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Equivalence between mirror-field-atom and ion-laser interactions

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Abstract: We show that the interaction between a movable mirror with a quantized field that interacts with a two-level atom may be simplified via a transformation that involves Susskind-Glogower operators (SGO). By using this transformation it is easy to show that we can cast the Hamiltonian, after a unitary transformation, into a Hamiltonian that is equivalent to the ion-laser Hamiltonian. We would like to stress that the transformation in terms of SGO already simplifies enough the Hamiltonian in the sense that, in an exact way, it "eliminates" one of the three-subsystems, namely the quantized field.

Keywords: Optomechanical interactions, small rotations

1. Introduction

Recently, special attention has been devoted to a system consisting of a cavity field and a movable mirror [1,2]. This is due to the fact that for such a system we can produce non-classical states [3], particularly the macroscopic superposition of at least two coherent states, i.e. Schrödinger cat-states. The concept of superposition of states plays a fundamental role in understanding the foundations of quantum mechanics, this is why the generation of non-classical states, such as squeezed states [4], and the particularly important limit of extreme squeezing, i.e. Fock or number states [5], has been widely studied in several systems. It is known that a non-linear interaction can generate Schrödinger cat-states. The non-linear interaction used to generate such states is the one produced by a Kerr medium [6,7] that corresponds to a quadratic Hamiltonian in the number field operator [8]. Our main motivation to make the field-mirror system interact with an atom is to look for the possibility to extract information about the mirror state by later measuring atomic properties, as it is well known that several quasiprobability reconstruction techniques [9,10] for the quantized field [11] or the vibrational motion of an ion [12,13], rely on the measurement of atomic properties. It would be possible also to reconstruct the autocorrelation function for the mirror state in this form [14]. Therefore, the passage of atoms through such systems, could give us information, not only about the states of the mirror or field, but also about their interaction. This is, the passage of a two-level atom through a cavity with a movable mirror may give us information about the entanglement between mirror and field. The purpose of this contribution is not to study this possibility, however, but to show how the total system may be simplified, by several rotations that can produce a well known Hamiltonian, namely the ion-laser Hamiltonian, in such a way that knowledge form the techniques used in this interaction may be borrowed to produce solutions in the total atom-field-mirror system.

2. Interaction between the cavity and the mirror

The interaction between an electromagnetic field and a movable mirror (treated quantum mechanically) has a relevant Hamiltonian given by [8] (we set $\hbar = 1$)

$$H_{fm} = \omega a^\dagger a + \nu b^\dagger b - g a^\dagger a (b^\dagger + b), \qquad (1)$$

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where a and $a^\dagger$ are the annihilation and creation operators for the cavity field, respectively. The field frequency is ω. b and $b^\dagger$ are the annihilation and creation operators for the mirror oscillating at a frequency ν and

$$g = \frac{\omega}{L}\sqrt{\frac{\hbar}{2m\nu}}, \tag{2}$$

with L and m the length of the cavity and the mass of the movable mirror.

3. Mirror-Field-Atom interaction

If we pass a two-level atom through a cavity with a movable mirror as the one described by equation (1), we have have to add the free Hamiltonian for the atom and the interaction with the quantized field, so we obtain [15]

$$H_{afm} = \frac{\omega_0}{2}\sigma_z + \lambda\left(a\sigma_+ + a^\dagger\sigma_-\right) + \omega a^\dagger a + \nu b^\dagger b \\ - g a^\dagger a(b^\dagger + b), \tag{3}$$

where λ is the atom-field interaction constant, ω_0 is the atomic transition frequency and σ_- (σ_+) is the lowering (raising) operator for the atom, with $[\sigma_+, \sigma_-] = 2\sigma_z$. We will pass from this notation to matrix notation at convenience. In matrix form the Pauli matrices read

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

We consider the on-resonant interaction between the field an the atom, i.e. $\omega = \omega_0$, and pass to an interaction picture, taking advantage that the operator $\omega(a^\dagger a + 2\sigma_z/2)$ commutes with all the other operators involved in the Hamiltonian, to obtain

$$H = \nu\hat{N} + \chi\hat{n}\left(b + b^\dagger\right) + \lambda\left(a\sigma_+ + a^\dagger\sigma_-\right). \tag{5}$$

The quantities $\hat{n} = a^\dagger a$ and $\hat{N} = b^\dagger b$ are the number operators for the field and mirror, respectively. We will use the Susskind-Glogower operators [16]

$$V = \frac{1}{\sqrt{\hat{n}+1}}a, \qquad V^\dagger = a^\dagger\frac{1}{\sqrt{\hat{n}+1}}, \tag{6}$$

that satisfy the commutation relation $[V, V^\dagger] = |0\rangle\langle 0|$ to transform the above Hamiltonian with the following matrix operator [17]

$$M = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}, \qquad M^\dagger = \begin{pmatrix} V^\dagger & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

such that we can rewrite the interaction Hamiltonian as

$$H = M H_V M^\dagger \tag{8}$$

where

$$H_V = \begin{pmatrix} \nu\hat{N} + \chi(\hat{n}-1)\left(b+b^\dagger\right) & \lambda\sqrt{\hat{n}} \\ \lambda\sqrt{\hat{n}} & \nu\hat{N} + \chi\hat{n}\left(b+b^\dagger\right) \end{pmatrix}. \tag{9}$$

Although $V^\dagger V = 1 - |0\rangle\langle 0|$, that makes $M^\dagger M \neq 1$, i.e. M a nonunitary matrix, it is not difficult to show that

$$H^k = M H_V^k M^\dagger, \tag{10}$$

allowing us to write the evolution operator as

$$U(t) = e^{-iHt} = M e^{-iH_V t} M^\dagger. \tag{11}$$

We then have achieved the following: it has been eliminated from Hamiltonian (9) the noncommuting field operators, such that this Hamiltonian may be viewed as a two-subsystems Hamiltonian, instead of the three-subsystems one with which we started, i.e. Hamiltonian (4). Therefore realizing a relevant simplification as the field operators (just number operators) may be treated from now on as classical numbers. Therefore, we have effectively and exactly eliminated one sub-system, namely the field, from the initial problem.

We now transform the Hamiltonian H_V with

$$\mathscr{D} = \begin{pmatrix} D_b\left[\frac{\chi}{\nu}(\hat{n}-1)\right] & 0 \\ 0 & D_b\left(\frac{\chi}{\nu}\hat{n}\right) \end{pmatrix}, \tag{12}$$

with

$$D_b(\varepsilon\hat{n}) = e^{\varepsilon\hat{n}(b^\dagger - b)}, \tag{13}$$

such that $\tilde{H} = \mathscr{D}^\dagger H_V \mathscr{D}$ is written as

$$\tilde{H} = \begin{pmatrix} \nu\hat{N} - \frac{\chi^2(\hat{n}-1)^2}{\nu} & \lambda\sqrt{\hat{n}}D_b(\chi/\nu) \\ \lambda\sqrt{\hat{n}}D_b^\dagger(\chi/\nu) & \nu\hat{N} - \frac{\chi^2\hat{n}^2}{\nu} \end{pmatrix}, \tag{14}$$

that may be further written as

$$\tilde{H} = \begin{pmatrix} \nu\hat{N} + \frac{\chi^2(\hat{n}-1/2)}{\nu} & \lambda\sqrt{\hat{n}}D_b(\chi/\nu) \\ \lambda\sqrt{\hat{n}}D_b^\dagger(\chi/\nu) & \nu\hat{N} - \frac{\chi^2(\hat{n}-1/2)}{\nu} \end{pmatrix} \\ + F(\hat{n})1_{2\times 2}, \tag{15}$$

with $F(\hat{n}) = \frac{\chi^2}{2\nu}(2\hat{n}^2 - 2\hat{n} + 1)$ and $1_{2\times 2}$ the 2×2 unity matrix.

The above Hamiltonian is equivalent to ion-laser interaction Hamiltonian

$$H_{\text{ion}} = \begin{pmatrix} \nu_t\hat{n} + \frac{\delta}{2} & \Omega\hat{D}(i\eta) \\ \Omega\hat{D}^\dagger(i\eta) & \nu_t\hat{n} - \frac{\delta}{2} \end{pmatrix} \tag{16}$$

except for the term $F(\hat{n})1_{2\times 2}$, that represents an overall phase. In the above Hamiltonian, ν_t is the (ion) trap frequency, δ is the detuning between the laser field and the ion transition frequencies and Ω is the Rabi frequency. The number operator $\hat{n}$ represent the harmonic oscillator Hamiltonian (the ion free oscillation) of the ion, and D is the displacement operator in the vibrational variables. Both Hamiltonians are equivalent simply by identifying

$\nu \to \nu_t$,
$\hat{N} \to \hat{n}$,
$\frac{2\chi^2(\hat{n}-1/2)}{\nu} \to \delta$
$\lambda\sqrt{\hat{n}} \to \Omega$,

and

$D_b(\chi/\nu) \to D(i\eta)$.

Hamiltonian (15) being similar to (5), means that both, mirror-atom-field and ion-laser interactions are equivalent, and therefore methods of solution and generations of non-classical states from one interaction may be borrowed by the other interaction.

4. Exact eigenstates

Let us return now to the Hamiltonian in equation (15). We can construct an *ansatz* that allows the determination of exact eigenstates of this system, provided certain relations are satisfied between the parameters ν, χ and λ . In order to motivate our general solution, let us consider first the possibility of finding a state of the form (we consider simply the mirror and atom states as the field state simply multiplies the eigenstate)

$$|\psi\rangle = |e\rangle\left(c_0|0\rangle + c_1|1\rangle\right) + |g\rangle|\phi\rangle \equiv \begin{pmatrix} c_0|0\rangle + c_1|1\rangle \\ |\phi\rangle \end{pmatrix}, \tag{17}$$

for each eigenstate of the mirror operator, $\hat{N}|n\rangle = n|n\rangle$. We have used in the above equation a notation where the atomic elements are written out explicitly (e.g., $|e\rangle = \binom{1}{0}$). Let us now see whether the eigenvalue equation

$$\tilde{H}|\psi\rangle = E|\psi\rangle, \tag{18}$$

can be satisfied. Equation (17) shows that it is required $|\phi\rangle$ be of the form

$$|\phi\rangle = D_b^\dagger(\chi)\left(d_0|0\rangle + d_1|1\rangle\right) = d_0|-\chi\rangle + d_1|-\chi;1\rangle, \tag{19}$$

where $|-\chi\rangle$ is a coherent state and $|\beta;k\rangle \equiv \hat{D}(\beta)|k\rangle$ is a displaced number state [18]. We thus require

$$\tilde{H}|\psi\rangle = \begin{pmatrix} (\beta_1|0\rangle + \beta_2|1\rangle) \\ (\beta_3|-\chi\rangle + \beta_4|-\chi;1\rangle) \end{pmatrix}. \tag{20}$$

with

$$\left(\beta_1 = c_0\frac{\chi^2(\hat{n}-1/2)}{\nu} + \lambda\sqrt{\hat{n}}d_0\right)$$
$$\beta_2 = \lambda\sqrt{\hat{n}}d_1 + c_1\left(\nu + \frac{\chi^2(\hat{n}-1/2)}{\nu}\right)$$
$$\beta_3 = c_0\lambda\sqrt{\hat{n}} + d_0\left(\nu\hat{n} - \frac{\chi^2(\hat{n}-1/2)}{\nu}\right)$$

and

$$\beta_4 = c_1\lambda\sqrt{\hat{n}} + +d_1\left(\nu\hat{n} - \frac{\chi^2(\hat{n}-1/2)}{\nu}\right).$$

Now, by using the well-known fact that $D_b^\dagger(\beta)bD_b(\beta) = a + \beta$ [19], it is easy to show that displaced number states satisfy the recursion relation

$$\hat{N}|\beta;k\rangle = (|\beta|^2 + k)|\beta;k\rangle + \beta\sqrt{k+1}|\beta;k+1\rangle + \beta^*\sqrt{k}|\beta;k-1\rangle. \tag{21}$$

Substituting then equations (17) and (21) into equation (18) gives the following eigenstate conditions:

$$d_1 = 0;\; c_0 = \frac{\lambda\sqrt{\hat{n}}}{\nu}\;;\; c_1 = \frac{\chi\nu}{\lambda\sqrt{\hat{n}}}$$
$$E = \nu + \frac{\chi^2(\hat{n}-1/2)}{\nu}, \tag{22}$$

which hold however *only if* the parameters λ, ν, χ satisfy the additional constraint

$$\left(\frac{\lambda\sqrt{\hat{n}}}{\nu}\right)^2 + \chi^2/\nu^2 = 1 + \frac{\chi^2(\hat{n}-1/2)}{\nu^2}. \tag{23}$$

Under these conditions the state

$$|\psi_{ion}^+\rangle = |e\rangle\left(\frac{\lambda\sqrt{\hat{n}}}{\nu}|0\rangle + \frac{\chi\nu}{\lambda\sqrt{\hat{n}}}|1\rangle\right) + |g\rangle|-\chi\rangle. \tag{24}$$

is an (unnormalised) eigenstate of $\tilde{H}$ with eigenvalue $\nu + \frac{\chi^2(\hat{n}-1/2)}{\nu}$. Condition (23) means that the *ansatz* in eq. (17) does not always succeed, as only two of the three parameters $\lambda\sqrt{\hat{n}}, \nu, \chi$ can be chosen independently. Nevertheless, the existence of solutions satisfying equation (17) leads us naturally to seek for other solutions using similar or slightly generalised *eigenstates*.

4.1. More general eigenstates

One can easily generalize Eq. (24) to obtain a more general eigenstate for $\tilde{H}$. It can be written as

$$|\psi\rangle = \frac{\lambda\sqrt{\hat{n}}}{\nu}\sum_{n=0}^{m+1} c_n|n\rangle|e\rangle + \sum_{n=0}^{m} d_n|-\chi,n\rangle|g\rangle,$$

where

$$c_n = \begin{cases} \frac{1}{m+1-n}d_n; 0 \le n \le m \\ \chi\frac{\nu^2}{\lambda^2\hat{n}}\sqrt{n+1}d_n; n = m+1 \end{cases}$$

and the d_n coefficients satisfy

$$\begin{bmatrix} \varepsilon_m & -\chi\nu & & & \\ \chi\nu & \varepsilon_{m-1} & -\chi\nu\sqrt{2} & & \\ & \chi\nu\sqrt{2} & \ddots & \ddots & \\ & & \ddots & \varepsilon_2 & -\chi\nu\sqrt{m-1} \\ & & & \chi\nu\sqrt{m-1} & \varepsilon_1 \end{bmatrix} \times \begin{bmatrix} d_0 \\ \vdots \\ d_m \end{bmatrix} = \mathbf{0}$$

where

$$\varepsilon_m = \nu\left(m+1-\chi/\nu^2\right) + \frac{\chi^2(\hat{n}-1/2)}{\nu} - \frac{\lambda^2\hat{n}}{(m+1)\nu}.$$

The corresponding eigenvalue is $(m+1)\nu + \frac{\chi^2(\hat{n}-1/2)}{\nu}$.

Note that the vector of coefficients $(d_0,\ldots,d_m)$ is an eigenvector of this tridiagonal matrix with zero eigenvalue. This is only possible if $\det M = 0$, which imposes a constraint on $\lambda\sqrt{\hat{n}},\chi,\nu$. This is the generalisation of equation (23).

5. Conclusions

We have shown that the Hamiltonian of a quantized field interacting simultaneously with a two-level atom and a movable mirror is equivalent to the ion-laser Hamiltonian. In order to produce this equivalency we have used a set of transformations, the main one, being a transformation that involves Susskind-Glogower operators, equation (7). This transformation, besides the fact that does not involve approximations, allows us to simplify the problem by “eliminating” the field operators to leave an effective interaction between atom and mirror. The Hamiltonian then may be further unitarily transformed to obtain an ion-laser-like interaction. It is important to note that these kind of systems now may be modelled in optical lattices [20,21] and quasiprobability distribution functions may be reconstructed in optical physics [22].

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