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Direct and Inverse Problems for a Samarskii-Ionkin Type Problem for a Two Dimensional Fractional Parabolic Equation

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Abstract: Samarskii-Ionkin type problems are known classes of problems that represent a generalization of classical ones. At the same time they are obtained in a natural way by constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology, etc. Here we investigate the ability to solve non-local problems of its type in 2D using the Fourier method of the separation of variables. We study the completeness of the root functions of the corresponding spectral problems in $L^2(0 < x,y < 1)$, when they are defined as products of two systems of functions, where one of them is an orthonormal basis, and another is a Riesz basis. Using the properties of biorthogonal systems, we also study the problem of identifying the source function in the spatial domain.

Keywords: Non-local problems, fractional differential operator, Samarskii-Ionkin type problem, eigenvalues, eigenfunctions, root functions, bi-orthonormal system, Riesz basis.

1 Introduction and Problem Statement

The interest in the study of problems of the Samarskii-Ionkin type began after the well-known classical work of N.I. Ionkin [1]. Such problems differ from the classical ones, as corresponding spatial differential operator is nonself-adjoint and hence the system of eigenfunctions is not complete. From this arise the problems of studying completeness and basicity of such systems, which play an important role in research of boundary value problems. At the present time there is an extensive list of scientific papers pertaining to study of such problems and all of the works relate mainly to partial differential equations of the second order ([2]- [6]).

We study similar problems for fourth-order partial differential equations with three variables. Direct problem is addressed with a time-dependent source term, while a time-independent one is used to address an inverse problem; the two problems are solved in non-local 2D fractional parabolic equations of the following form:

$$ {}_C D_{0t}^{\alpha} u + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x,y,t), \tag{1} $$

and

$$ {}_C D_{0t}^{\alpha} u + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x,y), \tag{2} $$

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respectively.
Here ${}_CD_{0t}^{\alpha}u = I_{0t}^{1-\alpha}(u_t)$, $\alpha \in (0,1]$ is the derivative of order α in the sense of Caputo [7] and for any integral function g, the left-sided Riemann-Liouville fractional integral I_{0t}^{β} of order $\beta > 0$ is defined by

$I_0^{\beta} g(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{g(\tau)}{(t-\tau)^{1-\beta}} d\tau$, where $\Gamma(\beta)$ is the Euler Gamma function.

Let

$$\Omega_{xy} = \{(x,y) : 0 < x,y < 1\},\ \Omega_{xt} = \{(x,t) : 0 < x < 1; 0 < t < T\},$$
$$\Omega_{yt} = \{(y,t) : 0 < y < 1; 0 < t < T\},\ \Omega = \Omega_{xy} \times (0,T), T > 0.$$

The direct and inverse methods are analyzed as stated below.

Direct problem. Given $f(x,y,t)$ and $\varphi(x,y)$, we find a function $u(x,y,t)$ such that:

(1)u is continuous in $\bar{\Omega}$, together with its derivatives, they appear in the boundary conditions,
(2)u satisfies equation (1) in Ω,
(3)u also satisfies the following boundary conditions

$$u(x,y,0) = \varphi(x,y),\ (x,y) \in \bar{\Omega}_{xy}, \tag{3}$$
$$\frac{\partial^k u}{\partial x^k}|_{x=0} = 0,\ \frac{\partial^\ell u}{\partial x^\ell}|_{x=0} = \frac{\partial^\ell u}{\partial x^\ell}|_{x=1},\ \ (y,t) \in \bar{\Omega}_{y,t}, k = 1,3, \ell = 0,2, \tag{4}$$
$$\frac{\partial^k u}{\partial y^k}|_{y=0} = \frac{\partial^k u}{\partial y^k}|_{y=1},\ \ (x,t) \in \bar{\Omega}_{x,t}, k = 0,2, \tag{5}$$

where $\bar{\Omega}$ denotes the closure of Ω .

Inverse problem. Given φ and ψ, we find a pair of functions $\{u(x,y,t), f(x,y)\}$ with the following properties:

(1)u is continuous in $\bar{\Omega}$, its derivatives alongside itself also appear in the boundary conditions and $f(x,y) \in C(\Omega_{xy})$,
(2)u satisfies equation (2) in Ω,
(3)u also satisfies boundary conditions (3)-(5) and the following terminal condition

$$u(x,y,T) = \psi(x,y),\ (x,y) \in \bar{\Omega}_{xy}. \tag{6}$$

Before we tackle our stated problems, we recall that the theory of boundary value problems for fractional differential equations has grown for the past years in two directions: on one hand, for its applications in real-life problems including viscoelasticity, dynamical processes, biosciences, signal processing, system control theory, electrochemistry, diffusion processes, and many others ([7]-[13]) and on the other hand for its intensive contribution in the general theory of differential equations. For our knowledge, very few papers have been devoted to initial-boundary value problems governed by fractional partial differential equations of fourth or higher order ([14]-[16]). In addition to the work devoted to direct problems involving fractional differential equations, driven and motivated by their applications, parallel studies investigated inverse problems for these equations that appear in different fields, for instance in quantum physics (inverse problems in quantum theory of scattering), geophysics (inverse problems of electric prospecting, seismology, and theory of potentials), biology, medicine, quality checking programs, and others ([17]-[23]).

The organisation of the manuscript is as follows. Section 2 deals with the investigation of the direct problem. The uniqueness and existence of a solution of this problem can be seen in Section 3. The investigation of the inverse problem is given in Section 4. The conclusion part is depicted in Section 5.

2 Investigation of the Direct Problem

The solution of the direct problem can be expressed as the sum of a solution for the homogeneous equation

$${}_CD_{0t}^{\alpha}u + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0, \tag{7}$$

with the boundary conditions (3)-(5), and a solution of the nonhomogeneous equation(1) subject to the boundary conditions (4)-(5), and the homogeneous initial condition $u(x,y,0) = 0$.

We seek a nontrivial particular solution of the homogeneous equation (7) in the separate form

$$u(x,y,t) = Z(x,y)C(t).$$

Using separation of variables and considering the boundary conditions (3)-(5), we obtain the spectral problem:

$$\frac{\partial^4 Z}{\partial x^4} + \frac{\partial^4 Z}{\partial y^4} = \sigma Z, \ (x,y) \in \Omega_{xy}, \tag{8}$$

$$\frac{\partial^k Z}{\partial x^k}\Big|_{x=0} = 0, \ \frac{\partial^\ell Z}{\partial x^\ell}\Big|_{x=0} = \frac{\partial^\ell Z}{\partial x^\ell}\Big|_{x=1}, \ \ y \in [0,1], k = 1,3, \ell = 0,2, \tag{9}$$

$$\frac{\partial^k Z}{\partial y^k}\Big|_{y=0} = \frac{\partial^k Z}{\partial y^k}\Big|_{y=1} = 0, \ x \in [0,1], k = 0,2; \tag{10}$$

its corresponding adjoint problem is

$$\frac{\partial^4 W}{\partial x^4} + \frac{\partial^4 W}{\partial y^4} = \sigma W, \ (x,y) \in \Omega_{xy}, \tag{11}$$

$$\frac{\partial^k W}{\partial x^k}\Big|_{x=1} = 0, \ \frac{\partial^\ell W}{\partial x^\ell}\Big|_{x=0} = \frac{\partial^\ell W}{\partial x^\ell}\Big|_{x=1}, \ \ y \in [0,1], k = 0,2, \ell = 1,3, \tag{12}$$

$$\frac{\partial^k W}{\partial y^k}\Big|_{y=0} = \frac{\partial^k W}{\partial y^k}\Big|_{y=1} = 0, \ x \in [0,1], k = 0,2, \tag{13}$$

where σ is the separation parameter.
In the last decade, there were several types of new non-classical mathematical-physical problems that are devoted to the study of the spectral properties of a non-self-adjoint differential operators, where the eigenfunctions are generally not orthogonal and complete [2,3,5,17].

Now, we look for a solution to the problem (8)-(10) in the form

$$Z(x,y) = X(x)Y(y). \tag{14}$$

Similarly as in the previous problem, using separation of variables and the boundary conditions (9)-(10), the problem (8)-(10) is split into two spectral problems governed by ordinary differential equations:

$$X^{''''}(x) = \lambda X(x), \quad 0 < x < 1, \tag{15}$$

$$X'(0) = X'''(0) = 0, \quad X(0) = X(1), \quad X''(0) = X''(1), \tag{16}$$

and

$$Y^{''''}(y) - \mu Y(y) = 0, \ Y(0) = Y''(0) = Y(1) = Y''(1) = 0, \ 0 < y < 1, \tag{17}$$

where $\mu = \sigma - \lambda$ and $'$ denotes the classical derivative of order one.

The eigenvalues and eigenfunctions of the problem (17) have the form

$$\mu_k = (k\pi)^4, \text{ and } \quad Y_k(y) = \sqrt{2}\sin(k\pi y), \quad k \in \mathbb{N}. \tag{18}$$

The characteristic equation of the problem (15)-(16) is given by

$$k^4 - \lambda = 0 \Leftrightarrow k^4 = \lambda.$$

We now investigate three cases for the values of λ and determine their corresponding eigenfunctions.

For $\lambda = 0$, the corresponding solution has the form

$$X(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4.$$

Substituting the obtained expression in equation (16), we obtain $C_1 = C_2 = C_3 = 0$, while C_4 is an arbitrary real number. Thus, the eigenfunction associated to the eigenvalue $\lambda_0 = 0$ can be chosen as $X_0(x) = 1$.

For $\lambda < 0$, say $\lambda = -4\mu^4, (\mu > 0)$, the characteristic equation $k^4 = -4\mu^4$ has the roots

$$k_1 = (1+i)\mu, \quad k_2 = (-1+i)\mu, \quad k_3 = (1-i)\mu, \quad k_4 = (-1-i)\mu$$

and the corresponding solution is of the form

$$X(x) = C_1 ch(\mu x)\cos(\mu x) + C_2 ch(\mu x)\sin(\mu x) + C_3 sh(\mu x)\cos(\mu x) + C_4 sh(\mu x)\sin(\mu x).$$

Substituting this expression into the boundary conditions (16), we obtain $C_1 = C_2 = C_3 = C_4 = 0$ and hence the problem (15)-(16) has only a trivial solution for all $\lambda < 0$.

Finally, for $\lambda > 0$, say $\lambda = \mu^4, (\mu > 0)$; the characteristic equation $k^4 = \mu^4$ has the roots $k_{1,2} = \pm\mu, k_{3,4} = \pm\mu i$, and consequently, the corresponding solution can be written as

$$X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} + C_3 \cos(\mu x) + C_4 \sin(\mu x).$$

Substituting this solution into equation (16), we obtain the following set of eigenvalues and eigenfunctions of the problem (15)-(16)

$$\lambda_n = (2\pi n)^4, \quad X_n(x) = \cos(2\pi n x),\ n \in \mathbb{N} \cup \{0\}.$$

Thereby, according to the representation of the solution given by equation (14), the eigenvalues and the corresponding eigenfunctions of the problem (8)-(10) have the form

$$\sigma_{nk} = \lambda_n + \mu_k = (2n\pi)^4 + (k\pi)^4,$$

and

$$Z_{nk}(x,y) = X_n(x) Y_k(y), \quad n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$$

respectively. Note that the problem (8)-(10) is not self-adjoint and the set of eigenfunctions $Z_{nk}(x,y)$ is not complete in the space $L^2(\Omega_{xy})$ in the sense of the inner product $< \xi, \eta >= \iint\limits_{\Omega_{xy}} \xi(x,y)\eta(x,y)dxdy$.

Following [5], we supplement the problem with the associated functions $\tilde{Z}_{nk}(x,y), n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$, to make the set complete on $L^2(\Omega_{xy})$.

The associated functions $\tilde{Z}_{nk}(x,y), n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$, are solutions of the following problem

$$\frac{\partial^4 \tilde{Z}_{nk}}{\partial x^4} + \frac{\partial^4 \tilde{Z}_{nk}}{\partial y^4} - \sigma_{nk}\tilde{Z}_{nk} = -4(2n\pi)^3 Z_{nk}, \quad (x,y) \in \Omega_{xy},$$

$$\frac{\partial^k \tilde{Z}}{\partial x^k}\Big|_{x=0} = 0, \ \frac{\partial^\ell \tilde{Z}}{\partial x^\ell}\Big|_{x=0} = \frac{\partial^\ell \tilde{Z}}{\partial x^\ell}\Big|_{x=1}, \quad y \in [0,1], k = 1,3, \ell = 0,2,$$

$$\frac{\partial^k \tilde{Z}}{\partial y^k}\Big|_{y=0} = \frac{\partial^k \tilde{Z}}{\partial y^k}\Big|_{y=1} = 0, \quad x \in [0,1], k = 0,2.$$

Note that for $n = 0, k \in \mathbb{N}$, corresponding to the eigenvalues $\sigma_{0k} = \mu_k$, the above problem has no solution, while for $n, k \in \mathbb{N}$, we obtain the following expression of the associated functions

$$\tilde{Z}_{nk}(x,y) = x\sin(2n\pi x)\sqrt{2}\sin(k\pi y).$$

For a notation convenience, the system of eigenfunctions and the corresponding associated functions of problem (8)-(10) are respectively expressed as follows

$$Z_{0k}(x,y) = Y_k(y), \quad Z_{2n-1\,k}(x,y) = X_{2n-1}(x) Y_k(y),$$

and

$$Z_{2n\,k}(x,y) = X_{2n}(x) Y_k(y),\ n, k \in \mathbb{N}, \tag{19}$$

where

$$X_0(x) = 1,\ X_{2n-1}(x) = \cos(2\pi nx),\ X_{2n}(x) = x\sin(2\pi nx),\ n \in \mathbb{N}. \tag{20}$$

(For further methods on spectral problems, check Refs. [1,4,5,17,24,25]). Here, we are using the scheme of [1] to construct a system of eigenfunctions and associated functions to problems under consideration.

The problem (11)-(13) has eigenvalues $\lambda_0 = 0,\ \lambda_n = (2\pi n)^4,\ n \in \mathbb{N}$; the corresponding eigenfunctions and associated functions are given by

$$W_{0k}(x,y) = X_0^*(x)Y_k(y), \quad W_{2n-1\,k}(x,y) = X_{2n-1}^*(x)Y_k(y),$$

and

$$W_{2n\,k}(x,y) = X_{2n}^*(x)Y_k(y),\ n,k \in \mathbb{N}, \tag{21}$$

respectively.
Here

$$X_0^*(x) = 2(1-x), X_{2n-1}^*(x) = 4(1-x)\cos(2\pi nx), X_{2n}^*(x) = 4\sin(2\pi nx). \tag{22}$$

It follows from [2] that the system of functions (20) and (22) are complete and forms a Riesz' basis in $L^2(0,1)$; the similar statement applies to systems (19) and (21).

Prior to the statement of our results, we recall the following lemma.

Lemma 1 ([26]). For any fixed number, let $n = 0,1,2,...,\ \{\varphi_n^k(x)\}_{k\geq 0}$ be a complete orthonormal system of functions in $[0,\pi]$ and the system of functions $\psi_n(y)$ forms a Riesz basis in $L^2[0,2\pi]$. Then the system of functions $u_{nk}(x,y) = \varphi_n^k(x)\psi_n(y), n,k = 0,1,2,...$ forms a Riesz basis in $L^2([0,\ \pi]\times[0,\ 2\pi])$.

Now, we are ready to state the following lemmas.

Lemma 2. The systems of functions (19) and (21) are bi-orthogonal.

The proof of Lemma 2 can be achieved by direct calculations of appropriate integrals.

Lemma 3. The systems of functions (19) and (21) form a Riesz basis in $L^2(\Omega_{xy})$.

The proof of Lemma 3 follows from the basis property of the set of functions (20) and (22), and lemma 1.

3 Uniqueness and Existence of a Solution for the Direct Problem

Since the set of functions (19) and (21) is complete and forms a Riesz basis in $L^2(\Omega_{xy})$, then the solution for equation (7) subject to the boundary conditions (3)-(5), can be represented by the bi-orthogonal series

$$u(x,y,t) = \sum_{k=1}^{\infty} C_{0\,k}(t)Z_{0\,k}(x,y) + \sum_{n,k=1}^{\infty} C_{2n-1\,k}(t)Z_{2n-1\,k}(x,y)$$

$$+ \sum_{n,k=1}^{\infty} C_{2n\,k}(t)Z_{2n\,k}(x,y), \tag{23}$$

where $C_{0\,k}(t), C_{2n-1\,k}(t), C_{2n\,k}(t)$ are unknown functions.

Using properties of the bi-orthogonal set of functions (19) and (21), we obtain from (23) the following representation of these unknowns

$$\begin{aligned} C_{0\,k}(t) &= < u(x,y,t), W_{0\,k}(x,y) >,\ C_{2n-1\,k}(t) = < u(x,y,t), W_{2n-1\,k}(x,y) >, \\ C_{2n\,k}(t) &= < u(x,y,t), W_{2n\,k}(x,y) > . \end{aligned} \tag{24}$$

Acting the operator ${}_CD_{0t}^{\alpha}$ on both sides of each equation in (24), considering equation (7) and boundary condition (3), we obtain the following time fractional differential equations for $C_{0\,k}(t), C_{2n\,k}(t)$ and $C_{2n-1\,k}(t)$:

$$\begin{cases} {}_CD_{0t}^{\alpha}C_{0\,k}(t) + \mu_k C_{0\,k}(t) = 0, \\ C_{0\,k}(0) = \varphi_{0k}, \end{cases}$$

$$\begin{cases} {}_C D_{0t}^{\alpha} C_{2n\,k} + \sigma_{nk} C_{2n\,k}(t) = 0, \\ C_{2n\,k}(0) = \varphi_{2n\,k}, \end{cases}$$

and

$$\begin{cases} {}_C D_{0t}^{\alpha} C_{2n-1\,k} + \sigma_{nk} C_{2n-1\,k}(t) = 4(2\pi n)^3 C_{2n\,k}(t), \\ C_{2n-1\,k}(0) = \varphi_{2n-1\,k}. \end{cases}$$

The solutions of the above initial value problems are given (Check [27] [p. 231] for details) by

$$C_{0\,k}(t) = \varphi_{0\,k} E_{\alpha}(-\mu_k t^{\alpha}), \tag{25}$$

$$C_{2n\,k}(t) = \varphi_{2n\,k} E_{\alpha}(-\sigma_{nk} t^{\alpha}), \tag{26}$$

and

$$C_{2n-1\,k}(t) = \varphi_{2n-1\,k} E_{\alpha}(-\sigma_{nk} t^{\alpha})$$

$$+4(2\pi n)^3 \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^{\alpha}) C_{2n\,k}(\tau) d\tau, \tag{27}$$

respectively, where

$$\varphi_{0\,k} =< \varphi(x,y), W_{0\,k}(x,y) >, \quad \varphi_{2n-1\,k} =< \varphi(x,y), W_{2n-1\,k}(x,y) >,$$

$$\varphi_{2n\,k} =< \varphi(x,y), W_{2n\,k}(x,y) > . \tag{28}$$

Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [7, **?**] defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; E_{\alpha,1}(z) = E_{\alpha}(z), \; z, \alpha, \beta \in \mathbb{C}, \; \mathrm{Re}(\alpha) > 0$$

and satisfies the following properties

1) for $\mu > 0, \alpha, \beta \in (0,1], \alpha \leq \beta$ the function $t^{\alpha-1} E_{\alpha,\beta}(-\mu t^{\alpha})$ is completely monotone, i.e. $(-1)^n \left[t^{\beta-1} E_{\alpha,\beta}(-\mu t^{\alpha})\right]^{(n)} \geq 0, n \in \mathbb{N} \cup \{0\}$. (See [6] and [27] [p. 118, 120] for details).

2) for $\alpha \in (0,2), \; \gamma \leq |\arg z| \leq \pi, \beta \in R, \; \gamma \in (\pi\alpha/2; \min\{\pi; \pi\alpha\})$

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{M}{1+|z|}, \tag{29}$$

where M is a constant that is independent of z;

$$3)\, E_{\alpha,\mu}(z) = \frac{1}{\Gamma(\mu)} + z E_{\alpha,\alpha+\mu}(z), \; \int_0^z t^{\mu-1} E_{\alpha,\mu}(\lambda t^{\alpha}) dt = z^{\mu} E_{\alpha,\mu+1}(\lambda z^{\alpha}). \tag{30}$$

Plugging the expressions of the functions $C_{0\,k}(t), C_{2n\,k}(t)$ and $C_{2n-1\,k}(t)$ in equation (23), the solution of problem (7) subject to the boundary conditions (3)-(5) can be represented by the series

$$u(x,y,t) = \sum_{k=1}^{\infty} u_{0\,k}(x,y,t) + \sum_{n,k=1}^{\infty} (u_{2n-1\,k}(x,y,t) + u_{2n\,k}(x,y,t)), \tag{31}$$

where

$$u_{0\,k}(x,y,t) = \varphi_{0\,k} E_{\alpha}(-\mu_k t^{\alpha}) Z_{0\,k}(x,y), \tag{32}$$

$$u_{2n-1\,k}(x,y,t) = \left(\varphi_{2n-1\,k} E_{\alpha}(-\sigma_{nk} t^{\alpha}) + 4(2\pi n)^3 \varphi_{2n\,k} F_{nk}(t)\right) Z_{2n-1\,k}(t), \tag{33}$$

$$u_{2n\,k}(x,y,t) = \varphi_{2n\,k} E_{\alpha}(-\sigma_{nk} t^{\alpha}) Z_{2n\,k}(x,y), \tag{34}$$

$$F_{nk}(t)=\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^{\alpha})E_{\alpha}(-\sigma_{nk}\tau^{\alpha})d\tau. \tag{35}$$

Afterwards, we prove the uniqueness and the existence of a solution to the problem (7) subject to the boundary conditions (3)-(5).

Theorem 1. Assume that $\varphi\in C^{4,1}_{x,y}(\bar{\Omega}_{xy})\cap C^{1,4}_{x,y}(\bar{\Omega}_{xy})$ satisfying the following conditions:

$$\varphi(0,y)=\varphi(1,y),\ \varphi_{xx}(0,y)=\varphi_{xx}(1,y),\ \varphi_{x}(0,y)=\varphi_{xxx}(0,y)=0,\ y\in[0,1],$$

$$\varphi(x,0)=\varphi(x,1)=\varphi_{yy}(x,0)=\varphi_{yy}(x,1)=0,\ x\in[0,1].$$

Then, the problem (7) subject to the boundary conditions (3)-(5) has a unique solution and it is represented by the series (31).

Proof.

• **Uniqueness of the solution**:
Let u_1 and u_2 be two solutions of the problem (7), (3)-(5) in the domain Ω. The function $u=u_1-u_2$ satisfies equation (7), conditions (4), (5), and $u(x,y,0)=0$, $(x,y)\in\Omega$. Taking (25), (26) and (27) into account, we have

$$C_{0\,k}(t)=0,\quad C_{2n-1\,k}(t)=0,\quad C_{2n\,k}(t)=0,$$

or

$$(u(x,y,t),W_{0\,k}(x,y))=0,\quad (u(x,y,t),W_{2n-1\,k}(x,y))=0,$$
$$(u(x,y,t),W_{2n\,k}(x,y))=0.$$

This shows that the function u is orthogonal to the set of functions (21) which is complete and forms a basis in $L^2(\Omega_{xy})$. Hence, $u(x,y,t)=0$ in the domain Ω. Since $u\in C(\bar{\Omega})$, we have $u=0$ in $\bar{\Omega}$.

• **Existence of the solution**:
By construction $u(x,y,t)$, the initial and boundary conditions are satisfied. What remains is to prove the validity of the differentiation of the series. We show that $u\in C(\bar{\Omega})$. The estimates of (19) and (21) can be easily obtained as follows

$$|Z_{n\,k}(x,y)|\le\sqrt{2},\quad |W_{n\,k}(x,y)|\le 4\sqrt{2},$$

while estimates for (32) and (34) are obtained by using the boundedness properties of the Mittag-Leffler function

$$|u_{0k}(x,y,t)|\le\sqrt{2}M_1|\varphi_{0k}|,\quad |u_{2n\,k}(x,y,t)|\le\sqrt{2}M_2|\varphi_{2n\,k}|,$$

where M_1,M_2 are positive constants.

Using the property given by equation (30), we estimate the expression (35) as follows:

$$|F_{nk}(t)|\le\frac{M_4}{\sigma_{nk}},\quad M_4=M_3(1+M_2),\ t\in[0,1],$$

where M_3 is positive constant.

From the (33), we have

$$|u_{2n-1\,k}(x,y,t)|\le M_5\left(|\varphi_{2n-1\,k}|+\varphi_{2n\,k}|\right),\ M_5=\max\{\sqrt{2}M_2,4\sqrt{2}M_4\}.$$

Thus, $u(x,y,t)$ can be estimated through (31)-(35) as follows:

$$|u(x,y,t)|\le\sqrt{2}M_1\sum_{k=1}^{\infty}|\varphi_{0\,k}|+M_5\sum_{n,k=1}^{\infty}|\varphi_{2n-1\,k}|+(\sqrt{2}M_2+M_5)\sum_{k=1}^{\infty}|\varphi_{0\,k}|.$$

Furthermore, applying Cauchy-Schwarz and Bessel's inequalities to each expression of (28), and taking into account the conditions imposed on the function φ, we obtain the estimations

$$\sum_{k=1}^{\infty}|\varphi_{0\,k}| \leq \frac{\sqrt{2}}{3}\|\varphi_y\|_{L_2(\Omega_{xy})}, \quad \sum_{n,k=1}^{\infty}|\varphi_{2n-1\,k}| \leq \frac{\sqrt{2}(30+\sqrt{15})}{180}\|\varphi_{xy}\|_{L_2(\Omega_{xy})},$$

$$\sum_{n,k=1}^{\infty}|\varphi_{2n\,k}| \leq \frac{\sqrt{2}}{6}\|\varphi_{xy}\|_{L_2(\Omega_{xy})}. \tag{36}$$

Consequently, the series (31) is dominated by the convergent series

$$\sum_{k=1}^{\infty}|\varphi_{0\,k}| + \sum_{n,k=1}^{\infty}(|\varphi_{2n-1\,k}|+|\varphi_{2n\,k}|)$$

and hence by the M-test of Weierstrass, the series is absolutely and uniformly convergent.

Similarly, one can prove that

$$\frac{\partial^i u}{\partial x^i} \in C(\bar{\Omega}), \quad \frac{\partial^j u}{\partial y^j} \in C(\bar{\Omega}), \quad i=1,2,3,\ j=1,2.$$

Now, we show that $\dfrac{\partial^4 u}{\partial x^4} \in C(\Omega)$. Acting the operator $\dfrac{\partial^4}{\partial x^4}$ on (33) and (34), we get

$$\frac{\partial^4 u_{2n-1\,k}}{\partial x^4} = \left(\varphi_{2n-1\,k}E_\alpha(-\sigma_{nk}t^\alpha)+4(2\pi n)^3\varphi_{2n\,k}F_{nk}(t)\right)\lambda_n\cos(2\pi nx)\sqrt{2}\sin(\pi ny),$$

and

$$\frac{\partial^4 u_{2n\,k}}{\partial x^4} = \sqrt{2}\varphi_{2n\,k}E_\alpha(-\sigma_{nk}t^\alpha)\sin(k\pi y)\left(\lambda_n\sin(2\pi nx)-4(2\pi n)^3\cos(2\pi nx)\right),$$

respectively, followed by the estimates

$$\left|\frac{\partial^4 u_{2n-1\,k}}{\partial x^4}\right| \leq \frac{\sqrt{2}M_2}{\varepsilon^\alpha}|\varphi_{2n-1k}| + 4\sqrt{2}(2\pi n)^3 M_4|\varphi_{2nk}|,$$

$$\left|\frac{\partial^4 u_{2nk}}{\partial x^4}\right| \leq \frac{2\sqrt{2}M_2}{\varepsilon^\alpha}|\varphi_{2nk}|.$$

The convergence of the series $\sum\limits_{n,k=1}^{\infty}\varphi_{2n-1\,k}$ follows from (36), while of the series $\sum\limits_{n,k=1}^{\infty}\varphi_{2n\,k}$ follows from the estimate

$$\sum_{n,k=1}^{\infty}\lambda_n^{3/4}|\varphi_{2n\,k}| \leq \frac{\sqrt{2}}{6}\left\|\frac{\partial^5\varphi}{\partial x^4\partial y}\right\|_{L_2(\Omega_{xy})}, \tag{37}$$

which is obtained by applying Cauchy-Schwartz and Bessel's inequalities to the integral representation of $\varphi_{2n\,k}$ given by equation (28).

Thus, from the above estimates, the series

$$\sum_{n,k=1}^{\infty}\frac{\partial^4 u_{2n-1\,k}(x,y,t)}{\partial x^4}, \quad \sum_{n,k=1}^{\infty}\frac{\partial^4 u_{2n\,k}(x,y,t)}{\partial x^4}$$

are dominates by a convergent series, and therefore they converge absolutely and uniformly in Ω.
In a similar manner, it can be proved that ${}_C D_{0t}^{\alpha} u \in C(\Omega)$. This ends the proof of the Theorem.

Remark. The solution of the direct problem can be represented as well (23), where the functions $C_{0\,k}(t), C_{2n-1\,k}(t), C_{2n\,k}(t)$ have the form

$$C_{0\,k}(t) = \varphi_{0\,k}E_\alpha(-\mu_k t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{0k}(t-\tau)^\alpha)f_{0\,k}(\tau)d\tau,$$

$$C_{2n\,k}(t)=\varphi_{2n\,k}E_{\alpha}(-\sigma_{nk}t^{\alpha})+\int\limits_{0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^{\alpha})f_{2n\,k}(\tau)d\tau,$$

$$C_{2n-1\,k}(t)=\varphi_{2n-1\,k}E_{\alpha}(-\sigma_{nk}t^{\alpha})$$

$$+\int\limits_{0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^{\alpha})\left(4(2\pi n)^{3}C_{2n\,k}(\tau)+f_{2n\,k}(\tau)\right)d\tau.$$

Here

$$f_{0\,k}(t)=\langle f(x,y,t),W_{0\,k}(x,y)\rangle\,,f_{2n-1\,k}(t)=\langle f(x,y,t),W_{2n-1\,k}(x,y)\rangle\,,$$

$$f_{2n\,k}(t)=\langle f(x,y,t),W_{2n\,k}(x,y)\rangle\,,$$

and the coefficients $\varphi_{0\,k}$, $\varphi_{2n-1\,k}$, $\varphi_{2n\,k}$ are defined by (28).

4 Investigation of the Inverse Problem

The source term $f(x,y)$ can be expanded by the series

$$f(x,y)=\sum_{k=1}^{\infty}f_{0\,k}Z_{0\,k}(x,y)+\sum_{n,k=1}^{\infty}f_{2n-1\,k}Z_{2n-1\,k}(x,y)$$

$$+\sum_{n,k=1}^{\infty}f_{2n\,k}Z_{2n\,k}(x,y), \tag{38}$$

and the corresponding solution $u(x,y,t)$ is represented by the bi-orthogonal series (23).
Together, equation (23) and (38) involve three unknown functions $C_{2n-1\,k}(t)$, $C_{2n\,k}(t)$, $C_{0\,k}(t)$ and three unknown constants f_{0k}, f_{2n-1k}, f_{2nk}.
Substituting (23) and (38) into (2), and using conditions(3)-(6), these unknowns satisfy the following problems

$$\begin{cases} {}_{C}D_{0t}^{\alpha}C_{0\,k}(t)+\mu_{k}C_{0\,k}(t)=f_{0k}, \\ C_{0\,k}(0)=\varphi_{0k}, C_{0\,k}(T)=\psi_{0k}, \end{cases} \tag{39}$$

$$\begin{cases} {}_{C}D_{0t}^{\alpha}C_{2n\,k}+\sigma_{nk}C_{2n\,k}(t)=f_{2nk}, \\ C_{2n\,k}(0)=\varphi_{2n\,k}, C_{2nk}(T)=\psi_{2nk}, \end{cases} \tag{40}$$

and

$$\begin{cases} {}_{C}D_{0t}^{\alpha}C_{2n-1\,k}+\sigma_{nk}C_{2n-1\,k}(t)=4(2\pi n)^{3}C_{2n\,k}(t)+f_{2nk}, \\ C_{2n-1\,k}(0)=\varphi_{2n-1\,k}, C_{2n-1\,k}(T)=\psi_{2n-1\,k}, \end{cases} \tag{41}$$

where φ_{0k}, $\varphi_{2n-1\,k}$, $\varphi_{2n\,k}$, ψ_{0k}, $\psi_{2n-1\,k}$, $\psi_{2n\,k}$ are the coefficients determined by (28), and

$$\psi_{0\,k}(t)=(\psi(x,y),W_{0\,k}(x,y))\,,\quad \psi_{2n-1\,k}=(\psi(x,y),W_{2n-1\,k}(x,y))\,,$$

$$\psi_{2n\,k}=(\psi(x,y),W_{2n\,k}(x,y))\,. \tag{42}$$

The solution of the equation (39), satisfying the first condition, has the form

$$C_{0k}(t)=\varphi_{0k}E_{\alpha}(-\mu_{k}t^{\alpha})+\int\limits_{0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\mu_{k}(t-\tau)^{\alpha})f_{0k}d\tau$$

or

$$C_{0k}(t)=\varphi_{0k}E_{\alpha}(-\mu_{k}t^{\alpha})+f_{0k}t^{\alpha}E_{\alpha,\alpha+1}(-\mu_{k}t^{\alpha}).$$

Now, using the second condition of (39), we obtain

$$\varphi_{0k}E_{\alpha}(-\mu_{k}T^{\alpha})+f_{0k}T^{\alpha}E_{\alpha,\alpha+1}(-\mu_{k}T^{\alpha})=\psi_{0k}.$$

It follows, from the last equation and the Mittag-Leffler function properties given in (30), that

$$f_{0k}=\frac{\mu_k}{1-E_\alpha(-\mu_k T^\alpha)}(\psi_{0k}-E_\alpha(-\mu_k T^\alpha)\varphi_{0k}). \tag{43}$$

Substituting the obtained values of f_{0k} into the representation of $C_{0k}(t)$ yields

$$C_{0k}(t)=\varphi_{0k}E_\alpha(-\mu_k t^\alpha)+\frac{\mu_k(\psi_{0k}-\varphi_{0k}E_\alpha(-\mu_k T^\alpha))}{1-E_\alpha(-\mu_k T^\alpha)}t^\alpha E_{\alpha,\alpha+1}(-\mu_k t^\alpha),$$

or equivalently

$$C_{0k}(t)=\frac{E_\alpha(-\mu_k t^\alpha)-E_\alpha(-\mu_k T^\alpha)}{1-E_\alpha(-\mu_k T^\alpha)}\varphi_{0k}+\frac{1-E_\alpha(-\mu_k t^\alpha)}{1-E_\alpha(-\mu_k T^\alpha)}\psi_{0k}. \tag{44}$$

The expressions of the function $C_{2nk}(t)$ and the value of f_{2nk} can be obtained from (40) by following the same approach used in solving problem (39):

$$C_{2nk}(t)=\frac{E_\alpha(-\sigma_{nk} t^\alpha)-E_\alpha(-\sigma_{nk} T^\alpha)}{1-E_\alpha(-\sigma_{nk} T^\alpha)}\varphi_{2nk}+\frac{1-E_\alpha(-\sigma_{nk} t^\alpha)}{1-E_\alpha(-\sigma_{nk} T^\alpha)}\psi_{2nk}, \tag{45}$$

and

$$f_{2nk}=\frac{\sigma_{nk}}{1-E_\alpha(-\sigma_{nk} T^\alpha)}(\psi_{2nk}-E_\alpha(-\sigma_{nk} T^\alpha)\varphi_{2nk}). \tag{46}$$

The solution of the problem (41) is given by the equation

$$C_{2n-1k}(t)=\varphi_{2n-1k}E_\alpha(-\sigma_{nk} t^\alpha)$$
$$+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^\alpha)\left(4(2\pi n)^3 C_{2nk}+f_{2n-1k}\right)d\tau$$

or explicitly by

$$C_{2n-1k}(t)=\varphi_{2n-1k}E_\alpha(-\sigma_{nk} t^\alpha)+f_{2n-1k}t^\alpha E_{\alpha,\alpha+1}(-\sigma_{nk} t^\alpha)$$
$$+4\,(2\pi n)^3\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^\alpha)C_{2nk}d\tau.$$

Now, using the second condition of (41), we have

$$\varphi_{2n-1k}E_\alpha(-\sigma_{nk} T^\alpha)+f_{2n-1k}T^\alpha E_{\alpha,\alpha+1}(-\sigma_{nk} T^\alpha)$$
$$+4\,(2\pi n)^3\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(T-\tau)^\alpha)C_{2nk}d\tau=\psi_{2n-1k},$$

or

$$f_{2n-1k}=\frac{1}{T^\alpha E_{\alpha,\alpha+1}(-\sigma_{nk} T^\alpha)}[\psi_{2n-1k}-\varphi_{2n-1k}E_\alpha(-\sigma_{nk} T^\alpha)$$
$$-4\,(2\pi n)^3\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(T-\tau)^\alpha)C_{2nk}\,d\tau.$$

Using properties of the Mittag-Leffler functions given by (32) and (30), it follows from (45) that $C_{2n-1k}(t)$ and f_{2n-1k} can be expressed respectively by the equations

$$C_{2n-1k}(t)=\frac{E_\alpha(-\sigma_{nk} t^\alpha)-E_\alpha(-\sigma_{nk} T^\alpha)}{1-E_\alpha(-\sigma_{nk} T^\alpha)}\varphi_{2n-1}+\frac{1-E_\alpha(-\sigma_{nk} t^\alpha)}{1-E_\alpha(-\sigma_{nk} T^\alpha)}\psi_{2n-1k}$$
$$+\frac{4(2\pi n)^3}{1-E_\alpha(-\sigma_{nk} T^\alpha)}[(1-E_\alpha(-\sigma_{nk} T^\alpha)F_{nk}(t)$$

$$-(1-E_\alpha(-\sigma_{nk}t^\alpha))F_{nk}(T)](\varphi_{2nk}-\psi_{2nk}), \tag{47}$$

and

$$f_{2n-1k}=\frac{\sigma_{nk}}{1-E_\alpha(-\sigma_{nk}T^\alpha)}(\psi_{2n-1k}-E_\alpha(-\sigma_{nk}T^\alpha)\varphi_{2n-1k})$$
$$-\frac{4(2\pi n)^3\sigma_{nk}F_{nk}(T)}{(1-E_\alpha(-\sigma_{nk}T^\alpha))^2}(\varphi_{2nk}-\psi_{2nk})$$
$$-\frac{4(2\pi n)^3\sigma_{nk}F_{nk}(T)}{1-E_\alpha(-\sigma_{nk}T^\alpha)}(\psi_{2nk}-E_\alpha(-\sigma_{nk}T^\alpha)\varphi_{2nk}). \tag{48}$$

Thus, the solution pair $\{u(x,y,t),f(x,y)\}$ of the inverse problem is represented by (23) and (38), where the functions $C_{0k}(t),C_{2n-1k}(t),C_{2nk}(t)$ and the coefficients $f_{0k},\ f_{2n-1k},\ f_{2nk}$ are determined by (44), (45), (47), and (43), (48), (46).

Theorem 2. Let the functions φ and ψ satisfy the conditions:

$$\varphi\in C^{4,1}_{x,y}(\bar{\Omega}_{xy})\cap C^{1,4}_{x,y}(\bar{\Omega}_{xy}),\qquad \psi\in C^{5,1}_{x,y}(\bar{\Omega}_{xy})\cap C^{1,5}_{x,y}(\bar{\Omega}_{xy}),$$

$$\frac{\partial^{2i}\varphi}{\partial x^{2i}}|_{x=0}=\frac{\partial^{2i}\varphi}{\partial x^{2i}}|_{x=1}=0,\ \frac{\partial\varphi}{\partial x}|_{x=0}=\frac{\partial^3\varphi}{\partial x^3}|_{x=0}=0,\ i=0,1,\ y\in[0,1],$$
$$\frac{\partial^{2j}\varphi}{\partial y^{2j}}|_{y=0}=\frac{\partial^{2j}\varphi}{\partial y^{2j}}|_{y=1}=0,\ j=0,1,\ x\in[0,1],$$
$$\frac{\partial\psi}{\partial x}|_{x=0}=\frac{\partial^3\psi}{\partial x^3}|_{x=0}=0,\ \frac{\partial^{2i}\psi}{\partial x^{2i}}|_{x=0}=\frac{\partial^{2i}\psi}{\partial x^{2i}}|_{x=1},\ \ i=\overline{0,2},\ y\in[0,1],$$

and

$$\frac{\partial^{2j}\psi}{\partial y^{2j}}|_{y=0}=\frac{\partial^{2j}\psi}{\partial y^{2j}}|_{y=1}=0,\ j=0,1,2,\ x\in[0,1].$$

Then the solution of the problem (2), (3)-(6) exists, it is unique and can be represented by the sum of series (23)), (38).

Proof. The uniqueness of the problem easily follows from the representations (23) and (38), and from the completeness of the system (19).

By construction, $u(x,y,t)$ and $f(x,y)$ satisfy equation (2) and conditions (3) and (6). One can similarly prove (as in direct problem) that $u\in C^{3,2,0}_{x,y,t}(\bar{\Omega})\cap C^{4,4,0}_{x,y,t}(\Omega)$, ${}_CD^\alpha_{0t}u\in C(\Omega)$ (see formulation of inverse problem). Hence, we have to show that $f(x,y)\in C(\Omega_{xy})$.

The series represented by (38) is bounded by the expression

$$\sum_{k=1}^{\infty}|f_{0k}|+\sum_{n,k=1}^{\infty}|f_{2n-1k}|+\sum_{n,k=1}^{\infty}|f_{2nk}|. \tag{49}$$

Let us consider the first series of (49). Let $\Delta_{0k}=1-E_\alpha(-\mu_k p^\alpha)$. Since $\Delta_{0k}\neq 0$, there exists $\delta>0$ such that $|\Delta_{0k}|\geq\delta>0$. Then from (29) and (43) we have

$$|f_{0k}|\leq C_1(|\varphi_{0k}|+\mu_k|\psi_{0k}|),\quad C_1=\max\left\{\frac{M_1}{\delta p^\alpha},\frac{1}{\delta}\right\}.$$

From (36), it follows that the series $\sum\limits_{k=1}^{\infty}|\varphi_{0k}|$ converges. Furthermore, by considering the imposed conditions on ψ, and applying the Cauchy-Schwarz and the Bessel inequalities, from (42) we deduce

$$\psi_{0k}=\frac{2}{\mu_k^{5/4}}\iint\limits_{\Omega_{xy}}\frac{\partial^5\psi}{\partial y^5}(1-x)\sqrt{2}\cos(k\pi y)dy,\ \sum_{k=1}^{\infty}\mu_k^{1/4}|\psi_{0k}|\leq\frac{\sqrt{2}}{3}\left\|\frac{\partial^5\psi}{\partial y^5}\right\|_{L_2(\Omega_{xy})}.$$

Hence, the series $\sum\limits_{k=1}^{\infty}|f_{0k}|$ is convergent.

Now, we consider the third series of (49). Since $\Delta_{nk} = 1 - E_\alpha(-\sigma_{nk}p^\alpha) \neq 0$, there exists $\delta > 0$ such that $|\Delta_{nk}| \geq \delta > 0$. Then from (29) and (46) we get

$$|f_{2nk}| \leq C_2(|\varphi_{2nk}| + \sigma_{nk}|\psi_{2nk}|), \quad C_2 = \max\left\{\frac{M_2}{\delta p^\alpha}, \frac{1}{\delta}\right\}.$$

From (36) it follows that series $\sum\limits_{n,k=1}^{\infty} |\varphi_{2nk}|$ converges. Considering the imposed conditions on the function ψ, from (42) we get

$$\sigma_{nk}\psi_{2nk} = \frac{2\sqrt{2}}{2\pi^2 nk}\left(\psi_{2nk}^{5,1} + \psi_{2nk}^{1,5}\right),$$

$$\psi_{2nk}^{5,1} = \iint\limits_{\Omega_{xy}} \frac{\partial^6\psi}{\partial x^5 \partial y}\sqrt{2}\cos(2n\pi x)\sqrt{2}\cos(k\pi y)dxdy,$$

and

$$\psi_{2nk}^{1,5} = \iint\limits_{\Omega_{xy}} \frac{\partial^6\psi}{\partial x \partial y^5}\sqrt{2}\cos(2n\pi x)\sqrt{2}\cos(k\pi y)dxdy.$$

We then deduce from applying the Cauchy-Schwartz and the Bessel inequalities that

$$\sum_{n,k=1}^{\infty} \sigma_{nk}|\psi_{2nk}| \leq \frac{\sqrt{2}}{6}\left(\left\|\frac{\partial^6\psi}{\partial x^5\partial y}\right\|_{L_2(\Omega_{xy})} + \left\|\frac{\partial^6\psi}{\partial x\partial y^5}\right\|_{L_2(\Omega_{xy})}\right). \tag{50}$$

From here, it follows that the series $\sum\limits_{n,k=1}^{\infty} \sigma_{nk}|\psi_{2nk}|$ converges as well.
As $|\Delta_{nk}| \geq \delta > 0$, from (48) we have

$$|f_{2n-1k}| \leq C\left(|\varphi_{2n-1k}| + \sigma_{nk}|\psi_{2n-1k}| + (2\pi n)^3(|\varphi_{2nk}| + |\psi_{2nk}|)\right).$$

Thus, the convergence of the second series of (49) follows from the last equation and (36), (37) and (50).
Whereupon, it follows from the M-test of Weierstrass that series (38) converges absolutely and uniformly in Ω_{xy}. The theorem is proved.
In order to illustrate obtained results, we consider some examples below.

Example 1. Let $\varphi(x,y) = \cos 2\pi x \sin \pi y$. Then by virtue of equations (28), we have

$$\varphi_{0k} = 0,\ \varphi_{0k} = 0,\ \varphi_{2nk} = 0,\ \varphi_{2n-1k} = \begin{cases} 0,\ n \neq 1 \text{ or } k \neq 1 \\ \dfrac{\sqrt{2}}{2},\ n = k = 1. \end{cases}$$

Since

$$\mu_1 = \pi^4,\ \lambda_1 = (2\pi)^4,\ \sigma_{11} = \lambda_1 + \mu_1 = 17\pi^4,$$

then by considering equations (25), (26) and (27), we obtain

$$C_{0k}(t) = 0, C_{2nk}(t) = 0,$$

$$C_{2n-1k}(t) = \begin{cases} 0,\ n \neq 1 \text{ or } k \neq 1 \\ \dfrac{\sqrt{2}}{2}E_\alpha(-17\pi^4 t^\alpha),\ n = k = 1. \end{cases}$$

By substituting these expressions of the functions $C_{0k}(t), C_{2n-1k}(t)$ and $C_{2nk}(t)$ in equation (23) , we have the solution of problem A in the form

$$u(x,y,t) = E_\alpha(-17\pi^4 t^\alpha)\cos 2\pi x \sin \pi y.$$

Example 2. Let $\alpha = 1$. Then from (25), (26) and (27), it follows that

$$C_{0k}(t) = \varphi_{0k}e^{-\mu_k t}, C_{2nk}(t) = \varphi_{2nk}e^{-\sigma_{nk}t},$$

$$C_{2n-1\,k}(t) = \left(\varphi_{2n-1\,k} + 4(2\pi n)^3 t \cdot \varphi_{2n\,k}\right) e^{-\sigma_{nk} t}.$$

Hence, the solution of the Problem A can be represented as

$$u(x,y,t) = \sum_{k=1}^{\infty} \varphi_{0\,k} e^{-\mu_k t} z_{0\,k}(x,y) + \sum_{n,k=1}^{\infty} \varphi_{2n\,k} e^{-\sigma_{nk} t} z_{2n\,k}(x,y)$$

$$+ \sum_{n,k=1}^{\infty} \left(\varphi_{2n-1\,k} + 4(2\pi n)^3 t \cdot \varphi_{2n\,k}\right) e^{-\sigma_{nk} t} z_{2n-1\,k}(x,y),$$

which coincides with the solution of the equation

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0$$

together with conditions (3) - (5). Here functions $z_{0\,k}(x,y)$, $z_{2n-1\,k}(x,y)$ and $z_{2n\,k}(x,y)$ are defined in formula (19).

5 Conclusion

We established the existence and uniqueness of regular solutions of problems for fractional parabolic equations, with nonlocal conditions given with respect to two spatial variables. Our method to prove that is based on expanding the solution using a bi-orthogonal set of functions. In addition, we explained the corresponding spectral problems, and we analyzed the eigenfunctions and associated functions of such problems, and finally, we established their completeness in $L^2(0 < x,y < 1)$.

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