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Hermite–Hadamard Type Inequalities for Harmonically Convex Functions on n-Coordinates

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Abstract: We introduce the notion of harmonically convex functions on n-coordinates and present some examples and properties of them. We also establish some Hermite-Hadamard type inequalities for the class of harmonically convex functions on n-coordinates which generalizes previous results.

Keywords: Harmonically convex functions, Hermite–Hadamard type inequality, harmonically convex functions on coordinates.

1 Introduction

Convex functions are important and provide a basis for constructing literature on mathematical inequalities. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if

$$f(tx+(1-t)y) \leq t f(x)+(1-t) f(y),$$

where $t \in [0,1]$ and $x,y \in I$.

A large number of inequalities are obtained by means of convex functions see [2,3,4,7,10]. A classical inequality for convex functions is the Hermite–Hadamard inequality, this is given as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

where $f : I \to \mathbb{R}$ is a convex function and $a,b \in I$ with $a < b$ (see [11]).

In this article we are dealing with a recent notion of generalized convexity, this notion was introduced by I. Iscan in [9], Iscan gave the following definition of harmonically convex functions:

Definition 1([9]). *Let I be an interval in $\mathbb{R} \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said to be* harmonically convex *on I if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y)+(1-t) f(x), \tag{1}$$

holds, for all $x,y \in I$ and $t \in [0,1]$.

The following result is immediate from the above definition.

Proposition 1([9]). *Let $I \subset (0,\infty)$ be a real function interval and $f : I \to \mathbb{R}$ is a function.*

(a) If f is convex and nondecreasing function, then f is harmonically convex.
(b) If f is harmonically convex and nonincreasing function, then f is convex.

The following result of the Hermite-Hadamard type for harmonically convex functions holds.

Theorem 1([9]). *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

In [14] authors gave the inequalities of Hermite-Hadamard type for rectangle in plane by defining harmonically convex functions on coordinates.

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Definition 2. *Let us consider the bidimensional interval $\Delta = [a,b]\times[c,d]$ in $(0,\infty)\times(0,\infty)$ with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ is said to be harmonically convex on Δ if the following inequality:*

$$f\left(\frac{xz}{tz+(1-t)x},\frac{yw}{tw+(1-t)y}\right)\le t\,f(x,y)+(1-t)f(z,w),$$

holds, for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$.

Definition 3. *Let us consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $(0,\infty)\times(0,\infty)$ with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ is said to be harmonically convex on the co-ordinates if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are harmonically convex where defined for all $y\in[c,d]$ and $x\in[a,b]$.*

The following inequalities of Hermite-Hadamard type hold.

Theorem 2. *Suppose that $f:\Delta\to\mathbb{R}$ is harmonically convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right) \qquad (3)\\
&\le \frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{f\left(x,\frac{2cd}{c+d}\right)}{x^2}dx+\frac{cd}{d-c}\int_c^d\frac{f\left(\frac{2ab}{a+b},y\right)}{x^2}dy\right]\\
&\le \frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{f(x,y)}{(xy)^2}dxdy\\
&\le \frac{1}{4}\left[\frac{ab}{b-a}\int_a^b\frac{f(x,c)}{x^2}dx+\frac{ab}{b-a}\int_a^b\frac{f(x,d)}{x^2}dx\right.\\
&\quad\left.+\frac{cd}{d-c}\int_c^d\frac{f(a,y)}{y^2}dy+\frac{cd}{d-c}\int_c^d\frac{f(b,y)}{y^2}dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities sharp.

We started with some notations and definitions that will be use throughout the rest of this paper, some of them were introduced by the authors in [1,5,6].
As usually, $\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes the set of all positive integers (resp. non-negative integers), typical point of $\mathbb{R}^n$ are denoted as $\mathbf{x}=(x_1,x_2,\dots,x_n)$ and $\mathbb{R}^n_+ := \{\mathbf{x}\in\mathbb{R}^n : x_i>0,\ i=1,\dots,n\}$. If $\mathbf{a}=(a_1,a_2,\dots,a_n)$, $\mathbf{b}=(b_1,b_2,\dots,b_n)\in\mathbb{R}^n$ we use the notation $\mathbf{a}<\mathbf{b}$ to denote that $a_i<b_i$ for each $i=1,\dots,n$ and similarly are defined $\mathbf{a}=\mathbf{b}$, $\mathbf{a}\le\mathbf{b}$ and $\mathbf{a}\ge\mathbf{b}$. If $\mathbf{a}<\mathbf{b}$, the set $\Delta^n := [\mathbf{a},\mathbf{b}] = \prod_{i=1}^n[a_i,b_i]$ will be called a n–dimensional closed interval. Furthermore, for $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$ and $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ will use the notations

$$|\alpha| := \alpha_1+\ldots+\alpha_n \qquad\text{and}\qquad \alpha x := (\alpha_1x_1,\dots,\alpha_nx_n).$$

Now we define two sets that will play an important role in this work:

$$\mathscr{E}(n) := \{\theta\in\mathbb{N}_0^n : \theta\le 1 \text{ and } |\theta| \text{ is even}\},$$
$$\mathscr{O}(n) := \{\theta\in\mathbb{N}_0^n : \theta\le 1 \text{ and } |\theta| \text{ is odd}\}.$$

Motivated by [1,5], we give the following definition

Definition 4. *Given $f:\Delta^n\to\mathbb{R}$ we define the n–dimensional Vitali permutation of f sobre an n–dimensional interval $[\mathbf{x},\mathbf{y}]\subseteq\Delta^n$, by*

$$\mathscr{J}_n(f,[\mathbf{x},\mathbf{y}]) := \sum_{\theta\in\mathscr{E}(n)} f(\theta x+(1-\theta y)) + \sum_{\theta\in\mathscr{O}(n)} f(\theta x+(1-\theta y)).$$

Note that in the case when $n=2$, we get $\mathscr{E}(2)=\{(0,0),(1,1)\}$ and $\mathscr{O}(2)=\{(0,1),(1,0)\}$, therefore

$$\mathscr{J}_2(f,[\mathbf{x},\mathbf{y}]) = f(x_1,x_2)+f(x_1,y_2)+f(y_1,y_2)+f(y_1,x_2).$$

In [8] Ghulam Farid and Atiq ur Rehman gave generalization of the work of S. S. Dragomir (see [12]) by defining convex functions on n–coordinates as follow:

Definition 5. *Let $(x_1,\dots,x_n)\in\Delta^n$. A mapping $f:\Delta^n\to\mathbb{R}$ is called convex on n–coordinates if the functions $f^i_{x_n}$, where $f^i_{x_n}(t) := f(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)$, are convex on $[a_i,b_i]$ for $i=1,\dots,n$.*

The following result of the Fejér–Hadamard's inequality for convex on functions n–coordinates holds.

Theorem 3([8]). *Let $(x_1,\dots,x_n)\in\Delta^n$ and $f:\Delta^n\to\mathbb{R}$ be a convex mapping on n–coordinates. Also, let $g_i:[a_i,b_i]\to\mathbb{R}$ be an integrable and symmetric function about $\dfrac{a_i+b_i}{2}$ for each $i=1,\dots,n$. Then we have*

$$\begin{aligned}
&\sum_{k=1}^n\frac{1}{G_k}\int_{a_k}^{b_k} f^{k+1}_{x_n}\left(\frac{a_{k+1}+b_{k+1}}{2}\right)g_k(x_k)dx_k\\
&\le \sum_{k=1}^n\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f^{k+1}_{x_n}(x_{k+1})g_{k+1}(x_{k+1})g_k(x_k)dx_{k+1}dx_k\\
&\le \sum_{k=1}^n\left[\frac{1}{G_k}\int_{a_k}^{b_k}\frac{f^{k+1}_{x_n}(a_{k+1})+f^{k+1}_{x_n}(b_{k+1})}{2}g_k(x_k)dx_k\right],
\end{aligned}$$

where

$$G_k=\int_{a_k}^{b_k} g_k(x_k)dx_k,$$

with $k=1,\dots,n$. These inequalities are sharp.

2 Main results

Motivated by [8,13,14], we introduce a new concept of n–coordinated convex functions which is called harmonically convex functions on the n–coordinates. Under this new concept, we present the Hermite–Hadamard inequalities for these new classes of functions.

Definition 6. *Let $\mathbf{a},\mathbf{b}\in\mathbb{R}^n_+$; such that $\mathbf{a}<\mathbf{b}$, then we will say that a function $f:\Delta^n\to\mathbb{R}$ is harmonically convex on Δ^n if the following inequality*

$$f\left(\frac{x_1y_1}{\alpha y_1+(1-\alpha)x_1},\ldots,\frac{x_ny_n}{\alpha y_n+(1-\alpha)x_n}\right) \tag{4}$$
$$\leq \alpha f(x_1,\ldots,x_n)+(1-\alpha)f(y_1,\ldots,y_n),$$

holds, for all $(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in\Delta^n$ and $\alpha\in[0,1]$.

We introduce the generalization of the work done by A. Set and I. Iscan (see [14]), by defining harmonically convex functions on n–coordinates as follow:

Definition 7. *A function $f:\Delta^n\subset\mathbb{R}^n_+\to\mathbb{R}$ is called harmonically convex on n-coordinates if the functions $f^i_{x_n}$, where $f^i_{x_n}(t)=f(x_1,x_2,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$, are harmonically convex on $[a_i,b_i]$ for $i=1,\ldots,n$.*

Example 1. Let us consider a function $f:[1,3]^n\to\mathbb{R}$ defined as:

$$f(x_1,\ldots,x_n)=\frac{1}{(x_1\cdots x_n)^p}$$

with $p\geq1$, then f is harmonically convex on n-coordinates.

Proof. In effect, let $x,y\in[1,3]$, $\alpha\in[0,1]$ and for each $i=1,\ldots,n$, we have

$$\begin{aligned}
&f^i_{x_n}\left(\frac{xy}{\alpha x+(1-\alpha)y}\right)\\
&=\frac{1}{\left[x_1\cdots x_{i-1}\cdot\frac{xy}{\alpha x+(1-\alpha)y}\cdot x_{i+1}\cdots x_n\right]^p}\\
&=\frac{[\alpha x+(1-\alpha)y]^p}{x_1^p\cdots x_{i-1}^p\cdot x^p\cdot y^p\cdot x_{i+1}^p\cdots x_n^p}\\
&\leq\frac{\alpha x^p+(1-\alpha)y^p}{x_1^p\cdots x_{i-1}^p\cdot x^p\cdot y^p\cdot x_{i+1}^p\cdots x_n^p}\\
&=\frac{\alpha x^p}{x_1^p\cdots x_{i-1}^p\cdot x^p\cdot y^p\cdot x_{i+1}^p\cdots x_n^p}+\frac{(1-\alpha)y^p}{x_1^p\cdots x_{i-1}^p\cdot x^p\cdot y^p\cdot x_{i+1}^p\cdots x_n^p}\\
&=\frac{\alpha}{x_1^p\cdots x_{i-1}^p\cdot y^p\cdot x_{i+1}^p\cdots x_n^p}+\frac{(1-\alpha)}{x_1^p\cdots x_{i-1}^p\cdot x^p\cdot x_{i+1}^p\cdots x_n^p}\\
&=\alpha f^i_{x_n}(y)+(1-\alpha)f^i_{x_n}(x).
\end{aligned}$$

Thus $f^i_{x_n}$ is harmonically convex function on $[1,3]$, for each $i=1,\ldots,n$. Hence f is harmonically convex on n-coordinates.

From definition 7 have the following important consequence.

Proposition 2. *Let $f:\subset\mathbb{R}^n_+\to\mathbb{R}$ a function and $f^i_{x_n}(t)=f(x_1,x_2,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$, for some $i=1,\ldots,n$, we have that*

(*a*) *If $f^i_{x_n}$ is convex and nondecreasing function for $i=1,\ldots,n$, then f is harmonically convex on n-coordinates.*

(*b*) *If $f^i_{x_n}$ is harmonically convex and nonincreasing function for $i=1,\ldots,n$, then f is convex on n-coordinates.*

Proof. It is immediate using the proposition 1.

The following theorem holds:

Theorem 4. *Every harmonically convex function $f:\Delta^n\subset\mathbb{R}^n_+\to\mathbb{R}$ is harmonically convex on n-coordinates.*

Proof. Suppose that f is harmonically convex on Δ^n. Consider $f^i_{x_n}:[a_i,b_i]\to\mathbb{R}$, defined as $f^i_{x_n}(t)=f(x_1,x_2,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$. Then for all $\alpha\in[0,1]$ and $u,v\in[a_i,b_i]$, we have

$$\begin{aligned}
&f^i_{x_n}\left(\frac{uv}{\alpha u+(1-\alpha)v}\right)\\
&=f\left(x_1,\ldots,x_{i-1},\frac{uv}{\alpha u+(1-\alpha)v},x_{i+1},\ldots,x_n\right)\\
&=f\left(\frac{x_1^2}{\alpha x_1+(1-\alpha)x_1},\frac{uv}{\alpha u+(1-\alpha)v},\ldots,\frac{x_n^2}{\alpha x_n+(1-\alpha)x_n}\right)\\
&\leq\alpha f(x_1,\ldots,x_{i-1},v,x_{i+1},\ldots,x_n)+(1-\alpha)f(x_1,\ldots,x_{i-1},u,x_{i+1},\ldots,x_n)\\
&=\alpha f^i_{x_n}(v)+(1-\alpha)f^i_{x_n}(u),
\end{aligned}$$

The converse of the previous theorem is not generally true, we give the following counter example:

Example 2. Consider the function $f:[2,5]\times[3,5]\times[4,5]\to[0,+\infty]$ defined as:

$$f(x_1,x_2,x_3)=(x_1-2)(x_2-3)(x_3-4),$$

then f is harmonically convex on 3-coordinates but it is not harmonically convex on $[2,5]\times[3,5]\times[4,5]$.

In effect, let $f^1_{x_3}:[2,5]\to[0,+\infty]$, defined as:
$f^1_{x_3}(t)=(t-2)(x_2-3)(x_3-4)$, then

$$\frac{d}{dt}f^1_{x_3}(t)=(x_2-3)(x_3-4)\geq0,\ \text{with}\ (x_2,x_3)\in[3,5]\times[4,5],$$

and

$$\frac{d^2}{dt^2}f^1_{x_3}(t)=0.$$

Thus, $f^1_{x_3}$ is a convex and nondecreasing function on $[2,5]$. Similarly it is proved that $f^2_{x_3}$ and $f^3_{x_3}$ are convex and nondecreasing functions on $[3,5]$ and $[4,5]$ respectively. Hence by the theorem 2, we get f is harmonically convex on 3-coordinates.

Now let's see that f is not harmonically convex on $[2,5]\times[3,5]\times[4,5]$.
Indeed, for $(2,5,5),(3,5,4)\in[2,5]\times[3,5]\times[4,5]$ and $\alpha\in(0,1)$, we have

$$\begin{aligned}
& f\left(\frac{2\cdot 3}{2\alpha+(1-\alpha)3},\frac{5\cdot 5}{5\alpha+(1-\alpha)5},\frac{5\cdot 4}{5\alpha+(1-\alpha)4}\right)\\
&= f\left(\frac{6}{3-\alpha},5,\frac{20}{4+\alpha}\right)\\
&= \left(\frac{6}{3-\alpha}-2\right)(5-3)\left(\frac{20}{4+\alpha}-4\right)\\
&= \frac{2\alpha}{3-\alpha}\cdot 2\cdot\frac{4-4\alpha}{4+\alpha}=\frac{16\alpha(1-\alpha)}{(3-\alpha)(4+\alpha)}>0,
\end{aligned}$$

and

$$\begin{aligned}
&\alpha f(3,5,4)+(1-\alpha)f(2,5,5)\\
&=\alpha(3-2)(5-3)(4-4)+(1-\alpha)(2-2)(5-3)(5-4)=0.
\end{aligned}$$

This for all $\alpha\in(0,1)$, we have

$$\begin{aligned}
& f\left(\frac{2\cdot 3}{2\alpha+(1-\alpha)3},\frac{5\cdot 5}{5\alpha+(1-\alpha)5},\frac{5\cdot 4}{5\alpha+(1-\alpha)4}\right)\\
&>\alpha f(3,5,4)+(1-\alpha)f(2,5,5),
\end{aligned}$$

therefore shows that f is not harmonically convex on $[2,5]\times[3,5]\times[4,5]$.

The following inequalities of Hermite-Hadamard type integral inequalities hold.

Theorem 5. *Let $f:\Delta^n\subset\mathbb{R}^n_+\to\mathbb{R}$ is harmonically convex function on n-coordinates on Δ^n. Then the following inequalities are hold,*

$$\begin{aligned}
&\sum_{i=1}^{n-1} f\left(x_1,\ldots,x_{i-1},\frac{2a_ib_i}{b_i-a_i},\frac{2a_{i+1}b_{i+1}}{b_{i+1}-a_{i+1}},\ldots,x_n\right) \qquad (5)\\
&\le \sum_{i=1}^{n-1}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}\left(\frac{2a_{i+1}b_{i+1}}{a_{i+1}+b_{i+1}}\right)}{x_i^2}dx_i\\
&\le \sum_{i=1}^{n-1}\frac{a_ib_ia_{i+1}b_{i+1}}{(b_i-a_i)(b_{i+1}-a_{i+1})}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}}\frac{f_{x_n}^{i+1}(x_{i+1})}{(x_ix_{i+1})^2}dx_{i+1}dx_i\\
&\le \sum_{i=1}^{n-1}\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1})}{2x_i^2}dx_i\\
&\le \frac{1}{4}\sum_{i=1}^{n-1}\left[f(x_1,\ldots,a_i,a_{i+1},\ldots,x_n)+f(x_1,\ldots,b_i,a_{i+1},\ldots,x_n)\right.\\
&\qquad\left.+f(x_1,\ldots,a_i,b_{i+1},\ldots,x_n)+f(x_1,\ldots,b_i,b_{i+1},\ldots,x_n)\right].
\end{aligned}$$

Proof. Since $f:\Delta^n\to\mathbb{R}$ is harmonically convex on n-coordinates, we have that the following functions $f_{x_n}^i:[a_i,b_i]\to\mathbb{R}$, $f_{x_n}^i(t)=f(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$ is harmonically convex on $[a_i,b_i]$ for all $x_n\in[a_n,b_n]$. Then by the inequality (1) of Hermite-Hadamard type for harmonically convex function $f_{x_n}^{i+1}$ on interval $[a_{i+1},b_{i+1}]$, we get

$$\begin{aligned}
f_{x_n}^{i+1}\left(\frac{2a_{i+1}b_{i+1}}{a_{i+1}+b_{i+1}}\right) &\le \frac{a_{i+1}b_{i+1}}{b_{i+1}-a_{i+1}}\int_{a_{i+1}}^{b_{i+1}}\frac{f_{x_n}^{i+1}(x_{i+1})}{x_{i+1}^2}dx_{i+1}\\
&\le \frac{f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1})}{2}.
\end{aligned}$$

Now by integrating on $[a_i,b_i]$, we obtain

$$\begin{aligned}
&\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}\left(\frac{2a_{i+1}b_{i+1}}{a_{i+1}+b_{i+1}}\right)}{x_i^2}dx_i \qquad (6)\\
&\le \frac{a_ia_{i+1}b_ib_{i+1}}{(b_i-a_i)(b_{i+1}-a_{i+1})}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}}\frac{f_{x_n}^{i+1}(x_{i+1})}{(x_ix_{i+1})^2}dx_{i+1}dx_i\\
&\le \frac{a_ib_i}{2(b_i-a_i)}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1})}{x_i^2}dx_i.
\end{aligned}$$

Again applying the inequality of Hermite-Hadamard, we get

$$\begin{aligned}
& f\left(x_1,\ldots,\frac{2a_ib_i}{a_i+b_i},\frac{2a_{i+1}b_{i+1}}{a_{i+1}+b_{i+1}},\ldots,x_n\right) \qquad (7)\\
&\le \frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}\left(\frac{2a_{i+1}b_{i+1}}{a_{i+1}+b_{i+1}}\right)}{x_i^2}dx_i,
\end{aligned}$$

for each $i\in\{1,\ldots,n-1\}$ and also

$$\begin{aligned}
&\frac{a_ib_i}{2(b_i-a_i)}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1})}{x_i^2}dx_i \qquad (8)\\
&=\frac{1}{2}\left[\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}(a_{i+1})}{x_i^2}dx_i+\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\frac{f_{x_n}^{i+1}(b_{i+1})}{x_i^2}dx_i\right]\\
&\le \frac{1}{2}\left[\frac{f(x_1,\ldots,a_i,a_{i+1},\ldots,x_n)+f(x_1,\ldots,b_i,a_{i+1},\ldots,x_n)}{2}\right.\\
&\quad\left.+\frac{f(x_1,\ldots,a_i,b_{i+1},\ldots,x_n)+f(x_1,\ldots,b_i,b_{i+1},\ldots,x_n)}{2}\right]\\
&=\frac{1}{4}\left[f(x_1,\ldots,a_i,a_{i+1},\ldots,x_n)+f(x_1,\ldots,b_i,a_{i+1},\ldots,x_n)\right.\\
&\quad\left.+f(x_1,\ldots,a_i,b_{i+1},\ldots,x_n)+f(x_1,\ldots,b_i,b_{i+1},\ldots,x_n)\right]
\end{aligned}$$

for each $i\in\{1,\ldots,n-1\}$.

Using the inequalities (7) and (8) in (6) and taking summation from 1 to $n-1$, we have (5).

Theorem 6. *Let $f:\Delta^n\subset\mathbb{R}^n_+\to\mathbb{R}$ is harmonically convex function on n-coordinates on Δ^n. Then the following inequalities are hold,*

$$\begin{aligned}
& f\left(\frac{2a_1b_1}{b_1-a_1},\ldots,\frac{2a_{n-1}b_{n-1}}{b_{n-1}-a_{n-1}},\frac{2a_nb_n}{b_n-a_n}\right) \qquad (9)\\
&\le \left(\prod_{i=1}^{n}\frac{a_ib_i}{b_i-a_i}\right)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{f_{x_n}^{n}(x_n)}{\left(\prod_{i=1}^{n}x_i\right)^2}dx_n\ldots dx_1\\
&\le \frac{1}{2^n}\mathscr{I}_n(f,[\boldsymbol{a},\boldsymbol{b}]).
\end{aligned}$$

Proof. Since $f^n_{x_n}$ is harmonically convex on $[a_n,b_n]$, then by the inequality (1) of Hermite-Hadamard type for harmonically convex function $f^n_{x_n}$ on interval $[a_n,b_n]$, we get

$$f^n_{x_n}\left(\frac{2a_nb_n}{a_n+b_n}\right)\le\frac{a_nb_n}{b_n-a_n}\int_{a_n}^{b_n}\frac{f^n_{x_n}(x_n)}{x_n^2}dx_n\le\frac{f^n_{x_n}(a_n)+f^n_{x_n}(b_n)}{2}.$$

Integrating this inequality on $[a_{n-1},b_{n-1}]$, we obtain

$$\begin{aligned}&\frac{a_{n-1}b_{n-1}}{b_{n-1}-a_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\frac{f^n_{x_n}\left(\frac{2a_nb_n}{a_n+b_n}\right)}{x_{n-1}^2}dx_{n-1} \qquad (10)\\ &\le\frac{a_{n-1}a_nb_{n-1}b_n}{(b_{n-1}-a_{n-1})(b_n-a_n)}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\frac{f^n_{x_n}(x_n)}{(x_{n-1}x_n)^2}dx_{n-1}dx_n\\ &\le\frac{a_{n-1}b_{n-1}}{b_{n-1}-a_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\frac{f^n_{x_n}(a_n)+f^n_{x_n}(b_n)}{2x_{n-1}^2}dx_{n-1}.\end{aligned}$$

By the inequality (7), we get

$$\begin{aligned}&f\left(x_1,\dots,\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}},\frac{2a_nb_n}{a_n+b_n}\right) \qquad (11)\\ &\le\frac{a_{n-1}b_{n-1}}{b_{n-1}-a_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\frac{f^n_{x_n}\left(\frac{2a_nb_n}{a_n+b_n}\right)}{x_{n-1}^2}dx_{n-1}.\end{aligned}$$

Now from (8), we have

$$\begin{aligned}&\frac{a_{n-1}b_{n-1}}{2(b_{n-1}-a_{n-1})}\int_{a_{n-1}}^{b_{n-1}}\frac{f^n_{x_n}(a_n)+f^n_{x_n}(b_n)}{x_{n-1}^2}dx_{n-1} \qquad (12)\\ &\le\frac{1}{2^2}\left[f(x_1,\dots,a_{n-1},a_n)+f(x_1,\dots,b_{n-1},a_n)\right.\\ &\quad\left.+f(x_1,\dots,a_{n-1},b_n)+f(x_1,\dots,b_{n-1},b_n)\right].\end{aligned}$$

From (10)-(12), we obtain

$$\begin{aligned}&f\left(x_1,\dots,\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}},\frac{2a_nb_n}{a_n+b_n}\right) \qquad (13)\\ &\le\frac{a_{n-1}b_{n-1}a_nb_n}{(b_{n-1}-a_{n-1})(b_n-a_n)}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\frac{f^n_{x_n}(x_n)}{(x_{n-1}x_n)^2}dx_{n-1}dx_n\\ &\le\frac{1}{2^2}\left[f(x_1,\dots,a_{n-1},a_n)+f(x_1,\dots,b_{n-1},a_n)\right.\\ &\quad\left.+f(x_1,\dots,a_{n-1},b_n)+f(x_1,\dots,b_{n-1},b_n)\right].\end{aligned}$$

Integrating the inequalities (13) above on $[a_{n-2},b_{n-2}]$,

$$\begin{aligned}&\frac{a_{n-2}b_{n-2}}{b_{n-2}-a_{n-2}}\int_{a_{n-2}}^{b_{n-2}}\frac{f\left(x_1,\dots,\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}},\frac{2a_nb_n}{a_n+b_n}\right)}{x_{n-2}^2}dx_{n-2} \qquad (14)\\ &\le\left(\prod_{i=n-2}^{n}\frac{a_ib_i}{b_i-a_i}\right)\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\frac{f^n_{x_n}(x_n)}{(x_{n-2}x_{n-1}x_n)^2}dx_{n-2}dx_{n-1}dx_n\\ &\le\frac{1}{2^2}\frac{a_{n-2}b_{n-2}}{b_{n-2}-a_{n-2}}\int_{a_{n-2}}^{b_{n-2}}\frac{1}{x_{n-2}^2}\left[f(x_1,\dots,a_{n-1},a_n)+f(x_1,\dots,b_{n-1},a_n)\right.\\ &\quad\left.+f(x_1,\dots,a_{n-1},b_n)+f(x_1,\dots,b_{n-1},b_n)\right]dx_{n-2}.\end{aligned}$$

By the inequality (7), we get

$$\begin{aligned}&f\left(x_1,\dots,\frac{2a_{n-2}b_{n-2}}{a_{n-2}+b_{n-2}},\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}},\frac{2a_nb_n}{a_n+b_n}\right) \qquad (15)\\ &\le\frac{a_{n-2}b_{n-2}}{b_{n-2}-a_{n-2}}\int_{a_{n-2}}^{b_{n-2}}\frac{f\left(x_1,\dots,\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}},\frac{2a_nb_n}{a_n+b_n}\right)}{x_{n-2}^2}dx_{n-2}.\end{aligned}$$

Again by the inequality (8), we have

$$\begin{aligned}&\frac{1}{2^2}\frac{a_{n-2}b_{n-2}}{b_{n-2}-a_{n-2}}\int_{a_{n-2}}^{b_{n-2}}\frac{1}{x_{n-2}^2}\left[f(x_1,\dots,a_{n-1},a_n)+f(x_1,\dots,b_{n-1},a_n)\right. \qquad (16)\\ &\quad\left.+f(x_1,\dots,a_{n-1},b_n)+f(x_1,\dots,b_{n-1},b_n)\right]dx_{n-2}.\\ &\le\frac{1}{2^3}\left[f(x_1,\dots,a_{n-2},a_{n-1},a_n)+f(x_1,\dots,b_{n-2},a_{n-1},a_n)\right.\\ &\quad+f(x_1,\dots,a_{n-2},b_{n-1},a_n)+f(x_1,\dots,b_{n-2},b_{n-1},a_n)\\ &\quad+f(x_1,\dots,a_{n-2},a_{n-1},b_n)+f(x_1,\dots,b_{n-2},a_{n-1},b_n)\\ &\quad\left.+f(x_1,\dots,a_{n-2},b_{n-1},b_n)+f(x_1,\dots,b_{n-2},b_{n-1},b_n)\right].\end{aligned}$$

Using the inequalities (14)-(16)

$$\begin{aligned}&\frac{a_{n-2}b_{n-2}}{b_{n-2}-a_{n-2}}\int_{a_{n-2}}^{b_{n-2}}\frac{f\left(x_1,\dots,\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}},\frac{2a_nb_n}{a_n+b_n}\right)}{x_{n-2}^2}dx_{n-2} \qquad (17)\\ &\le\left(\prod_{i=n-2}^{n}\frac{a_ib_i}{b_i-a_i}\right)\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\frac{f^n_{x_n}(x_n)}{(x_{n-2}x_{n-1}x_n)^2}dx_{n-2}dx_{n-1}dx_n\\ &\le\frac{1}{2^3}\left[f(x_1,\dots,a_{n-2},a_{n-1},a_n)+f(x_1,\dots,b_{n-2},a_{n-1},a_n)\right.\\ &\quad+f(x_1,\dots,a_{n-2},b_{n-1},a_n)+f(x_1,\dots,b_{n-2},b_{n-1},a_n)\\ &\quad+f(x_1,\dots,a_{n-2},a_{n-1},b_n)+f(x_1,\dots,b_{n-2},a_{n-1},b_n)\\ &\quad\left.+f(x_1,\dots,a_{n-2},b_{n-1},b_n)+f(x_1,\dots,b_{n-2},b_{n-1},b_n)\right].\end{aligned}$$

Doing this procedure successively we obtain the desired inequalities.

Theorem 7. *Let $f:\Delta^n\subset\mathbb{R}^n_+\to\mathbb{R}$ is harmonically convex function on n-coordinates on Δ^n. Then the following inequalities holds:*

$$\begin{aligned}&\sum_{i=1}^{n}\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\left[f^i_{a_n}(x_i)+f^i_{b_n}(x_i)\right]dx_i \qquad (18)\\ &\le\frac{n}{2}\left[f(\boldsymbol{a})+f(\boldsymbol{b})\right]+\frac{1}{2}\sum_{i=1}^{n}\left[f^i_{a_n}(b_i)+f^i_{b_n}(a_i)\right].\end{aligned}$$

Proof. Since $f:\Delta^n\subset\mathbb{R}^n_+\to\mathbb{R}$ is harmonically convex function on n-coordinates, then $f^i_{x_n}:[a_i,b_i]\to\mathbb{R}$ is harmonically convex function on $[a_i,b_i]$, for each $i=1,\dots,n$. From the right inequality of Hermite–Hadamard type (2), for each $i=1,\dots,n$, we get

$$\begin{aligned}\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}f^i_{a_n}(x_i)dx_i&\le\frac{f^i_{a_n}(a_i)+f^i_{a_n}(b_i)}{2} \qquad (19)\\ &=\frac{f(\mathbf{a})+f^i_{a_n}(b_i)}{2}, \qquad (20)\end{aligned}$$

and

$$\begin{aligned}\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}f^i_{b_n}(x_i)dx_i&\le\frac{f^i_{b_n}(a_i)+f^i_{b_n}(b_i)}{2} \qquad (21)\\ &=\frac{f^i_{b_n}(a_i)+f(\mathbf{b})}{2}. \qquad (22)\end{aligned}$$

Adding the inequalities (19) and (21), we obtain

$$\begin{aligned}&\frac{a_ib_i}{b_i-a_i}\int_{a_i}^{b_i}\left[f^i_{a_n}(x_i)+f^i_{b_n}(x_i)\right]dx_i \qquad (23)\\ &\le\frac{1}{2}\left[f(\mathbf{a})+f(\mathbf{b})\right]+\frac{1}{2}\left[f^i_{a_n}(b_i)+f^i_{b_n}(a_i)\right]\end{aligned}$$

where $i=1,\dots,n$. Taking sum from 1 to n, we have (18).

A particular case of the inequalities (5), (9) and (18) is indicated in the following result,

Corollary 1. *Let $\Delta^2 := [a,b]\times[c,d]$ and $f:\Delta^2\to\mathbb{R}$ be a harmonically convex function on* 2*–coordinates. Then* (3) *is valid.*

Proof. Putting $n=2$ in the theorems 5, 6 and 7, and taking $a_1=a$, $b_1=b$, $a_2=c$, and $b_2=d$, we obtain the required result.

3 Conclusions

The principal contribution of this paper has been the introduction of a new class of function of generalized convexity on coordinates, we present some examples and properties. We have shown that these class contain some previously known classes as special cases as well as Hermite–Hadamard's inequalities type for these functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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