Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.12785/jsap/020207

A Test of Homogeneity Against Umbrella Scale Alternative Based on Gini's Mean Difference

Anil Gaur

Department of Statistics, Panjab University, Chandigarh, India
Email Address: anilgaur8@gmail.com

Received: 4 Sep. 2012 , Revised: 16 Oct. 2012, Accepted: 30 Oct. 2012
Published online: 1 Jul. 2013

Abstract: In this paper, a test procedure based on Gini's mean difference for testing homogeneity of scale parameters against umbrella ordered alternative with at least one strict inequality, when peak of the umbrella is known, is proposed. The exact critical points through simulation are computed for the proposed test in case of standard exponential, standard logistic and standard uniform distributions; however the proposed test can be applied even in case of other distributions like Laplace, Pareto, Weibull, etc. Construction of Simultaneous one-sided confidence intervals (SOCIs) is proposed along with an illustration. Power of the proposed test is computed and some power comparisons are also made.

Keywords: Gini's mean difference; Umbrella ordering; Critical point computation; Simultaneous one-sided confidence intervals; Statistical simulation.

1. Introduction

Let $\pi_1, \pi_2, ...,\pi_k$ be k ($k \geq 3$) independent populations such that an observation from population π_i follows a distribution with cumulative distribution function (*cdf*) $F_i(x) = F[(x-\mu_i)/\theta_i]$. Here μ_i $(-\infty < \mu_i < \infty)$ is the location parameter, θ_i $(\theta_i > 0)$ is the scale parameter and $F(.)$ is any absolutely continuous *cdf i.e.*, $F(.)$ is a member of location-scale family, $i = 1, 2, ..., k$. The problem of testing the null hypothesis of homogeneity of scale parameters $H_0 : \theta_1 = ... = \theta_k$ against simple ordered alternative hypothesis $H_A : \theta_1 \leq ...... \leq \theta_k$ with at least one strict inequality has received considerable attention in the literature. For some earlier work on this problem, one may refer to Rao [11], Shanubhogue [12], Kusum and Bagai [8], Gill and Dhawan [6], Singh and Gill [14] among others. Except Gill and Dhawan [6] and Singh and Gill [14] tests, all the above tests are based on *U*-statistics. Gill and Dhawan [6] proposed the test procedure based on isotonic estimator of exponential scale parameter. Singh and Gill [14] proposed the test based on sample quasi ranges for testing homogeneity of the scale parameters against the simple ordered alternative with at least one strict inequality. These procedures have important applications for problems where the treatments can be assumed to satisfy a simple ordering, such as for a sequence of increasing dose-level of a drug.

To find the optimal dosage of a new drug, subjects can be randomly assigned to several promising dosage levels suggested by past experience or medical history. It is conceivable that in many situations the dose-response will increase with an increase in the dosage level up to a point, then decrease with further increase in the dosage level due to toxicity and side effects. In the literature this up-then-down response pattern has been called as an umbrella ordering (Mack and Wolfe [9]). The dosage level corresponding to the turning point of the ordering is called the peak of the umbrella. Such umbrella orderings can be observed with many physical and biological phenomena in a wide variety of scientific research areas. Umbrella ordering is important in dose-response experiment (e.g., see Simpson and Margolin [13]). In case where mode of action of a drug is related to its toxic effects, e.g., in case of life saving therapy of heart failure, life-saving digitalization therapy of heart failure, umbrella behavior is anticipated and careful dosage planning is required.

For testing the null hypothesis of homogeneity of scale parameters $H_0 : \theta_1 = ... = \theta_k$ against the umbrella ordered alternative $H_A : \theta_1 \le .... \le \theta_h \ge .... \ge \theta_k$ with at least one strict inequality where h is the known peak of umbrella, Singh and Liu [15] proposed a test procedure based on isotonic estimator of the scale parameter. Carpenter and Singh [2] gave a test based on sample quasi ranges to test homogeneity of scale parameters against umbrella alternative. Singh and Liu [15] and Carpenter and Singh [2] also provided one-sided simultaneous confidence intervals for all the ordered pairwise scale ratios, and critical points for the two parameter exponential probability distribution when $n_1 = n_2 = ... = n_k$ is assumed. Recently, Gaur et al. [5] and Gaur [4] provided three test statistics based on linear combination of two-sample U-statistics for testing homogeneity of scale parameters against umbrella alternative, with at least one strict inequality, when the peak of the umbrella, h is known.

David and Nagaraja [3] and Budescu [1] mentioned that Gini’s mean difference can be considered as a good measure of dispersion. With this motivation, in this article, a multi-sample test for testing homogeneity of scale parameters against umbrella ordered alternative based on Gini’s mean difference is proposed. The critical points for two-parameter exponential distribution, logistic distribution and uniform distribution have been computed. Using certain transformations, these critical points can be applied even in case of other distributions like Laplace, Pareto, Weibull and Gamma etc.

This article is organized as follows. In section 2, a nonparametric test based on Gini’s mean difference for testing the scale parameters against the umbrella alternative with at least one strict inequality is proposed. The details for calculation of the critical points for the test procedure in case of standard exponential, standard logistic and standard uniform distribution are provided through simulation in section 3. Construction of simultaneous one-sided confidence intervals (SOCIs) along with an illustration is shown in section 4, whereas power of the proposed test along with some power comparison is given in section 5.

2. Proposed Test Procedure

Let $X_{i1}, X_{i2}, ..., X_{in}$ be a random sample of size n from the population π_i and $X_{i[1]} \le X_{i[2]} \le ...... \le X_{i[n]}$ be the corresponding order statistics, $i = 1, 2, ..., k$. Gini’s mean difference as a measure of dispersion based on a random sample of size n from population π_i, denoted by G_i, is defined as

$$G_i = \frac{1}{n(n-1)} \sum_{u=1}^{n} \sum_{v=1}^{n} \left| X_{iu} - X_{iv} \right|, \; i = 1, 2, ..., k. \tag{1}$$

We propose a test procedure for testing the null hypothesis of homogeneity of scale parameters

$$H_0 : \theta_1 = \theta_2 = ... = \theta_k$$

against the umbrella ordered alternative

$$H_A : \theta_1 \leq .... \leq \theta_h \geq .... \geq \theta_k \,,$$

with at least one strict inequality, where h is the peak of umbrella. For a given h, $(1 \leq h \leq k)$, We define the test statistic as

$$T_h = max\left(\max_{1 \leq i < j \leq h} (G_j / G_i), \max_{h \leq j < i \leq k} (G_j / G_i) \right). \tag{2}$$

Test is reject H_0 at level α *iff*

$$T_h \geq g_{k,h,\alpha,n} \,, \tag{3}$$

where $g_{k,h,\alpha,n}$ are the critical points for k samples each of size n defined by:

$$P_0[T_h \geq g_{k,h,\alpha,n}] = \alpha \tag{4}$$

where $P_0(A)$ indicates that the probability of event A is computed under the null hypothesis $H_0 : \theta_1 = \theta_2 = ... = \theta_k$ at level of significance α .

Intuitively, one may feel that if H_0 is true, all the scale parameters are equal so each of these ratios in the test statistic must be one and in case the scale parameters are umbrella ordered (as under alternative) these ratios should be greater than one and maximum of such ratios should lead to rejection of null hypothesis. The procedure for obtaining the critical points for the above test is given in the following section.

3. Calculation of Critical Points for Some Specific Distributions: Simulation Method

The exact distribution of ratio of two G_i'*s* (as defined in eq. (1)) is quite involved and even the exact distribution of G_i in a single dimension is not available in precise form for different distributions, although a few results can be found in David and Nagaraja [3]. The exact critical points for the proposed test involving ratio of two G_i'*s* (see eq (2) and (3)) can be evaluated either by Hayter's technique (Hayter [7]) or by statistical simulation for any configuration of n_i'*s* (sample sizes). However, to apply Hayter's technique one needs to have the exact distribution of ratio of G_i'*s* which is not readily available in precise form for different distributions.

Therefore, we have adopted the other technique, *i.e.*, the technique of statistical simulation. The exact simulated critical points of the proposed test for the above said problem are obtained through statistical simulation taking $2x10^5$ repetitions for different configuration of n_i'*s* (sample sizes). These critical points along with the proposed test statistic give the exact test for the above said multi-sample problem.

The method for calculation of simulated critical points is explained for k $(k \geq 3)$ two-parameter exponential, logistic and uniform distribution. We have taken standard exponential, standard logistic and standard uniform distribution as the basis respectively and have computed the critical points. One may note that these critical points can also be used for other distributions as well like Laplace, Pareto, Weibull and Gamma using certain transformations.

3.1 Critical Points for Standard Exponential, Logistic and Uniform Distributions

Let there be k ($k \geq 3$) independent populations. Using the method of simulation we generated $2\text{x}10^5$ values of test statistic $T_h = max\left(\max_{1\leq i<j\leq h} (G_j / G_i), \max_{h\leq j<i\leq k} (G_j / G_i) \right)$ by taking samples of equal size n = 6(1)25(5)40 from k = 3(1)10 standard exponential, standard logistic and standard uniform distributions, assuming h = 2 & 3. The critical points were then calculated using simulation technique for α = 0.05, where α is level of significance.

Table 1 Simulated critical points for Exponential distribution when h = 2 and α = 0.05

n/k	3	4	5	6	7	8	9	10
6	4.0727	5.0235	5.7540	6.4274	7.0894	7.5723	8.0475	8.5625
7	3.5992	4.3261	4.9496	5.4701	5.9015	6.2547	6.6849	6.9776
8	3.3039	3.8604	4.3459	4.8476	5.1327	5.4798	5.8265	6.0728
9	3.0725	3.5324	3.9455	4.3324	4.6712	4.8583	5.1931	5.3630
10	2.8235	3.3197	3.6840	3.9851	4.2347	4.4687	4.6888	4.9054
11	2.7305	3.0793	3.4346	3.6944	3.9416	4.1544	4.2995	4.4738
12	2.6073	2.9223	3.2480	3.5061	3.6982	3.8864	4.0115	4.2260
13	2.4431	2.8073	3.0988	3.3213	3.4694	3.6432	3.7977	3.9326
14	2.3777	2.7202	2.9687	3.1827	3.3234	3.4951	3.6188	3.7195
15	2.3117	2.5898	2.8142	3.0128	3.1981	3.2916	3.4728	3.5679
16	2.2607	2.5306	2.7401	2.9065	3.0669	3.1972	3.3599	3.4273
17	2.1906	2.4225	2.6572	2.7889	2.9698	3.0962	3.1763	3.3121
18	2.1403	2.3880	2.5706	2.7139	2.8594	2.9594	3.0783	3.1761
19	2.1151	2.3237	2.4847	2.6659	2.7908	2.8894	2.9719	3.0629
20	2.0521	2.2618	2.4432	2.5765	2.7126	2.8010	2.916	2.9821
21	2.0012	2.2127	2.3904	2.5350	2.6355	2.7432	2.8559	2.8888
22	1.9812	2.1811	2.3473	2.4629	2.5711	2.6624	2.7569	2.8141
23	1.9636	2.1421	2.2943	2.4134	2.5288	2.6182	2.6936	2.7513
24	1.9210	2.1161	2.2590	2.3623	2.4580	2.5561	2.6232	2.6942
25	1.8987	2.0624	2.2234	2.3283	2.4136	2.5073	2.6001	2.6523
30	1.7907	1.9671	2.0738	2.1560	2.2412	2.3025	2.3679	2.4195
35	1.7139	1.8419	1.9554	2.0259	2.1006	2.1593	2.2142	2.2562
40	1.6437	1.7842	1.8587	1.9312	2.0020	2.0591	2.1118	2.1332

Table 2 Simulated critical points for Exponential distribution when h = 3 and α = 0.05

n/k	3	4	5	6	7	8	9	10
6	4.6136	4.9404	5.5942	6.2370	6.8014	7.3597	7.8231	8.2409
7	3.9669	4.2918	4.8225	5.2494	5.6776	6.0326	6.4556	6.8318
8	3.5742	3.8202	4.2592	4.6337	5.0573	5.3284	5.6450	5.8753
9	3.3357	3.5602	3.8804	4.2041	4.4548	4.7696	5.0772	5.3319
10	3.1457	3.2731	3.5651	3.8616	4.1656	4.3491	4.5818	4.8116
11	2.9250	3.0952	3.3394	3.6204	3.8324	4.0650	4.2567	4.3802
12	2.7625	2.9390	3.1634	3.3572	3.6458	3.7808	3.9476	4.1302
13	2.6853	2.8187	3.0191	3.2518	3.4179	3.5962	3.6811	3.9241
14	2.5896	2.6786	2.8781	3.0884	3.2390	3.3883	3.5126	3.7188
15	2.4619	2.6031	2.7782	2.9554	3.0889	3.2671	3.3651	3.4949
16	2.3836	2.5149	2.6949	2.8277	3.0097	3.1574	3.2625	3.3706
17	2.3326	2.4365	2.5914	2.7516	2.8774	3.0178	3.1344	3.2175
18	2.2845	2.3763	2.5355	2.6584	2.7873	2.9193	2.9988	3.1224
19	2.2243	2.3179	2.4385	2.6051	2.7178	2.8256	2.9433	3.0201
20	2.1684	2.2442	2.4052	2.5163	2.6388	2.7538	2.8455	2.9324
21	2.1349	2.2148	2.3440	2.4496	2.5861	2.6994	2.7687	2.8594

22	2.1027	2.1838	2.2895	2.4228	2.5381	2.5944	2.7046	2.7981
23	2.0491	2.1544	2.2646	2.3805	2.4757	2.5655	2.6384	2.7208
24	2.0153	2.0909	2.2363	2.3292	2.4181	2.5097	2.5945	2.6499
25	1.9932	2.0762	2.1751	2.2816	2.3816	2.4607	2.5308	2.588
30	1.8750	1.9434	2.0382	2.1173	2.1979	2.2829	2.3431	2.3932
35	1.8015	1.8384	1.9261	2.0031	2.0695	2.1369	2.1894	2.2339
40	1.7202	1.7748	1.8437	1.9029	1.9684	2.0204	2.0799	2.1175

Table 3 Simulated critical points for Logistic distribution when h = 2 and α = 0.05

n/k	3	4	5	6	7	8	9	10
6	2.9369	3.3902	3.8793	4.174	4.4872	4.7848	5.0777	5.3387
7	2.6310	3.0615	3.4033	3.6240	3.8815	4.0969	4.3186	4.4647
8	2.4666	2.7748	3.0231	3.2631	3.4733	3.6295	3.8254	3.9628
9	2.2825	2.5944	2.8446	3.0254	3.1956	3.3316	3.4735	3.6090
10	2.1655	2.4378	2.6488	2.8266	2.9707	3.0957	3.2171	3.3416
11	2.0791	2.3136	2.5015	2.6720	2.7899	2.9027	3.0012	3.1174
12	2.0231	2.2502	2.4101	2.5589	2.6652	2.7943	2.8633	2.9553
13	1.9429	2.1501	2.3043	2.4428	2.5588	2.6473	2.7246	2.7921
14	1.9155	2.0821	2.2347	2.3459	2.4501	2.5396	2.6057	2.6837
15	1.8706	2.0312	2.1601	2.2761	2.3614	2.4288	2.5031	2.5902
16	1.8284	1.9849	2.1053	2.2098	2.2944	2.3757	2.4411	2.4865
17	1.7819	1.9535	2.0565	2.1636	2.2267	2.3166	2.3656	2.4081
18	1.7631	1.9057	2.0131	2.1025	2.1847	2.2354	2.3087	2.3539
19	1.7209	1.8652	1.9747	2.0691	2.1382	2.1901	2.2556	2.2949
20	1.6978	1.8376	1.9331	2.0224	2.0845	2.1404	2.2081	2.2483
21	1.6862	1.8013	1.9039	1.9771	2.0387	2.1029	2.1622	2.2013
22	1.6545	1.7801	1.8761	1.9535	2.0059	2.0681	2.1236	2.1555
23	1.6346	1.7514	1.8505	1.9086	1.9721	2.0314	2.0723	2.1065
24	1.6284	1.7328	1.8213	1.8861	1.9502	2.0022	2.0501	2.0764
25	1.5921	1.7011	1.8048	1.8623	1.9129	1.9754	2.0169	2.0373
30	1.5315	1.6267	1.7102	1.7632	1.8051	1.8519	1.8811	1.9147
35	1.4768	1.5599	1.6252	1.6789	1.7291	1.7650	1.7919	1.8213
40	1.4435	1.5233	1.5822	1.6260	1.6647	1.6921	1.7236	1.7489

Table 4 Simulated critical points for Logistic distribution when h = 3 and α = 0.05

n/k	3	4	5	6	7	8	9	10
6	3.1791	3.407	3.7473	4.0911	4.3843	4.6939	4.8567	5.1746
7	2.8647	3.0177	3.3271	3.5753	3.7789	3.9499	4.2016	4.3816
8	2.6077	2.7996	3.0122	3.1799	3.3752	3.5868	3.7056	3.9111
9	2.4685	2.6098	2.7782	2.9369	3.1223	3.2562	3.4128	3.5169
10	2.3151	2.4831	2.5961	2.7477	2.8933	3.0364	3.1243	3.2296
11	2.2305	2.3362	2.4452	2.6040	2.7632	2.8293	2.9552	3.0503
12	2.1236	2.2136	2.3673	2.4771	2.6137	2.6821	2.8042	2.8812
13	2.0740	2.1371	2.2733	2.3791	2.4775	2.6014	2.6617	2.7582
14	1.9847	2.0876	2.2123	2.2954	2.4076	2.4772	2.5468	2.6236
15	1.9341	2.0366	2.1332	2.2369	2.3323	2.3945	2.4928	2.5282
16	1.9161	1.9732	2.0770	2.1546	2.2734	2.3170	2.4041	2.4553
17	1.8621	1.9467	2.0277	2.1235	2.2036	2.2546	2.3376	2.3905
18	1.8415	1.9056	1.9852	2.0663	2.1428	2.2165	2.2651	2.3147
19	1.8054	1.8603	1.9558	2.0278	2.1008	2.1685	2.2191	2.2521
20	1.7575	1.8412	1.9153	1.9864	2.0455	2.1149	2.1678	2.2025
21	1.7465	1.7947	1.8866	1.9498	2.0182	2.0815	2.1242	2.1717

22	1.7297	1.7815	1.8551	1.9195	1.9765	2.0251	2.0894	2.1258
23	1.7071	1.7466	1.8227	1.8867	1.9518	2.0104	2.0522	2.0939
24	1.6725	1.7251	1.7973	1.8575	1.9118	1.9813	2.0161	2.0493
25	1.6637	1.7135	1.7810	1.8459	1.8931	1.9474	1.9757	2.0241
30	1.5889	1.6243	1.6868	1.7347	1.7842	1.8196	1.8652	1.8982
35	1.5203	1.5668	1.6192	1.6678	1.7108	1.7413	1.7728	1.8132
40	1.4869	1.5259	1.5645	1.6061	1.6427	1.6775	1.7129	1.7371

Table 5 Simulated critical points for Uniform distribution when $h = 2$ and $\alpha = 0.05$

n/k	3	4	5	6	7	8	9	10
6	2.1885	2.4767	2.741	2.9361	3.1435	3.2426	3.4152	3.5282
7	2.0068	2.2303	2.3846	2.5438	2.6912	2.8007	2.8751	3.0053
8	1.8475	2.0188	2.1571	2.2974	2.3913	2.5045	2.5483	2.6203
9	1.7411	1.8953	2.0174	2.1232	2.1955	2.2689	2.3374	2.3905
10	1.6669	1.7943	1.8945	1.9818	2.0543	2.1122	2.1596	2.2297
11	1.6039	1.7317	1.8190	1.8931	1.9513	2.0057	2.0588	2.1061
12	1.5612	1.6751	1.7515	1.8205	1.8719	1.9298	1.9612	1.9986
13	1.5245	1.6197	1.6955	1.7576	1.8072	1.8416	1.8791	1.9202
14	1.4953	1.5836	1.6412	1.7083	1.7510	1.7880	1.8243	1.8474
15	1.4665	1.5415	1.6039	1.6564	1.6966	1.7414	1.7754	1.7981
16	1.4420	1.5216	1.5794	1.6186	1.6674	1.6968	1.7290	1.7582
17	1.4298	1.4863	1.5474	1.5931	1.6353	1.6591	1.6913	1.7110
18	1.4071	1.4699	1.5226	1.5642	1.5955	1.6325	1.6566	1.6758
19	1.3748	1.4409	1.4965	1.5411	1.5722	1.6018	1.6279	1.6446
20	1.3667	1.4279	1.4755	1.5164	1.5477	1.5803	1.5979	1.6286
21	1.3527	1.4158	1.4582	1.5043	1.5262	1.5525	1.577	1.5992
22	1.3484	1.4027	1.4466	1.4817	1.5075	1.5319	1.5552	1.5745
23	1.3360	1.3923	1.4359	1.4598	1.4915	1.5196	1.5385	1.554
24	1.3225	1.3744	1.4179	1.4486	1.4747	1.5044	1.5211	1.5371
25	1.3171	1.3602	1.4058	1.4373	1.4650	1.4875	1.5009	1.5180
30	1.2792	1.3182	1.3601	1.3847	1.4066	1.4195	1.4369	1.4523
35	1.2489	1.2884	1.3234	1.3437	1.3641	1.3775	1.3938	1.4035
40	1.2308	1.2675	1.2947	1.3159	1.3309	1.3486	1.3582	1.3716

Table 6 Simulated critical points for Uniform distribution when $h = 3$ and $\alpha = 0.05$

n/k	3	4	5	6	7	8	9	10
6	2.3101	2.4531	2.7104	2.8535	3.0918	3.2490	3.3461	3.4872
7	2.0978	2.1968	2.3673	2.4792	2.6154	2.7492	2.8671	2.9261
8	1.9043	2.0285	2.1253	2.2503	2.3391	2.4436	2.5019	2.5723
9	1.8089	1.8873	1.9841	2.0899	2.1701	2.2548	2.3022	2.3701
10	1.7187	1.7899	1.8841	1.9646	2.0286	2.0858	2.1583	2.1965
11	1.6613	1.7057	1.8029	1.8686	1.9374	1.9812	2.0295	2.0813
12	1.6092	1.6633	1.7292	1.7905	1.8448	1.8881	1.9408	1.9854
13	1.5752	1.6198	1.6827	1.7267	1.7834	1.8232	1.8684	1.9019
14	1.5281	1.5788	1.6391	1.6867	1.7229	1.7655	1.8109	1.8329
15	1.5052	1.5482	1.5896	1.6425	1.6838	1.724	1.7669	1.7825
16	1.4709	1.5109	1.5639	1.6083	1.6477	1.6801	1.7117	1.7415
17	1.4545	1.4880	1.5359	1.5810	1.6180	1.6533	1.6636	1.6994
18	1.4366	1.4706	1.5109	1.5602	1.5861	1.6169	1.6393	1.6644
19	1.4142	1.4569	1.4935	1.5251	1.5605	1.5841	1.6154	1.6342
20	1.4039	1.4302	1.4752	1.5075	1.5378	1.5632	1.5871	1.6109
21	1.3856	1.4118	1.4534	1.4905	1.5145	1.5492	1.5666	1.5826

22	1.3806	1.4028	1.4371	1.4722	1.4972	1.5206	1.5432	1.5627
23	1.3587	1.3921	1.4212	1.4556	1.4815	1.5040	1.5255	1.5419
24	1.3517	1.3762	1.4132	1.4389	1.4674	1.4888	1.5108	1.5254
25	1.3431	1.3620	1.3959	1.4261	1.4488	1.4683	1.4961	1.5082
30	1.3014	1.3247	1.3517	1.3748	1.3946	1.4082	1.4282	1.4476
35	1.2730	1.2883	1.3154	1.3352	1.3554	1.3722	1.3853	1.4002
40	1.2522	1.2667	1.2870	1.3119	1.3231	1.3373	1.3528	1.3662

In Table 1-2 (Table 3-4, Table 5-6), the simulated critical points are calculated for standard exponential (standard logistic, standard uniform) distribution for h = 2 & 3 and different samples, *i.e.*, k = 3(1)10.

Remark 3.1 The critical constants of the proposed test are computed for α = 0.01 and 0.10. The size of the test is also computed for the three distributions to check the validity of the critical points $g_{k,h,\alpha,n}$ and it was observed that the actual sizes were less than the nominal level α. These tables are available with author and can be obtained upon request. These tables are not given just to control the length of manuscript.

Remark 3.2 The programs for calculation of critical points are available with the author and can be obtained on request.

Remark 3.3 The critical points given in Table 1-2 under standard exponential distribution can also be used for Pareto, Weibull, Laplace and Gamma distributions using some appropriate transformations. For example, let the random variable X follows Weibull distribution with cumulative distribution function $F_i(x) = 1 - \exp[-(x/\lambda_i)^{k_i}]$, $\lambda_i > 0, k_i > 0$. Then the random variable $Y = X + \mu_i$ follows two parameter exponential distribution with location parameter μ_i and scale parameter $\theta_i = \lambda_i$ and $k_i = 1$, where i = 1, 2, …., k.

4. Simultaneous One-Sided Confidence Intervals (SOCI's) of the Proposed Test

One may also construct the SOCI*s* for the ordered pair wise ratios of scale parameters. The statistical analysis may not be further necessary if the null hypothesis of homogeneity, H_0 is not significant. However, if the null hypothesis is significant then one may wish to determine which θ_i'*s* differ and by how much. The test statistic $T_h = max\left(\max_{1 \le i < j \le h} (G_j / G_i), \max_{h \le j < i \le k} (G_j / G_i) \right)$ and using eq. (4) allows us to construct $(1-\alpha)$ level SOCIs for the ordered pairwise ratios θ_j / θ_i for $1 \le i < j \le h$, $h \le j < i \le k$, as follows (Hayter [7], Miller[10]):

$$1-\alpha = P_0(T_h \le g_{k,h,\alpha,n})$$

$$= P\left(\max_{1 \le i < j \le h} \left(\frac{G_j}{G_i} \right) \le g_{k,h,\alpha,n} \text{ and } \max_{h \le j < i \le k} \left(\frac{G_j}{G_i} \right) \le g_{k,h,\alpha,n} \right)$$

$$= P\left(\max_{1 \le i < j \le h} \left(\frac{G_j / \theta_j}{G_i / \theta_i} \right) \le g_{k,h,\alpha,n} \text{ and } \max_{h \le j < i \le k} \left(\frac{G_j / \theta_j}{G_i / \theta_i} \right) \le g_{k,h,\alpha,n} \right)$$

as under $H_0 : \theta_1 = \theta_2 = ... = \theta_k$, $\forall i = 1, 2, ..., k$ or $\frac{\theta_j}{\theta_i} = 1$ for $1 \le i < j \le h$, $h \le j < i \le k$ where $P_0(A)$ indicates that the probability of event A is computed under H_0 at level of significance α and G_i, for $i = 1, 2, ..., k$ is defined in eq. (1).

$$1-\alpha = P\left(\left(\frac{G_j/\theta_j}{G_i/\theta_i}\right) \le g_{k,h,\alpha,n}\ ,1 \le i < j \le h \text{ and } \left(\frac{G_j/\theta_j}{G_i/\theta_i}\right) \le g_{k,h,\alpha,n}\ , h \le j < i \le k\right)$$

$$= P\left(\frac{\theta_j}{\theta_i} \ge \frac{G_j}{G_i}\frac{1}{g_{k,h,\alpha,n}}\ ,\ 1 \le i < j \le h,\ h \le j < i \le k\right). \quad (5)$$

It may be noted that the validity of the above simultaneous confidence intervals given in (5) does not depend on the assumption whether heterogeneity among θ_i 's follow the umbrella ordering and are valid only if this ordering is specified independently without any examination of data. Although, if the knowledge of the likely ordering $\theta_1 \le .... \le \theta_h \ge .... \ge \theta_k$ is known in advance, then this information may be used to improve the confidence intervals given by (5). One can see that the lower end of the confidence intervals given by (5) with a value less than one, in this case, be non-informative, and may be truncated at one. Close values of the lower end points of the SOCIs given by (5) indicate closeness among the scale parameters. These SOCIs are of interest to the experimenter in agriculture, engineering, quality control etc. to see which of the treatments are heterogeneous.

4.1 Simulated Example to Compute Test Statistic and SOCIs

Computation of test statistic and construction of SOCIs for the ordered pairwise ratios of scale parameters with the help of the following data generated from four logistic distributions with h = 3:

$L(0, 1)$: 0.93296, 0.12686, -1.63394, -1.67137, -1.44878, -4.84377, 4.03685, 0.70232, -1.28142, 3.55501, -2.28274, 3.88147, 0.30309, 2.45439, -0.47616

$L(1, 3)$: -2.21116, 0.72162, -5.41170, 12.46777, 2.40378, -1.63655, 3.17123, 5.05441, 10.24663, 5.50854, 1.73313, 7.89126, 4.74046, -3.79062, 0.65063

$L(3, 6)$: -12.04252, 15.62463, 3.84707, -13.17145, -1.11966, 25.25414, -4.77628, -5.86493, -2.54881, 8.26375, -5.12989, -8.43720, 21.32898, -2.12956, 3.29223

$L(2, 4)$: 0.71926, 1.66014, 2.87359, 6.55311, -2.00146, 1.66417, -3.58431, 7.97312, 11.03168, 16.98328, -3.47318, 9.26520, 3.76935, 20.65122, -15.34923

where $L(a, b)$ denotes the logistic distribution with location (scale) parameter $a(b)$. Here, h = 3, G_1 = 2.9243, G_2 = 5.9268, G_3 = 13.1529, G_4 = 10.0314. The critical value $g_{k,h,\alpha,n} = g_{4,\,3,\,0.05,\,15}$, from Table 4, is 2.0366. Using these values, we can get T_h = 4.4977 which is significant at 5% level of significance. The set of 95% SOCIs, using (5) with lower end of the confidence interval truncated at one if it is less than one, for G_2/G_1, G_3/G_1, G_3/G_2, G_3/G_4 are computed as:

$$[[1,\infty),[2.2084,\infty),[1.0896,\infty),[1,\infty)].$$

5. Power of the Proposed Test

Statistical simulation was carried out for power computation of the test proposed in section 2. We computed power under some particular configuration of scale parameter of the exponential, logistic and uniform distributions for h = 2 & 3 and k = 3 & 5 at different levels of α and n_i*'s*. These tables are available with author and can be obtained upon request. One can see from these tables that the power of the test is quite high and increases with the increasing sample size.

For testing the homogeneity of scale parameters against the umbrella ordered alternative, Carpenter and Singh [2] proposed the test by using the test statistics based on sample quasi ranges. The test statistics considered was $W_{r,h} = max\left(\max_{1 \le i < j \le h} (Q_{jr} / Q_{ir}), \max_{h \le j < i \le k} (Q_{jr} / Q_{ir}) \right)$, where Q_{ir} is r^{th} order quasi range from population π_i, $r = 0, 1, ..., [n/2]-1$.

Gaur et al. [5] proposed three nonparametric test procedures *A*, *B* and *C* for testing the homogeneity of scale parameters against umbrella ordered alternative based *U*-statistic, when *k* ($k \ge 3$) populations differing in scale parameters only. We take five samples (k = 5) with h = 3 from exponential, logistic and uniform distributions with different parameters as

I ~ *E*(0, 1); II ~ *E*(0, 2.1); III ~ *E*(0, 3.2); IV ~ *E*(0, 2.4) and V~ *E*(0, 1.6)

I ~ *L*(0, 1); II ~ *L*(0, 2.2); III ~ *L*(0, 3.2); IV ~ *L*(0, 2.3) and V~ *L*(0, 1.5)

I ~ *U*(0, 1); II ~ *U*(0, 1.5); III ~ *U*(0, 2.1); IV ~ *U*(0, 1.7) and V~ *U*(0, 1.4)

where *E*(*a*, *b*), *L*(*a*, *b*) and *U*(*a*, *b*) are exponential, logistic and uniform distribution respectively.

The power comparison of our test (T_h) with that of Carpenter and Singh [2] test ($W_{r,h}$, r = 0, 1, 2), Gaur et al. tests (*A*, *B* and *C*) is done and presented in Table-7 and we infer that the power of the new proposed test T_h is considerably higher than the other tests.

Table 7. Power comparison of the proposed test when α = 0.05, h = 3 and k = 5

Distribution	*n*	T_3	$W_{0,3}$	$W_{1,3}$	$W_{2,3}$	*A*	*B*	*C*
Exponential	6	0.3509	0.3078	0.2257	0.0943	0.0914	0.0916	0.1090
	8	0.4794	0.3976	0.3473	0.2196	0.1182	0.1274	0.1454
	10	0.5381	0.4353	0.4475	0.3481	0.1558	0.1711	0.2026
	12	0.6475	0.5062	0.5233	0.4607	0.1981	0.2306	0.2846
Logistic	6	0.5255	0.4814	0.2767	0.0959	0.2810	0.2760	0.1124
	8	0.6848	0.6244	0.4787	0.2718	0.3207	0.3764	0.1642
	10	0.7885	0.6908	0.6582	0.4708	0.3862	0.4144	0.1858
	12	0.8955	0.7674	0.7530	0.6369	0.4414	0.5176	0.2800
Uniform	6	0.3237	0.1973	0.2048	0.0627	0.1534	0.1436	0.2056
	8	0.5356	0.3391	0.3524	0.1137	0.1938	0.1542	0.2743
	10	0.7249	0.5625	0.5733	0.1662	0.2342	0.1810	0.3292
	12	0.8409	0.7463	0.7625	0.2307	0.2654	0.2138	0.4166

Remark 5.1 The power and size of the test T_h can also be computed when sample size (n) varies, also when location parameter varies. But to control the number of tables, only the equal sample size and equal location parameter cases are considered.

References

[1] Budescu D.V. (1980). Approximate confidence intervals for a robust scale parameter. Psychometrika **45**, 397-402.

[2] Carpenter M., Singh P. (2009). On tests based on sample quasi ranges for ordered alternative (scale case of exponential distribution). Communications in Statistics Simulation and Computation **38**, 846-855.

[3] David H.A., Nagaraja H.N. (2003). Order Statistics. 3rd ed. Hoboken, NJ: John Wiley and Sons Inc.

[4] Gaur A. (2012). A note on test of homogeneity against umbrella scale alternative based on *U*-statistics. Journal of Statistics Applications and Probability **1**, 193-200.

[5] Gaur A., Mahakan K.K., Arora S. (2012). New nonparametric tests for testing homogeneity of scale parameters against umbrella alternative. Statistics and Probability Letters **82**, 1681-1689.

[6] Gill A.N., Dhawan A.K. (1999). A one-sided test for testing homogeneity of scale parameters against ordered alternatives. Communications in Statistics Theory and Methods **28**, 2417-2439.

[7] Hayter A.J. (1990). A one-sided studentized range test for testing against a simple ordered alternative. Journal of American Statistical Association **85**, 778-785.

[8] Kusum K., Bagai I. (1988). A new class of distribution free procedures for testing homogeneity of scale parameters against ordered alternatives. Communications in Statistics Theory and Methods **17**, 1365-1376.

[9] Mack G.A., Wolfe D.A. (1981). *k*-sample rank tests for umbrella alternatives. Journal of American Statistical Association **76**, 175-181.

[10] Miller R.G. (1981). Simultaneous Statistical Inference, 2nd ed., Springer-Verlag, New York.

[11] Rao K.S.M. (1982). Non-parametric tests for homogeneity of scale against ordered alternatives. Annals of Institute of Statistical Mathematics **34**, 57-58.

[12] Shanubhogue A. (1988). Distribution-free test for homogeneity against stochastic ordering. Metrika **35**, 109-119.

[13] Simpson D.G., Margolin B.H. (1986). Recursive nonparametric testing for dose-response relationships subject to down-turns at high dose. Biometrika **73**, 589-596.

[14] Singh P., Gill A.N. (2004). A one-sided test based on sample quasi ranges. Communications in Statistics Theory and Methods **33**(4), 835-849.

[15] Singh P., Liu W. (2006). A test against an umbrella ordered alternative. Computational Statistics and Data Analysis **51**, 1957-1964.