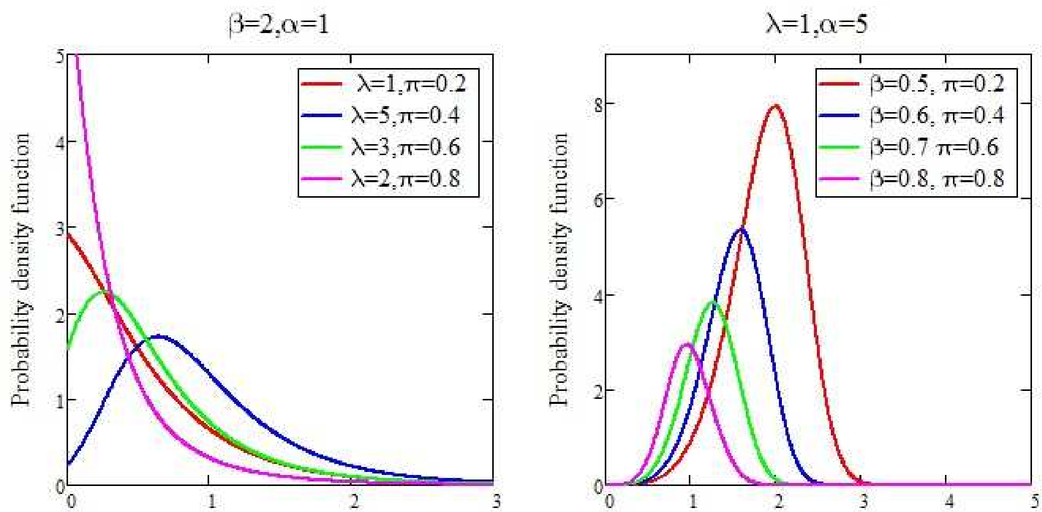
We shall refer to the distribution given by (8) and (9) as the geometric weibull Poisson (GWP1). The parameters, λ and π, control the shape. The parameter, β, controls the scale.
Ghitany et al. [6] investigated the properties of the zero-truncated Poisson-Lindley distribution. There exist some parametric models which are obtained in a compounding or mixing way with decreasing failure rate (DFR) such as the exponential geometric (EG) distribution (Adamidis and Loukas, [2]), modified Weibull geometric distribution (Wang and Elbatal, [12]), the extended exponential geometric (EEG) distribution has also been introduced by Adamidis et al. ([3]) which is an extension of the EG model with DFR and increasing failure rate (IFR) functions. Silva et al. [11] defined the generalized exponential geometric (GEG) distribution and showed that its failure rate function can be increasing, decreasing or unimodal. A Weibull geometric (WG) extension of the GE distribution was proposed by Barreto-Souza et al. for modeling monotone or unimodal failure rates. Mahmoudi and Shiran ([7]) introduced the exponentiated Weibull geometric distribution. Nadarajah et al.[8] proposed a geometric exponential Poisson distribution (GEP) and provided a comprehensive account of its mathematical properties. The failure rate function associated with (6) is given by

$$h(x)=\frac{\lambda\alpha\beta^{\alpha}(1-e^{-\lambda})x^{(\alpha-1)}exp\left(-(\beta x)^{\alpha}-\lambda e^{-(\beta x)^{\alpha}}\right)}{\left(1-exp(-\lambda e^{-(\beta x)^{\alpha}})\right)\left(1-e^{-\lambda}-\pi[1-exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)]\right)}, \quad (10)$$

Also, the reversed failure rate function

$$rh(x)=\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})x^{(\alpha-1)}exp\left(-(\beta x)^{\alpha}-\lambda e^{-(\beta x)^{\alpha}}\right)}{\left(exp(-\lambda e^{-(\beta x)^{\alpha}})-e^{-\lambda}\right)\left(1-e^{-\lambda}-\pi[1-exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)]\right)},$$

Figure 1 (a) and (b) provide some plots of the GWP density curves for different values of the parametersλ, β, α and$p$.



(a) (b)

Fig. 1: Plots of the GWP density function for some parameter values.

Figure 2 does the same for the associated hazard rate function, showing that it is quite flexible for modelling survival data.

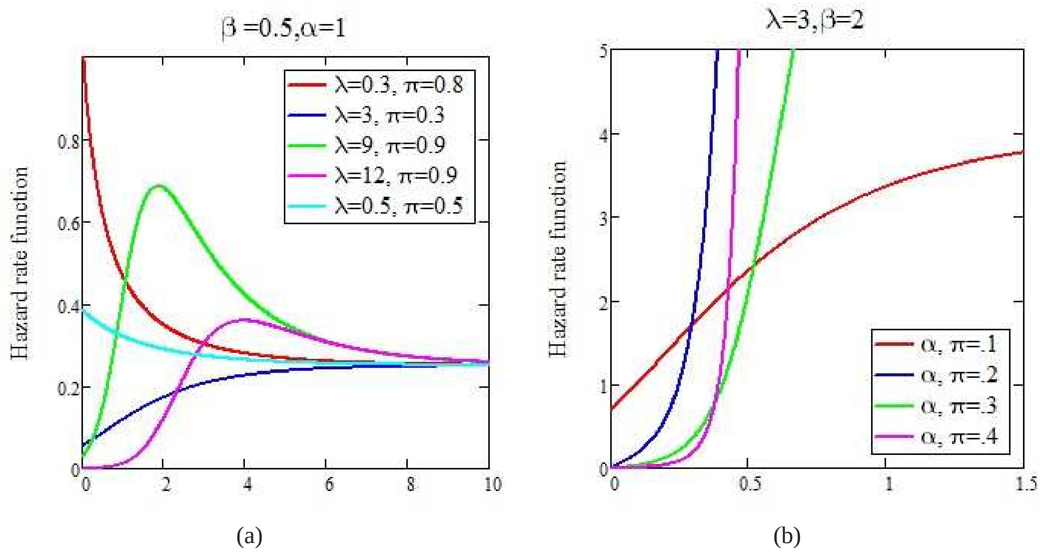


Fig. 2: Plots of the *GWP* hazard rate function for some parameter values.

Figure 2 illustrates possible shapes of (10) for selected parameter values. The shape appears monotonically increasing for a and p sufficiently small. The shape appears monotonically decreasing $\alpha = 1$andβsufficiently small. The shape appears upside down bathtub for both $\beta$and λ sufficiently small.

The rest of the paper is organized as follows: In Section.2, we derive an expansion to the pdf and the cdf functions. In Section.3, gives the quantile function for the new model. In Section 4, some properties of the new distribution are given. Bonferroni and Lorenz Curves and mean deviations are discussed In Section 5. In Section 6, we introduce the method of likelihood estimation as point estimation of the unknown parameters. . In Section 7, contains measures of uncertainty. In Section 8, we fit the distribution to a real data set to examine it and to suitability it with nested and non-nested models.

2 Expansion for the pdf and the cdf Functions

In this section we give another expression for the pdf and the cdf functions using the Maclaurin and Binomial expansions for simplifying the pdf and the cdf forms.

2.1 Expansion for the pdf Function

From (8), using the expansions

$$(1-z)^{-b} = \sum_{i=0}^{\infty} \binom{-b}{i} (-z)^i, \ \ |z| < 1, \tag{11}$$

and

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!}, \tag{12}$$

Using (11), we can write (6) as

$$g(x) = \frac{\lambda \alpha \beta^{\alpha} (1-\pi)(1-e^{-\lambda}) x^{(\alpha-1)} exp\left(-(\beta x)^{\alpha}\right)}{\left(1-e^{-\lambda}-\pi\right)^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[\frac{\pi}{(1-e^{-\lambda}-\pi)}\right]^k exp\left[-\lambda (k+1) e^{-(\beta x)^{\alpha}}\right]. \tag{13}$$

Applying (12) to (13) for the term $exp\left[-\lambda (k+1) e^{-(\beta x)^{\alpha}}\right]$,(13)can be written as:

$$g(x)=\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})x^{(\alpha-1)}exp\left(-(\beta x)^{\alpha}\right)}{\left(1-e^{-\lambda}-\pi\right)^{2}}\sum_{k,j=0}^{\infty}\binom{-2}{k}\frac{(-1)^{j}}{j!}\left[\frac{\pi}{(1-e^{-\lambda}-\pi)}\right]^{k}\lambda^{j}(k+1)^{j}exp\left(-j(\beta x)^{\alpha}\right). \quad (14)$$

(14)can be written as:

$$g(x)=\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})\sum_{k,j=0}^{\infty}\binom{-2}{k}\frac{(-1)^{j}}{j!}}{\left(1-e^{-\lambda}-\pi\right)^{2}}\left[\frac{\pi}{(1-e^{-\lambda}-\pi)}\right]^{k}\lambda^{j}(k+1)^{j}x^{\alpha-1}exp\left(-(j+1)(\beta x)^{\alpha}\right). \quad (15)$$

The pdf of GWP distribution can then be represented as:

$$g(x)=\sum_{k,j=0}^{\infty}A_{k,j}x^{\alpha-1}exp\left(-(j+1)(\beta x)^{\alpha}\right), \quad (16)$$

Where $A_{k,j}$ is a constant term given by:

$$A_{k,j}=\binom{-2}{k}\frac{\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})(-1)^{j}\pi^{k}\lambda^{j+1}(k+1)^{j}}{\left(1-e^{-\lambda}-\pi\right)^{k+2}j!}.$$

2.2 Expansion for the cdf Function

Applying the expansion in (11) to (5), the cdf function of the GWP distribution can be written as:

$$G(x)=\sum_{k=0}^{\infty}\binom{-1}{k}\frac{\pi^{k}\left[exp\left[-\lambda(k+1)e^{-(\beta x)^{\alpha}}\right]-exp\left[-\lambda-k\lambda e^{-(\beta x)^{\alpha}}\right]\right]}{\left(1-e^{-\lambda}-\pi\right)^{k+1}}.$$

3 Quantile Function

The quantile function is obtained by inverting the cumulative distribution (5), where the p-th quantile x_p of the GWP model is the real solution of the following equation:

$$x_p=\left\{\frac{-1}{\beta^{\alpha}}\ln\left[\frac{-1}{\lambda}\ln\left(\frac{p(1-\pi)+(1-p)e^{-\lambda}}{1-\pi p}\right)\right]\right\}^{\frac{1}{\alpha}}. \quad (17)$$

An expansion for the median$M$follows by taking $p=0.5$.

$$x_{0.5}=\left\{\frac{-1}{\beta^{\alpha}}\ln\left[\frac{-1}{\lambda}\ln\left(\frac{1-\pi+e^{-\lambda}}{2-\pi}\right)\right]\right\}^{\frac{1}{\alpha}}.$$

4 Statistical Properties

In this section, moments, conditional moments, Moment Generating Function of the GWP distribution.

4.1 Moments

The r^{th} non-central moments or (moments about the origin) of the GWP under using equation (16) is given by:

$$\mu_r' = E(X^r) = \int_0^\infty X^r g(x) dx,$$

$$\mu_r' = \int_0^\infty X^r \sum_{k,j=0}^\infty A_{k,j} x^{\alpha-1} exp\left(-(j+1)(\beta x)^\alpha\right) dx,$$

then

$$\mu_r' = \sum_{k,j=0}^\infty A_{k,j} \frac{\Gamma\left(\frac{r+\alpha}{\alpha}\right)}{\alpha\beta^{\alpha+r}(j+1)^{\frac{\alpha+1}{\alpha}}} \quad (18)$$

$$\mu = \mu_1' = \sum_{k,j=0}^\infty A_{k,j} \frac{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}{\alpha\beta^{\alpha+1}(j+1)^{\frac{\alpha+1}{\alpha}}}$$

4.2 Conditional and Reversed Conditional —Moments

For lifetime models, it is useful to know the conditional moments defined as$E(x^r|x>t)$

$$E(x^r|x>t) = \frac{1}{[1-G(t)]} \int_t^\infty x^r g(x) dx,$$

using equation (18) the conditional moments is,

$$E(x^r|x>t) = \frac{1}{[1-G(t)]} \sum_{k,j=0}^\infty A_{k,j} \frac{\Gamma_t\left(\frac{r+\alpha}{\alpha}\right)}{\alpha\beta^{\alpha+r}(j+1)^{\frac{\alpha+1}{\alpha}}}$$

where $\Gamma_t(a) = \int_t^\infty x^{a-1} e^{-x} dx$ is the upper incomplete gamma function.

Reversed conditional moments defined as $\mathrm{E}(x^r|x \le t)$

$$E(x^r|x \le t) = \frac{1}{[G(t)]} \int_0^t x^r g(x) dx,$$

using equation (18) the conditional moments is,

$$E(x^r|x \le t) = \frac{1}{[G(t)]} \sum_{k,j=0}^\infty A_{k,j} \frac{\Gamma\left(\frac{r+\alpha}{\alpha}, t\right)}{\alpha\beta^{\alpha+r}(j+1)^{\frac{\alpha+r}{\alpha}}},$$

where $\Gamma(a,t) = \int_0^t x^{a-1} e^{-x} dx$is the lower incomplete gamma function.

4.3 The Moment Generating Function

The moment generating function, $M_x(t)$, can be easily obtained as:

$$M_x(t) = \int_0^\infty e^{tx} g(x) dx,$$

$$M_x(t) = \int_0^\infty e^{tx} \sum_{k,j=0}^{\infty} A_{k,j} x^{\alpha-1} exp\left(-(j+1)(\beta x)^\alpha\right) dx,$$

then, the moment generating function of the GWP distribution is given by,

$$M_x(t) = \sum_{k,j,i=0}^{\infty} \frac{A_{k,j} t^i \Gamma\left(\frac{i+\alpha}{\alpha}\right)}{\alpha \beta^{\alpha+i} (j+1)^{\frac{\alpha+i}{\alpha}}}$$

where

$$E(e^x) = \sum_{i=0}^{\infty} \frac{E(x^i)}{i!}.$$

4.4 Probability Weighted Moments

The $PWMs$ are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. ThePWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,r) th $PWM$of following theGWPdistribution, say $\tau_{s,r}$, is formally defined by

$$\tau_{s,r} = E\left[X^r G(x)^s\right] = \int_0^\infty X^r G(x)^s g(x) dx,$$

$$\tau_{s,r} = \frac{\alpha\lambda\beta^\alpha (1-e^{-\lambda})(1-\pi)}{(1-e^{-\lambda}-\pi)^{s+2}} \int_0^\infty \frac{X^{r+\alpha-1}\left[exp\left(-\lambda e^{-(\beta x)^\alpha}\right) - e^{-\lambda}\right]^s exp\left[-(\beta x)^\alpha - \lambda e^{-(\beta x)^\alpha}\right]}{\left[1 + \frac{\pi\left[1-exp\left(-\lambda e^{-(\beta x)^\alpha}\right)\right]}{1-e^{-\lambda}-\pi}\right]^{s+2}} dx, \quad (19)$$

Using the expansions in (11), (12), and Binomial, and applying in (19) we get

$$\tau_{s,r} = \sum_{k=0}^{\infty} \sum_{i=0}^{s} \sum_{j=0}^{\infty} M_{k:j} \int_0^\infty X^{r+\alpha-1} exp\left[-(j+1)(\beta x)^\alpha\right] dx,$$

where

$$M_{k:j} = \frac{\binom{-s-2}{k}\binom{s}{i}(-1)^{j+s-i}\alpha(1-e^{-\lambda})(1-\pi)e^{-\lambda(s-i)}\pi^k\beta^\alpha\lambda^{j+1}(k+i+1)^j}{\left(1-e^{-\lambda}-\pi\right)^{s+k+2} j!}$$

By using gamma function we get

$$\tau_{s,r} = \sum_{k=0}^{\infty} \sum_{i=0}^{s} \sum_{j=0}^{\infty} M_{k:j} \frac{\Gamma\left(\frac{r+\alpha}{\alpha}\right)}{\alpha\beta^{\alpha+r}(j+1)^{\frac{\alpha+r}{\alpha}}}.$$

If $s = 0, \tau_{0,r} = E(X^r) = \mu_r'$ given in (18).

5 Lorenz Curves, Bonferroni and Mean Deviations

In this section, we present Lorenz curves, Bonferroni and the mean deviation about the mean, the mean deviation about the median. Bonferroni and Lorenz curves are income inequlity measures that are also useful and applicable in other areas including reliability, demography, medicine and insurance.

5.1 Lorenz Curves and Bonferroni

The Lorenz curves $L(G)$and Bonferroni $B(G)$are defined by

$$L(G) = \frac{1}{\mu}\int_{0}^{x} tg(t)dt,$$

$$L(G) = \sum_{k,j=0}^{\infty} A_{k,j}\frac{\Gamma\left(\frac{\alpha+1}{\alpha},x\right)}{\mu\alpha\beta^{\alpha+1}(j+1)^{\frac{\alpha+1}{\alpha}}}$$

where $\Gamma(a,x) = \int_{0}^{x} t^{a-1}e^{-t}dt$is the lower incomplete gamma function and

$$B(G) = \frac{1}{\mu G(x)}\int_{0}^{x} tg(x)dt,$$

$$B(G) = \sum_{k,j=0}^{\infty} A_{k,j}\frac{\Gamma\left(\frac{\alpha+1}{\alpha},x\right)}{\mu G(x)\alpha\beta^{\alpha+1}(j+1)^{\frac{\alpha+1}{\alpha}}}$$

5.2 The Mean Deviation

In statistics, mean deviation about the mean and mean devation about the median measure the amount of scatter in a population. For random variable with pdf $g(x)$, distribution function $G(x)$, mean and Median , the mean deviation about the mean and mean deviation about the median, are defined by,

$$\delta_1(x) = \int_{0}^{\infty} |x-\mu|g(x)dx$$

$$\delta_1(x) = 2\mu G(\mu) - 2\int_{0}^{\mu} xg(x)dx,$$

and

$$\delta_2(x) = \int_{0}^{\infty} |x-M|g(x)dx$$

$$\delta_2(x) = \mu - 2\int_{0}^{M} xg(x)dx,$$

respectively, if X is GWP random variable then,

$$\delta_1(x) = 2\mu G(\mu) - 2\sum_{k,j=0}^{\infty} A_{k,j}\frac{\Gamma\left(\frac{\alpha+1}{\alpha},\mu\right)}{\mu\alpha\beta^{\alpha+1}(j+1)^{\frac{\alpha+1}{\alpha}}},$$

and

$$\delta_2(x) = \mu - 2\sum_{k,j=0}^{\infty} A_{k,j}\frac{\Gamma\left(\frac{\alpha+1}{\alpha},M\right)}{\mu\alpha\beta^{\alpha+1}(j+1)^{\frac{\alpha+1}{\alpha}}}.$$

6 Measures of Uncertainty

In this section, we present Shannon entropy [10], as well as Renyi entropy [9] for the GWP distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and is a good measure of randomness or uncertainty.

6.1 Shannon Entropy

Shannon entropy [10], is defined by $H[g(x)] = E[-ln?[g(x)]]$. Thus, we have

$$H[g(x)] = -\ln\left[\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})}{(1-e^{-\lambda}-\pi)^2}\right] - (\alpha+1)E\ln x + \beta^{\alpha}Ex^{\alpha} + \lambda Ee^{-(\beta x)^{\alpha}} + 2E\ln\left[1+\frac{\pi exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)}{(1-e^{-\lambda}-\pi)}\right]. \tag{20}$$

Note that,

$$E\ln(1+x^a) = -\sum_{q=1}^{\infty}\frac{(-1)^q}{q}E(x^{qa}), \tag{21}$$

using (21)

$$E\ln\left[1+\frac{\pi exp\left(-\lambda e^{-(\beta x)^{\alpha}}\right)}{(1-e^{-\lambda}-\pi)}\right] = -\sum_{q=1}^{\infty}\frac{(-\pi)^q E\left[exp\left(-\lambda q e^{-(\beta x)^{\alpha}}\right)\right]}{q(1-e^{-\lambda}-\pi)^q}$$

$$= -\sum_{q=1}^{\infty}-\sum_{s=0}^{\infty}-\sum_{i=1}^{\infty}\frac{(-1)^{q+s+i}\pi^q\lambda^s s^i\beta^{i\alpha}Ex^{i\alpha}}{s!i!q^{1-s}(1-e^{-\lambda}-\pi)^q}$$

and

$$E\ln x = -\sum_{q=1}^{\infty}\frac{(-1)^q}{q}E(x-1)^q = -\sum_{q=1}^{\infty}\frac{(-1)^q}{q}\sum_{i=1}^{q}\binom{q}{i}(-1)^{q-i}Ex^i$$

$$E(e^{-x}) = \sum_{i=0}^{\infty}\frac{(-1)^i E(x^i)}{i!}.$$

then

$$H[g(x)] = -\ln\left[\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})}{(1-e^{-\lambda}-\pi)^2}\right] - (\alpha+1)E\ln x + \beta^{\alpha}Ex^{\alpha} + \lambda Ee^{-(\beta x)^{\alpha}} - 2\sum_{q=1}^{\infty}-\sum_{s=0}^{\infty}-\sum_{i=1}^{\infty}\frac{(-1)^{q+s+i}\pi^q\lambda^s s^i\beta^{i\alpha}Ex^{i\alpha}}{s!i!q^{1-s}(1-e^{-\lambda}-\pi)^q}. \tag{22}$$

Now, we obtain Shannon entropy for the GWP distribution as follows:

$$H[g(x)] = -\ln\left[\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})}{(1-e^{-\lambda}-\pi)^2}\right] - (\alpha+1)E\ln x + \beta^{\alpha}Ex^{\alpha} + \lambda Ee^{-(\beta x)^{\alpha}}$$

$$+(\alpha-1)\sum_{k,j=0}^{\infty}A_{k,j}\sum_{q=1}^{\infty}\frac{(-1)^q}{q}\sum_{i=1}^{q}\binom{q}{i}(-1)^{q-i}\frac{\Gamma\left(\frac{\alpha+i}{\alpha}\right)}{\mu\alpha\beta^{\alpha+i}(j+1)^{\frac{\alpha+i}{\alpha}}}$$

$$+\beta^{\alpha}\sum_{k,j=0}^{\infty}A_{k,j}\frac{1}{\alpha\beta^{2\alpha}(j+1)^2} + \lambda\sum_{k,j,i=0}^{\infty}A_{k,j}\frac{(-1)^i\beta^{i\alpha}\Gamma(i+1)}{i!\alpha\beta^{\alpha(i+1)(j+1)^{(i+1)}}}$$

$$-2\sum_{q=1}^{\infty}\sum_{s=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^{q+s+i}\pi^q\lambda^s s^i\beta^{i\alpha}}{s!i!q^{1-s}(1-e^{-\lambda}-\pi)^q}\sum_{k,j=0}^{\infty}A_{k,j}\frac{\Gamma(i+1)}{\alpha\beta^{\alpha(i+1)(j+1)^{(i+1)}}}.$$

6.2 Renyi Entropy

Renyi entropy [9] is an extension of Shannon entropy. Renyi entropy is defined to be

$$I_R(r) = \frac{1}{1-r}\ln\left[\int_0^\infty g^r(x)dx\right], \; r>0, \; r\neq 1, \tag{23}$$

where

$$g^r(x) = \left[\frac{\lambda\alpha\beta^\alpha(1-\pi)(1-e^{-\lambda})}{\left(1-e^{-\lambda}-\pi\right)^2}\right]^r x^{r(\alpha-1)}exp\left[-r(\beta x)^\alpha - \lambda r e^{-(\beta x)^\alpha}\right]\left[1+\frac{\pi exp\left(-\lambda e^{-(\beta x)^\alpha}\right)}{\left(1-e^{-\lambda}-\pi\right)}\right]^{-2r}. \tag{24}$$

can be write (24) as:

$$g^r(x) = \left[\frac{\lambda\alpha\beta^\alpha(1-\pi)(1-e^{-\lambda})}{\left(1-e^{-\lambda}-\pi\right)^2}\right]^r \sum_{k,j=0}^\infty \binom{-2r}{k}\frac{\pi^k}{\left(1-e^{-\lambda}-\pi\right)^k} \times \frac{(-1)^j\lambda^j(r+k)^j}{j!}x^{r(\alpha-1)}e^{-(r+j)(\beta x)^\alpha}$$

The $g^r(x)$ of GWP distribution can then be represented as:

$$g^r(x) = \sum_{k,j=0}^\infty B_{k,j}x^{r(\alpha-1)}e^{-(r+j)(\beta x)^\alpha} \tag{25}$$

where $B_{k,j}$ is a constant term given by

$$B_{k,j} = \left[\frac{\lambda\alpha\beta^\alpha(1-\pi)(1-e^{-\lambda})}{\left(1-e^{-\lambda}-\pi\right)^2}\right]^r \binom{-2r}{k}\frac{\pi^k}{\left(1-e^{-\lambda}-\pi\right)^k}\frac{(-1)^j\lambda^j(r+k)^j}{j!}.$$

Then

$$\int_0^\infty g^r(x)dx = \sum_{k,j=0}^\infty B_{k,j}\int_0^\infty x^{r(\alpha-1)}e^{-(r+j)(\beta x)^\alpha dx}$$

$$= \sum_{k,j=0}^\infty B_{k,j}\frac{\Gamma\left(\frac{r(\alpha-1)+1}{\alpha}\right)}{\alpha\beta^{r(\alpha-1)+1}(r+j)^{\frac{r(\alpha-1)+1}{\alpha}}}, \tag{26}$$

by substituting (26) in (25),

$$I_R(r) = \frac{1}{1-r}\ln\left[\sum_{k,j=0}^\infty B_{k,j}\frac{\Gamma\left(\frac{r(\alpha-1)+1}{\alpha}\right)}{\alpha\beta^{r(\alpha-1)+1}(r+j)^{\frac{r(\alpha-1)+1}{\alpha}}}\right], \; r>0, \; r\neq 1. \tag{27}$$

6.3 s-Entropy

The s-entropy for the GWP distribution is defined by

$$I_s(r) = \frac{1}{s-1}\left[1-\int_0^\infty g^r(x)dx\right], \; s>0, \; s\neq 1.$$

Now, using the same procedure that was used to derive Equation (27),

$$I_s(r) = \frac{1}{s-1}\left[1 - \sum_{k,j=0}^{\infty} D_{k,j} \frac{\Gamma\left(\frac{s(\alpha-1)+1}{\alpha}\right)}{\alpha\beta^{s(\alpha-1)+1}(s+j)^{\frac{s(\alpha-1)+1}{\alpha}}}\right], \; s>0, \; s\neq 1.$$

where $D_{k,j}$ is a constant term given by

$$D_{k,j} = \left[\frac{\lambda\alpha\beta^{\alpha}(1-\pi)(1-e^{-\lambda})}{(1-e^{-\lambda}-\pi)^2}\right]^s \binom{-2s}{k} \frac{\pi^k}{(1-e^{-\lambda}-\pi)^k} \frac{(-1)^j\lambda^j(s+k)^j}{j!}.$$

7 Parameter Estimation

In this section, the maximum likelihood estimation is used to estimate the unknown parameters. Let $X_1, X_2, \ldots, X_n$ be a sample of size n from a GWP distribution. Then the likelihood function (ι) is given by:

$$\iota = \frac{\lambda^n\alpha^n\beta^{\alpha n}(1-\pi)^n(1-e^{-\lambda})^n\prod_{i=1}^n x_i{}^{(\alpha-1)} exp\left(-\beta^{\alpha}\sum_{i=1}^n x_i{}^{\alpha} - \lambda\sum_{i=1}^n e^{-(\beta x_i)^{\alpha}}\right)}{\prod_{i=1}^n\left(1-e^{-\lambda}-\pi[1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)]\right)^2}, \quad (28)$$

Hence, the log-likelihood function, L, becomes:

$$L = n\ln\lambda + n\ln\alpha + \alpha n\ln\beta + n\ln(1-e^{-\lambda}) + \sum_{i=1}^n x_i{}^{(\alpha-1)} + \left(-\beta^{\alpha}\sum_{i=1}^n x_i{}^{\alpha} - \lambda\sum_{i=1}^n e^{-(\beta x_i)^{\alpha}}\right)$$
$$-2\sum_{i=1}^n \ln\left(1-e^{-\lambda}-\pi\left[1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)\right]\right). \quad (29)$$

Therefore, the MLEs of λ, β, α andπ must satisfy the following equations:

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} + \frac{n}{e^{-\lambda}-1} - \sum_{i=1}^n e^{-(\beta x_i)^{\alpha}} - 2\sum_{i=1}^n \frac{e^{-\lambda} - \pi exp\left(-(\beta x_i)^{\alpha} - \lambda e^{-(\beta x_i)^{\alpha}}\right)}{\left(1-e^{-\lambda}-\pi\left[1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)\right]\right)}, \quad (30)$$

$$\frac{\partial L}{\partial\beta} = \frac{n\alpha}{\beta} + \alpha\beta^{\alpha-1}\sum_{i=1}^n x_i{}^{\alpha} + \lambda\alpha\beta^{\alpha-1}\sum_{i=1}^n x_i{}^{\alpha}e^{-(\beta x_i)^{\alpha}} - \sum_{i=1}^n \frac{2\lambda\pi\alpha\beta^{\alpha-1}exp\left(-(\beta x_i)^{\alpha} - \lambda e^{-(\beta x_i)^{\alpha}}\right)}{\left(1-e^{-\lambda}-\pi\left[1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)\right]\right)}, \quad (31)$$

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} + n\ln\beta + \sum_{i=1}^n x_i{}^{\alpha-1}\ln x_i - \sum_{i=1}^n(\beta x_i)^{\alpha}\ln(\beta x_i) + \lambda\sum_{i=1}^n e^{-(\beta x_i)^{\alpha}}(\beta x_i)^{\alpha}\ln(\beta x_i)$$
$$-2\pi\lambda\sum_{i=1}^n \frac{exp\left(-(\beta x_i)^{\alpha} - \lambda e^{-(\beta x_i)^{\alpha}}\right)(\beta x_i)^{\alpha}\ln(\beta x_i)}{\left(1-e^{-\lambda}-\pi\left[1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)\right]\right)}, \quad (32)$$

and

$$\frac{\partial L}{\partial\pi} = \frac{-n}{(1-\pi)} + 2\sum_{i=1}^n \frac{1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)}{\left(1-e^{-\lambda}-\pi\left[1-exp\left(-\lambda e^{-(\beta x_i)^{\alpha}}\right)\right]\right)}. \quad (33)$$

The maximum likelihood estimator $\underline{\hat{\vartheta}} = (\hat{\lambda}, \hat{\beta}, \hat{\alpha}, \hat{\pi})$ of $\underline{\vartheta} = (\lambda, \beta, \alpha, \pi)$ is obtained by solving the nonlinar system of equations (30) through (33). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

Table 1: MLEs the measures $AIC, AIC_C, BIC, andKS$ test to failure time data for the models

Model	Parameter Estimates	$-LogL$	AIC	AIC_C	BIC	KS
GWP	$\hat{\lambda} = 0.990$ $\hat{\beta} = 0.030$ $\hat{\alpha} = 2.00$ $\hat{\pi} = 0.099$	234.12	476.24	477.12	483.88	0.14
GEP	$\hat{\lambda} = 0.0301$ $\hat{\beta} = 2.00$ $\hat{\pi} = 0.099$	241.28	488.57	489.10	494.31	0.147
EMW	$\hat{\theta} = 0.0186$ $\hat{\gamma} = 0.0018$ $\hat{\beta} = 0.0105$ $\hat{\alpha} = 0.703$	238.81	481.30	482.19	488.95	0.161
AW	$\hat{\theta} = 0.0002$ $\hat{\nu} = 1.85$ $\hat{\gamma} = 0.016$ $\hat{\beta} = 0.947$	237.75	483.51	484.40	491.16	0.15
MW	$\hat{\theta} = 1.82$ $\hat{\beta} = 1.00$ $\hat{\gamma} = 1.80$	241.02	488.05	488.57	493.79	0.17
W	$\hat{\beta} = 5.78$ $\hat{\lambda} = 0.614$	240.979	485.95	486.2145	489.7832	0.1729

8 Application

In this section, we use a real data set to to see how the new model works in practice. compare the fits of the GWP distribution with others models. In each case, the parameters are estimated by maximum likelihood as described in Section 7, using the R code.

In order to compare the two distribution models, we consider criteria like KS (Kolmogorov Smirnov),-2L, AIC (Akaike information criterion), BIC and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller KS,-2L, AIC and AICC values:

$$AIC = -2L + 2k,$$

$$AIC_C = -2L + \frac{2kn}{n-k-1},$$

and

$$BIC = -2L + k\log(n),$$

where L denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size, where L denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size. The data set represents failure time of 50 items reported in Aarset ([1]).

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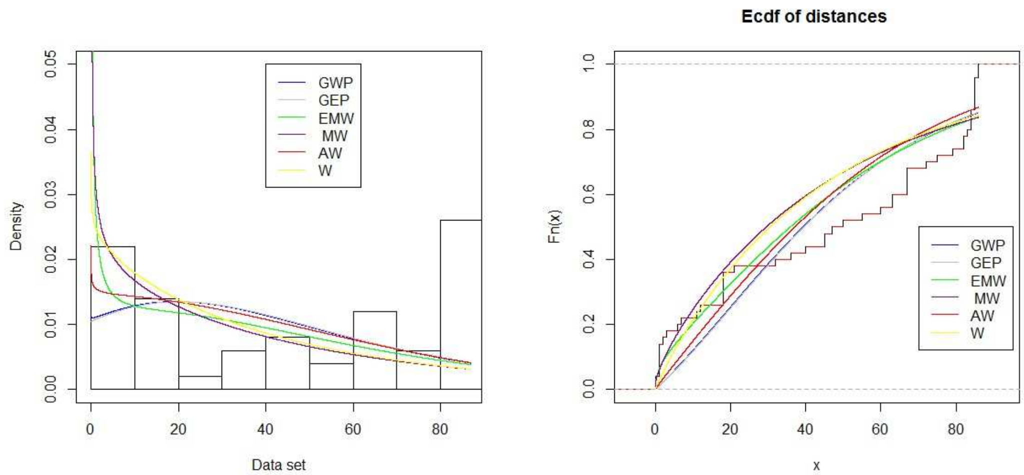
These results indicate that the GWP model has the lowest AIC and AICC and BIC values amang the fitted models. The values of these statistics indicate that the GWP model provides the best fit to this data.



(a) (b)

Fig. 3: (a) Estimated densities of the GWP, GEP, EMW, MW, AW and W distributions for the data set. (b) Estimated cdf function from the fitted the GWP, GEP, EMW, MW, AW and W distributions and the empirical cdf for the data set.

As we can see from Tables (1) , our model with smallest values of AIC, AICC, BIC and K-S test statstic best fits the data. Figures (3) shows the empirical distribution compared to the rival models and 8 compares the fitted densities against the data.

9 Concluding remarks

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modelling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this paper, we introduce a new four-parameter distribution. We refer to the new model as the GWP distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model, explicit expressions for the moments. The model parameters are estimated by maximum likelihood. The new model is compared with nested and non-nested models and provides consistently better fit than other classical lifetime models.

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References

[1] M.V Aarset. How to identify bathtub hazard rate, IEEE Transactions on Reliability **36**, 106-108(1987).
[2] A Adamidis, S Loukas. A lifetime distribution with decreasing failure rate. Stat Probab Lett **39**, 3542 (1998).
[3] A Adamidis , T Dimitrakopoulou, S Loukas. On a generalization of the exponential-geometric distribution. Statistics & Probability Letters, textbf73, 259-269(2005).
[4] C.E Bonferroni. Elmenti di statistica generale. Libreria Seber, Firenze (1930).
[5] I Elbatal Exponentiated modified Weibull distribution. Economic Quality Control. textbf26, 189200 (2011).
[6] M E Ghitany , D K Al-Mutairi, S Nadarajah. Zero-truncated Poisson-Lindley distribution and its Applications. Math. Comput. Simul. **79**(3),279-287(2008).
[7] Mahmoudi and Shiran. introduced the exponentiated Weibull geometric distribution. arXiv:1206.4008v1 [stat.ME](2012).

[8] S Nadarajah, V. G Cancho,E. M Ortega. The geometric exponential Poisson distribution. Statistical Methods & Applications, **22**(3), 355-380(2013).
[9] A Renyi. On measures of entropy and information, Proceedings of the 4thBerkeley Symposium on Mathematical Statistics and Probability , Berkeley(CA):University of California Press, I, 547 - 561(1961).
[10] E A Shannon. A Mathematical Theory of Communication, The Bell System Technical Journal, **27**(10), 379 - 423 (1948).
[11] R B Silva, W Barreto-Souza, G M Cordeiro. A new distribution with decreasing, increasing and upside-down bathtub failure rate. Computational Statistics and Data Analysis, **54**, 935-944 (2010).
[12] M Wang , I Elbatal (2014) The modified Weibull geometric distribution.METRON (In press).
[13] W A Weibull. Statistical distribution function of wide applicability. Journal of Applied Mechanics. textbf18, 293- 297 (1951).
[14] H Zakerzadah ,A Dolati. Generalized Lindley distribution. J. Math. Ext. **3**(2):13-25 (2010).

Mahmoud M. Mansour is Assistant Professor of statistics at Department of statistics, mathematics and insurance Benha University. His research interests are in the areas of statistics including the mathematical statistics (nonparametric statistics and probability distributions) and applied statistics (randomized response techniques and Statistica1 quality control). He has published research articles in reputed international journals of Statistics.

Enayat M. Abd Elrazik is Assistant Professor of statistics at Department of statistics, mathematics and insurance Benha University. Her research interests are in the areas of statistics including the mathematical statistics and applied statistics. she has published research articles in reputed international journals of Statistics.