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# New Perspectives on Nonlinear Multi-Atoms Interacting with a Cavity Field

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**Abstract:** We investigate the entanglement dynamics of two two-level atoms interacting with a quantized cavity field, considering the level shifts produced by the Stark effect. We perform comparisons of the entanglement evolution of this system when the cavity is filled with a Kerr-like medium. The presence of entanglement sudden death and sudden birth using extended Werner-like states is observed. Entanglement is guaranteed to survive and can be easily adapted to allow for periodic behavior. The similar dynamic relations between both the Kerr-like medium and the Stark-shift models are shown.

**Keywords:** Entanglement, energy, multi-atom

## 1 Introduction

One of the most interesting and useful phenomena in quantum optics in general and in quantum information processing specifically is entanglement [1,2,3,4,5]. Cavity QED is a useful tool to generate entangled states [6] and can be used to create the entanglement between atoms in cavities and to establish quantum communications between different optical cavities [7,8,9]. Significant efforts have been devoted to the study of the entanglement between two qubits [10]-[31]. It has been shown that [10] the entanglement of a two-qubit system interacting with uncorrelated reservoirs may disappear within a finite time during dynamic evolution (entanglement sudden death, ESD). ESD has been experimentally observed [32,33]. In addition, Yönac et al. [28,34] have discussed what is called entanglement sudden birth (ESB) in cavity QED. The entanglement properties of a quantum system consisting of two cavities interacting with two independent reservoirs have been discussed [35]. It has been shown that entanglement sudden death in a bipartite system independently coupled to reservoirs is related to entanglement sudden birth and that the cavity coherent state can be used as a controller of both ESB and ESD [36].

The creation of entanglement in an atomic ensemble located inside a high-Q ring cavity has been discussed [14]-[16]. The continuous variable approach to studying the entangled dynamics of coupled harmonic oscillators interacting with a thermal reservoir has been investigated [36]. It has been shown that a suitable unitary transformation of the position and momentum operators transform the system to a set of independent harmonic oscillators, with only one of them coupled to the reservoir. In addition, much effort has been devoted to the investigation of quantum discord, which can be used to study a family of two-qubit states [19]-[25]. For more general quantum states, such as the so-called $X$ states, the authors in [20] -[25] have derived explicit expressions for quantum discord and have found that quantum discord, entanglement and classic correlation are independent measures of correlations with no simple relative ordering.

In this paper, we investigate the entanglement dynamics of a quantum system formed by two two-level atoms within two spatially separated cavities with the Stark shift and a Kerr-like medium. We consider an initial state of the two atoms in the following form: $\mu|ee\rangle + \nu|gg\rangle$, where $\mu$ and $\nu$ are two complex numbers and $|\mu|^2 + |\nu|^2 = 1$. In addition, we consider extended

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Werner-like states [37] of the following form

$$A:\rho_{\phi}(0)=x|\phi\rangle\langle\phi|+\frac{1-x}{4}I,|\phi\rangle$$
$$=\cos(\theta)|ge\rangle+\sin(\theta)|eg\rangle,$$
$$B:\rho_{\psi}(0)=x|\psi\rangle\langle\psi|+\frac{1-x}{4}I,|\psi\rangle$$
$$=\cos(\theta)|ee\rangle+\sin(\theta)|gg\rangle, \qquad (1)$$

where $x$ is a real number that indicates the purity of the initial states. We discuss the entanglement through the concurrence of two different values of the parameter $x$ (such as $x=\frac{2}{3}$ (Werner-like states) and $x=1$ (Bell-states)) and for different values of the Kerr-like medium.

The present paper is organized as follows. In Section 2, we obtain an explicit analytical solution of the quantum system formed by two atoms interacting with two vacuum fields of two spatially separated cavities with Stark-shift effects. Then, we investigate the entanglement dynamics of two atoms, ESD, and two fields, ESB, by employing the concurrence. In Section 3, we discuss the influence of the Kerr-like medium on the system and observe the behavior of the entanglement. Finally, we summarize our results in Section 4.

## 2 Model and its solution

The system consists of two noninteracting two-level atoms, where each atom is trapped inside a single cavity (see Fig. 1).

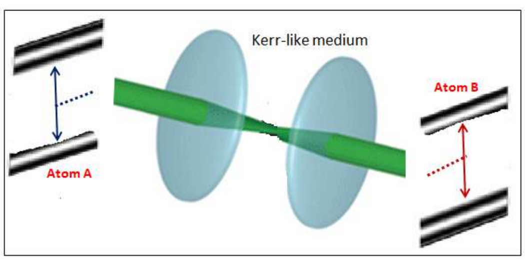


**Fig. 1:** Model for the nonlinear atomic coupling.

The Hamiltonian of the whole system can be written as

$$H=\sum_{i=1}^{2}[\omega_c a_i^{\dagger}a_i+\frac{\omega_0}{2}\sigma_i^z+\chi a_i^{\dagger 2}a_i^2+g(a_i^{\dagger 2}\sigma_i^-+a_i^2\sigma_i^+)]$$
$$+\sum_{i=1}^{2}[a_i^{\dagger}a_i+(\beta_2|e_i\rangle\langle e_i|+\beta_1|g_i\rangle\langle g_i|)], \qquad (2)$$

where $a^{\dagger}$ and $a$ are the creation and annihilation operators of the cavity mode, with the field frequency $\omega_c$ between the excited state $|e\rangle$ and the ground state $|g\rangle$; $\sigma^z$ and $\sigma^{\pm}$ are the atomic spin flip operators; $\omega_0$ is the atomic transition frequency; $\chi$ is the Kerr coupling constant; and $g$ is the coupling constant between the atom and the field. The coupling constants $g_1$, $g_2$ and $\Delta$ determine the Stark-shift parameters $\beta_1$ and $\beta_2$ between two atoms, where

$$\beta_1=\frac{g_1}{\Delta},\ \beta_2=\frac{g_2}{\Delta},\ g=\frac{\beta_1\beta_2}{\Delta^2}. \qquad (3)$$

One of the most effective methods for measuring the entanglement is using the concurrence. To study the entanglement of the above system, which is described by a density matrix $\rho$, we adopt the concurrence (see [38]), which is defined by

$$C=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}, \qquad (4)$$

where $\lambda_i$ $(i=1,2,3,4)$ are the square roots of the eigenvalues in decreasing order of magnitude of the "spin-flipped " density matrix operator $R=\rho(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$ and $\sigma_y$ is the Pauli matrix. $\rho^*$ is the complex conjugate of $\rho$.

### 2.1 Stark shift

Now, we study the influence of the Stark shift on the entanglement sudden death and sudden birth, and we ignore the Kerr-like medium ($i.e.$, $\chi=0$). We consider that the initial state of the two atoms is $\mu|ee\rangle+\nu|gg\rangle$, where $\mu$ and $\nu$ are two complex numbers and $|\mu|^2+|\nu|^2=1$. The fields are prepared in the vacuum states $|00\rangle$. Thus, the initial state of the system is

$$|\Psi(0)\rangle=\Big(\mu|ee\rangle+\nu|gg\rangle\Big)\otimes|00\rangle, \qquad (5)$$

Using the results of [35], we find that the state of the system at any time is given by

$$|\Psi(t)\rangle=\mu\Big[A(t)^2|ee\rangle\otimes|00\rangle+B(t)^2|gg\rangle\otimes|22\rangle$$
$$+A(t)B(t)|eg\rangle\otimes|02\rangle+A(t)B(t)|ge\rangle\otimes|20\rangle\Big]$$
$$+\nu\left(e^{-i\omega_0 t}|gg\rangle\otimes|00\rangle\right), \qquad (6)$$

with

$$A(t)=\sin^2(\alpha)e^{-i\zeta^+t}+\cos^2(\alpha)e^{-i\zeta^-t},$$
$$B(t)=\sin\left(\frac{2\alpha}{2}\right)\left(e^{-i\zeta^+t}-e^{-i\zeta^-t}\right),$$
$$\alpha=\tan^{-1}\left(\frac{\eta^+-2\beta}{\sqrt{2}g}\right),$$
$$\eta^{\pm}=g\left(r\pm\sqrt{r^2+2g}\right),$$
$$\zeta^{\pm}=\omega\pm\eta^{\pm} \qquad (7)$$

The reduced density matrix of the two atoms can be obtained by tracing out the variables of the fields from

Eq. (6)

$$\rho_{a_1a_2}(t) = \mu^2|A(t)|^4|ee\rangle\langle ee| + [\mu^2|B(t)|^4 + \nu^2]|gg\rangle\langle gg| + [\mu^2|A(t)B(t)|^2(|eg\rangle\langle eg| + |ge\rangle\langle ge|) + [\mu\nu A^2(t)e^{i\omega_0 t}|ee\rangle\langle gg| + h.c.] \quad (8)$$

Therefore, the concurrence of the above density matrix is given by

$$C_{a_1a_2}(t) = \max\left\{0, |A(t)|^2[2\mu\nu - 2\mu^2(1-|A(t)|^2)]\right\} \quad (9)$$

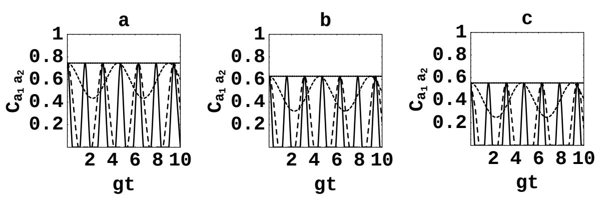


**Fig. 2:** The concurrence of two atoms as a function of the scaled time $gt$. The Stark-shift parameters are $r = 0$ (dotted line), $r = 0.7$ (short-dashed line), $r = 1$ (long-dashed line) and $r = 2$ (solid line), with $\omega = 4$, $\Delta = 1$, $g_2 = 1, \mu = 0.9$ and different values of $\nu$, where (a) $\nu = 0.4$, (b) $\nu = 0.3$ and (c) $\nu = 0.2$.

In Fig. 2, we plot the concurrence of two atoms as a function of the scaled time $gt$ for different values of the Stark-shift parameter $r$. In Fig. 2a, where $\mu = 0.9$ and $\nu = 0.4$, we observe that when $r = 0$, the entanglement exhibits no change, which means that the coupling constant between the atom and the field $g = \frac{\beta_1\beta_2}{\Delta^2} = 0$; in addition, there is no interaction between the atom and the field. When $r = 0.7$, a periodic behavior is exhibited for the entanglement without reaching a value of zero. This effect is not found when we consider $r = 1$, where the entanglement shows a periodic behavior but which remains zero for a finite time, which is called entanglement sudden death, ESD. By increasing the value of the Stark-shift parameter to $r = 2$, we find that the ESD period exhibits a greater periodicity than in the previous case ($r = 1$), but the length of each period of the ESD is less than that in the previous case when $r = 1$.

In Fig. 2b, where $\mu = 0.9$ and $\nu = 0.3$, we observe the same behavior shown by the entanglement in Fig. 2a but with a smaller amplitude with the ESD phenomenon. In this case, the ESD period is greater than that of Fig. 2a. When we set $\nu = 0.2$, the amplitude of the entanglement decreases to a greater extent with an increasing ESD period. This means that the Stark-shift parameter $r$, $\mu$ and $\nu$ play important roles in the entanglement sudden death phenomenon of the two-atom system.

Similarly, the reduced density matrix of the field can be obtained by tracing out the atomic variables from Eq. (6)

$$\rho_{f_1f_2}(t) = [\mu^2|A(t)|^4 + \nu^2]|00\rangle\langle 00| + \mu^2|B(t)|^4|22\rangle\langle 22| + [\mu^2|A(t)B(t)|^2(|02\rangle\langle 02| + |20\rangle\langle 20|) + [\mu\nu A^2(t)e^{i\omega_0 t}|22\rangle\langle 00| + h.c.]. \quad (10)$$

The concurrence of the above reduced density matrix is given by

$$C_{f_1f_2}(t) = \max\left\{0, |B(t)|^2[2\mu\nu - 2\mu^2(1-|B(t)|^2)]\right\} \quad (11)$$

In Fig. 3, we plot the concurrence of the two fields as a

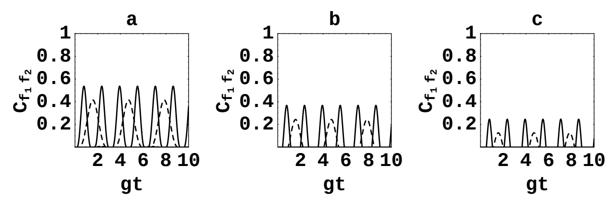


**Fig. 3:** The concurrence of two cavities as a function of the scaled time $gt$. The Stark-shift parameters are $r = 0$ (dotted line), $r = 0.7$ (short-dashed line), $r = 1$ (long-dashed line) and $r = 2$ (solid line), with $\omega = 4$, $\Delta = 1$, $g_2 = 1, \mu = 0.9$ and different values of $\nu$, where (a) $\nu = 0.4$, (b) $\nu = 0.3$, and (c) $\nu = 0.2$.

function of the scaled time $gt$. In each figure, we use four values of the Stark-shift parameter $r$ and different values of $\mu$ and $\nu$. We find that the concurrence $C_{f_1f_2}(t)$ is a periodic function of time. From Fig. 3, we see that the two disentangled fields will initially be entangled after a finite time due to the initial entanglement of the atoms and due to the interaction between the atoms and the fields. The larger the parameter $r$ is (and the larger the atom-field coupling constant $g$ is), the more rapid the oscillations of the entanglement of the two fields that are observed. We note that in Fig. 3, there is no entanglement between the two fields when the value of the parameter $r$ is less than one ( $r < 1$ ), confirming that the role that the Stark-shift parameter plays is to increase the entanglement between the two fields. From Figs. 2 and 3, we find that the entanglement between the two atoms and the entanglement between the two fields can be controlled by adjusting the Stark-shift parameter and the initial state of atoms. This means that the Stark-shift parameter plays an important role in controlling both ESD and ESB.

### *2.2 Kerr-like medium*

Here, we study the influence of the Kerr-like medium parameter on the entanglement sudden death and sudden birth. We ignore the influence of the Stark shift and consider that the initial states of the atoms are prepared in the extended Werner-like states [37]

$$A: \rho_\phi(0) = x|\phi\rangle\langle\phi| + \frac{1-x}{4}I,\ |\phi\rangle = \cos(\theta)|ge\rangle + \sin(\theta)|eg\rangle, \tag{12}$$

$$B: \rho_\psi(0) = x|\psi\rangle\langle\psi| + \frac{1-x}{4}I,\ |\psi\rangle = \cos(\theta)|ee\rangle + \sin(\theta)|gg\rangle, \tag{13}$$

where $(0 < \theta < \pi)$, $x$ is a real number that indicates the purity of initial states, and $I$ is a $4\times 4$ identity matrix. The fields are prepared in a vacuum state $|00\rangle$.
In the first case (A), the density matrix at $t=0$ is given by

$$\rho(0) = \rho_\phi(0)\otimes|0_{f_1}\rangle\langle 0_{f_1}|\otimes|0_{f_2}\rangle\langle 0_{f_2}|. \tag{14}$$

Using the results of [39], the density matrix of the system at any time can be written in the following form:

$$\begin{aligned}\rho(t) = {}& \frac{1-x}{4}[A^2(t)|ee,00\rangle + A(t)B(t)|eg,02\rangle \\ &+B(t)A(t)|ge,20\rangle + B^2(t)|gg,22\rangle][A^{*2}(t)\langle ee,00| \\ &+A^*(t)B^*(t)\langle eg,02| \\ &+B^*(t)A^*(t)\langle ge,20| + B^{*2}(t)\langle gg,22|] \\ &+(x|\sin(\theta)|^2 + \frac{1-x}{4})[A(t)|eg,00\rangle \\ &+B(t)|gg,20\rangle][A^*(t)\langle eg,00| \\ &+B^*(t)\langle gg,20|] \\ &+x\frac{\sin(2\theta)}{2}[A(t)|eg,00\rangle + B(t)|gg,02\rangle][A^*(t)\langle ge,00| \\ &+B^*(t)\langle gg,02|] \\ &+x\frac{\sin(2\theta)}{2}[A(t)|ge,00\rangle + B(t)|gg,02\rangle][A^*(t)\langle eg,00| \\ &+B^*(t)\langle gg,20|] \\ &+(x|\cos(\theta)|^2 + \frac{1-x}{4})[A(t)|ge,00\rangle \\ &+B(t)|gg,02\rangle][A^*(t)\langle ge,00| \\ &+B^*(t)\langle gg,02|] \\ &+\frac{1-x}{4}|gg,00\rangle\langle gg,00|,\end{aligned} \tag{15}$$

where

$$\begin{aligned}A(t) &= e^{-i\chi t}[\cos(\Omega t) + i\sin(\Omega t)], \\ B(t) &= e^{-i\chi t}[\frac{-i\sqrt{2}g\sin(\Omega t)}{\Omega}]\end{aligned} \tag{16}$$

and $\Omega = \sqrt{2g^2+\chi^2}$. The reduced density matrix of the field can be obtained by tracing out the variables of the atoms from Eq. (15)

$$\begin{aligned}\rho_{f_1f_2}(t) = {}& \frac{1-x}{4}[|A^2(t)|^2|00\rangle\langle 00| + |A(t)B(t)|^2(|02\rangle\langle 02| \\ &+|B(t)A(t)|^2|20\rangle\langle 20| + |B^2(t)|^2|22\rangle\langle 22|] \\ &+(x|\sin(\theta)|^2 + \frac{1-x}{4})[|A(t)|^2|00\rangle\langle 00| \\ &+|B(t)|^2|20\rangle\langle 20|] \\ &+x\frac{\sin(2\theta)}{2}|B(t)|^2[|20\rangle\langle 02| + |02\rangle\langle 20|] \\ &+(x|\cos(\theta)|^2 + \frac{1-x}{4})[|A(t)|^2|00\rangle\langle 00| \\ &+|B(t)|^2|02\rangle\langle 02|] \\ &+\frac{1-x}{4}|00\rangle\langle 00|.\end{aligned} \tag{17}$$

Thus, the concurrence between the two fields is given by

$$\begin{aligned}C_{f_1f_2}(t) = {}& 2\max\Big\{0, |x\frac{\sin(2\theta)}{2}||B(t)|^2 - \sqrt{\frac{1-x}{4}}|B(t)|^2 \\ &\times\Big[\frac{1-x}{4}|A^2(t)|^2 + (x|\sin(\theta)|^2 + \frac{1-x}{4})|A(t)|^2 \\ &+(x|\cos(\theta)|^2 + \frac{1-x}{4})|A(t)|^2 + \frac{1-x}{4}\Big]^{1/2}\Big\}\end{aligned} \tag{18}$$

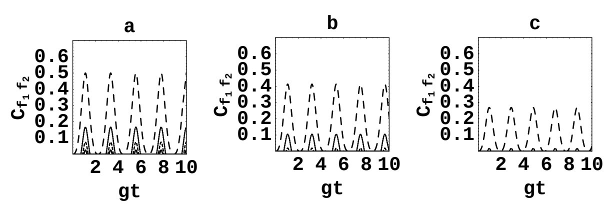


**Fig. 4:** The concurrence of two cavities as a function of the scaled time $gt$, for $\theta = \pi/21.5$ (solid line), $\theta = \pi/17$ (short-dashed line), $\theta = \pi/12$ (dotted line) and $\theta = \pi/4$ (long-dashed line) with $g = 1$, where the initial states of the atoms are as follows: $\rho_\phi(0) = x|\phi\rangle\langle\phi| + \frac{1-x}{4}I, |\phi\rangle = \cos(\theta)|ge\rangle + \sin(\theta)|eg\rangle$, $x = 2/3$, (a)$\chi/g = 0$, (b)$\chi/g = 0.4$ and (c)$\chi/g = 0.8$.

In Fig. 4, we plot the concurrence between the two fields as a function of the scaled time and for different values of the Kerr-like medium parameter ($\chi/g = 0$, $\chi/g = 0.4$, and $\chi/g = 0.8$). We consider the initial state of our system as in case (A), where $\theta$ takes on five different values and $x = 2/3$. In Fig. 4a, we observe that the concurrence of the two fields when $\chi/g = 0$ appears rapidly for different values of $\theta$. We see that the concurrence between the two fields is changing rapidly and strongly and is almost equal to the concurrence of the two atoms. Specifically, the ESB phenomenon occurs but only after a period of time for small values of $\theta$ (for

example, $\theta = \pi/21.5$). In Fig. 4b, when we consider the effect of the Kerr-like medium parameter $\chi/g = 0.4$, we find that the concurrence is smaller than in the previous case. In this case, the general behavior of the concurrence is similar to the effect of the parameter $\theta$, although the ESB phenomenon no longer appears. In Fig. 4c, with increased Kerr-like medium parameter $\chi/g = 0.8$, one can find that the concurrence is smaller than in the previous case, and when $\theta = \pi/21.5$ or $\theta = \pi/17$, the ESB phenomenon disappears. This means that the effect of the Kerr-like medium parameter on the ESB phenomenon plays an important role.
In what follows, we consider the same initial state of atoms as in case (A) but with $x = 1$, namely, the "Bell state". The reduced density matrix of the two atoms is given by

$$\begin{aligned}\rho_{a_1a_2}(t) = {} & x|\sin(\theta)|^2[|A(t)|^2|eg\rangle\langle eg| + |B(t)|^2|gg\rangle\langle gg|] \\ & + x\frac{\sin(2\theta)}{2}|A(t)|^2[|eg\rangle\langle ge| + |ge\rangle\langle eg|] \\ & + x|\cos(\theta)|^2[|A(t)|^2|ge\rangle\langle ge| + |B(t)|^2|gg\rangle\langle gg|].\end{aligned} \tag{19}$$

Thus, the concurrence of the two atoms is given by

$$C_{a_1a_2}(t) = 2\max\left\{0, |\frac{\sin(2\theta)}{2}||A(t)|^2\right\}. \tag{20}$$

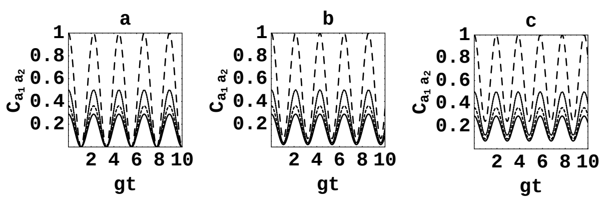


**Fig. 5:** The concurrence of two atoms as a function of the scaled time $gt$, for $\theta = \pi/21.5$ (solid line), $\theta = \pi/17$ (short-dashed line), $\theta = \pi/12$ (dotted line) and $\theta = \pi/4$ (long-dashed line) with $g = 1$, where the initial states of the atoms are as follows: $\rho_\phi(0) = x|\phi\rangle\langle\phi| + \frac{1-x}{4}I, |\phi\rangle = \cos(\theta)|ge\rangle + \sin(\theta)|eg\rangle$, $x = 1$, (a)$\chi/g = 0$, (b)$\chi/g = 0.4$ and (c)$\chi/g = 0.8$.

In Fig. 5, we plot the concurrence of the two atoms as a function of the scaled time and for different values of the Kerr-like medium parameter ($\chi/g = 0$, $\chi/g = 0.4$ and $\chi/g = 0.8$). We consider that the initial states of the two atoms are "Bell states" by setting the parameter $x = 1$ in case (B), and we consider five different values of the parameter $\theta$. In Fig. 5a, it can be observed that the concurrence is a periodic function of time, but the ESD phenomenon disappears for all values of $\theta$, although the concurrence tends to zero periodically. This can be interpreted as the new initial state: the "pure Bell state". In Fig. 5b, the periodic behavior of the concurrence and that it is not zero can be observed. This means that there is no entanglement sudden death. In Fig. 5c, by increasing the Kerr-like medium parameter ($\chi/g = 0.8$), the concurrence oscillates far from zero.

Now, we consider the same initial state for the atoms in case (A), with $x = 1$, the "Bell state", by setting $x = 1$ in Eq. (17). The reduced density matrix of the two cavities is given by

$$\begin{aligned}\rho_{f_1f_2}(t) = {} & |\sin(\theta)|^2[|A(t)|^2|00\rangle\langle 00| + |B(t)|^2|20\rangle\langle 20|] \\ & + \frac{\sin(2\theta)}{2}|B(t)|^2[|20\rangle\langle 02| + |02\rangle\langle 20|] \\ & + |\cos(\theta)|^2[|A(t)|^2|00\rangle\langle 00| + |B(t)|^2|02\rangle\langle 02|].\end{aligned} \tag{21}$$

Thus, the concurrence of the two fields is given by

$$C_{f_1f_2}(t) = 2\max\left\{0, |\frac{\sin(2\theta)}{2}||B(t)|^2\right\}. \tag{22}$$

In Fig. 6a, we see that the concurrence of the two cavities

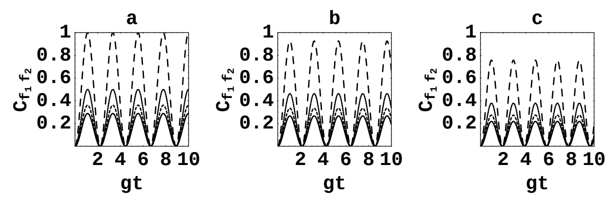


**Fig. 6:** The concurrence of two cavities as a function of the scaled time $gt$, for $\theta = \pi/21.5$ (solid line), $\theta = \pi/17$ (short-dashed line), $\theta = \pi/12$ (dotted line) and $\theta = \pi/4$ (long-dashed line) with $g = 1$, where the initial states of the atoms are as follows: $\rho_\phi(0) = x|\phi\rangle\langle\phi| + \frac{1-x}{4}I, |\phi\rangle = \cos(\theta)|ge\rangle + \sin(\theta)|eg\rangle$, $x = 1$, (a)$\chi/g = 0$, (b)$\chi/g = 0.4$ and (c)$\chi/g = 0.8$.

when $\chi/g = 0$ is a periodic function of time, but the ESB phenomenon disappears for all values of $\theta$, although the concurrence reaches zero periodically. This is quite similar to the previous figures when $\chi/g = 0$, and this is due to the use of the pure Bell state as the initial state. In Fig. 6b, the concurrence of the two cavities when $\chi/g = 0.4$ is similar to the behavior shown in Fig. 6(a), but the entanglement amplitude is less than that in Fig. 6a. The entanglement decreases further as the Kerr parameter is further increased (see Fig. 6c).

In the second case of the initial state (B), the density matrix at $t = 0$ is given by

$$\rho(0) = \rho_\psi(0) \otimes |0_{f_1}\rangle\langle 0_{f_1}| \otimes |0_{f_2}\rangle\langle 0_{f_2}|. \tag{23}$$

Using the results of Eq. [39], the density matrix of the system at any time is given by

$$\begin{aligned}\rho(t) = {} & (x|\cos(\theta)|^2 + \frac{1-x}{4})[A^2(t)|ee,00\rangle + A(t)B(t)|eg,02\rangle \\ & +B(t)A(t)|ge,20\rangle + B^2(t)|gg,22\rangle][A^{*2}(t)\langle ee,00| \\ & +A^*(t)B^*(t)\langle eg,02| + B^*(t)A^*(t)\langle ge,20| \\ & +B^{*2}(t)\langle gg,22|] \\ & +\frac{1-x}{4}[A(t)|eg,00\rangle + B(t)|gg,20\rangle][A^*(t)\langle eg,00| \\ & +B^*(t)\langle gg,20|] \\ & +\frac{1-x}{4}[A(t)|ge,00\rangle + B(t)|gg,02\rangle][A^*(t)\langle ge,00| \\ & +B^*(t)\langle gg,02|] \\ & +(x|\sin(\theta)|^2 + \frac{1-x}{4})|gg,00\rangle\langle gg,00| \\ & +x\frac{\sin(2\theta)}{2}[A^2(t)|ee,00\rangle + A(t)B(t)|eg,02\rangle \\ & +B(t)A(t)|ge,20\rangle \\ & +B^2(t)|gg,22\rangle]\langle gg,00| \\ & +x\frac{\sin(2\theta)}{2}|gg,00\rangle[A^{*2}(t)\langle ee,00| \\ & +A^*(t)B^*(t)\langle eg,02| + B^*(t)A^*(t)\langle ge,20| \\ & +B^{*2}(t)\langle gg,22|]. \end{aligned} \quad (24)$$

The reduced density matrix of the field can be obtained by tracing out the variables of the atoms from Eq. (24)

$$\begin{aligned}\rho_{f_1f_2}(t) = {} & (x|\cos(\theta)|^2 + \frac{1-x}{4})[|A^2(t)|^2|00\rangle\langle 00| \\ & +|A(t)B(t)|^2(|02\rangle\langle 02| \\ & +|B(t)A(t)|^2|20\rangle\langle 20| + |B^2(t)|^2|22\rangle\langle 22|] \\ & +\frac{1-x}{4}[|A(t)|^2|00\rangle\langle 00| + |B(t)|^2|20\rangle\langle 20|] \\ & +\frac{1-x}{4}[|A(t)|^2|00\rangle\langle 00| + |B(t)|^2|02\rangle\langle 02|] \\ & +(x|\sin(\theta)|^2 + \frac{1-x}{4})|00\rangle\langle 00| \\ & +x\frac{\sin(2\theta)}{2}[B(t)^2|22\rangle\langle 00|] \\ & +x\frac{\sin(2\theta)}{2}[B^*(t)^2|00\rangle\langle 22|]. \end{aligned} \quad (25)$$

The concurrence is given by

$$\begin{aligned}C_{f_1f_2}(t) = {} & 2\max\left\{0, |x\frac{\sin(2\theta)}{2}||B(t)|^2 \right. \\ & -[(x|\cos(\theta)|^2 + \frac{1-x}{4}) \\ & \left. \times|B(t)A(t)|^2 + \frac{1-x}{4}|B(t)|^2]\right\}. \end{aligned} \quad (26)$$

In Fig. 7, we plot the concurrence of the two fields as a function of the scaled time with different values of the

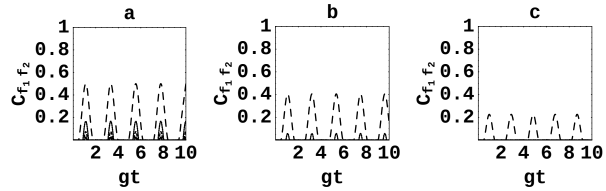


**Fig. 7:** The concurrence of two cavities as a function of the scaled time $gt$, for $\theta = \pi/21.5$ (solid line), $\theta = \pi/17$ (short-dashed line), $\theta = \pi/12$ (dotted line) and $\theta = \pi/4$ (long-dashed line) with $g = 1$, where the initial states of the atoms are as follows: $\rho_\psi(0) = x|\psi\rangle\langle\psi| + \frac{1-x}{4}I, |\psi\rangle = \cos(\theta)|ee\rangle + \sin(\theta)|gg\rangle$, $x = 2/3$, (a)$\chi/g = 0$, (b)$\chi/g = 0.4$ and (c)$\chi/g = 0.8$.

Kerr-like medium parameter $\chi/g = 0$, $\chi/g = 0.4$, and $\chi/g = 0.8$. In Fig. 7a, the ESB phenomenon occurs after a longer period of time compared with Fig. 4a, and the period of time of concurrence disappearance observed in Fig. 7a is longer than that in Fig. 4a for different values of $\theta$. In Fig. 7b, when we consider the effect of the Kerr-like medium parameter $\chi/g = 0.4$, the concurrence is smaller than in the previous case. In Fig. 7c, when increasing the value of the Kerr-like medium parameter, that is, for $\chi = 0.8$, the concurrence is smaller than in the previous case. For $\theta = \pi/21.5$, $\theta = \pi/17$, and $\theta = \pi/12$, the ESB phenomenon disappears. This means that the Kerr-like medium has an important effect on the production of the ESB phenomenon.

Here, let us consider the initial state as a Bell state, and we find the reduced density matrix of the two atoms as

$$\begin{aligned}\rho_{a_1a_2}(t) = {} & |\cos(\theta)|^2[|A^2(t)|^2|ee\rangle\langle ee| + |A(t)B(t)|^2(|eg\rangle\langle eg| \\ & +|B(t)A(t)|^2|ge\rangle\langle ge| + |B^2(t)|^2|gg\rangle\langle gg|] \\ & +|\sin(\theta)|^2|gg\rangle\langle gg| \\ & +\frac{\sin(2\theta)}{2}[A(t)^2|ee\rangle\langle gg|] \\ & +\frac{\sin(2\theta)}{2}[A^*(t)^2|gg\rangle\langle ee|]. \end{aligned} \quad (27)$$

Using Eq. 27, the concurrence of the two atoms is given by

$$\begin{aligned}C_{a_1a_2}(t) = {} & 2\max\left\{0, |\frac{\sin(2\theta)}{2}||A(t)|^2 \right. \\ & \left. -|\cos(\theta)|^2|B(t)A(t)|^2\right\}. \end{aligned} \quad (28)$$

In Fig. 8a, it can be observed that the concurrence is a periodic function of time. The ESD phenomenon is observed with different values of the ESD periods for different values of the parameter $\theta$. Comparing the ESD phenomenon in this case with that observed in Fig. 5a, we find that ESD is clearly shown in Fig. 8a. In Fig. 8b, the concurrence when $\chi/g = 0.4$ is a periodic function of time and the ESD phenomenon decreases with increasing

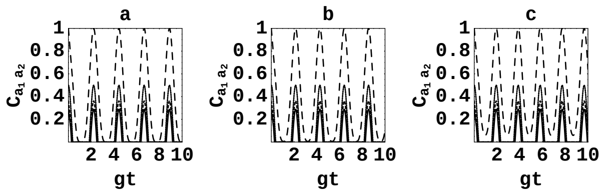


**Fig. 8:** The concurrence of two atoms as a function of the scaled time $gt$, for $\theta = \pi/21.5$ (solid line), $\theta = \pi/17$ (short-dashed line), $\theta = \pi/12$ (dotted line) and $\theta = \pi/4$ (long-dashed line) with $g = 1$, where the initial states of the atoms are as follows: $\rho_\psi(0) = x|\psi\rangle\langle\psi| + \frac{1-x}{4}I, |\psi\rangle = \cos(\theta)|ee\rangle + \sin(\theta)|gg\rangle$, $x = 1$, (a)$\chi/g = 0$, (b)$\chi/g = 0.4$ and (c)$\chi/g = 0.8$.

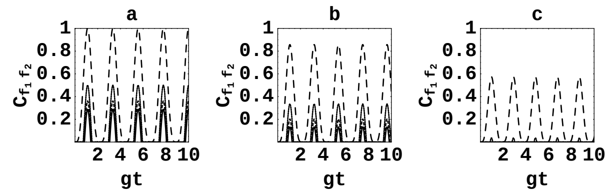


**Fig. 9:** The concurrence of two cavities as a function of the scaled time $gt$, for $\theta = \pi/21.5$ (solid line), $\theta = \pi/17$ (short-dashed line), $\theta = \pi/12$ (dotted line) and $\theta = \pi/4$ (long-dashed line) with $g = 1$, where the initial states of the atoms are as follows: $\rho_\psi(0) = x|\psi\rangle\langle\psi| + \frac{1-x}{4}I, |\psi\rangle = \cos(\theta)|ee\rangle + \sin(\theta)|gg\rangle$, $x = 1$, (a)$\chi/g = 0$, (b)$\chi/g = 0.4$ and (c)$\chi/g = 0.8$.

$\theta$; when $\theta = \pi/4$, there is no entanglement sudden death, although the concurrence becomes zero. This means that the Kerr-like medium parameter can be used to prevent ESD. In Fig. 8c, by increasing the Kerr-like medium parameter value to $\chi/g = 0.8$, the concurrence is far from zero more so than in the previous case when $\chi/g = 0.4$ for $\theta = \pi/4$, but entanglement sudden death appears for the other values of $\theta$.

We consider the same initial state of atoms as in case (B) but with $x = 1$, i.e., the Bell state. The reduced density matrix of the two cavities is given by

$$\begin{aligned}\rho_{f_1 f_2}(t) = & |\cos(\theta)|^2[|A^2(t)|^2|00\rangle\langle00| + |A(t)B(t)|^2(|02\rangle\langle02| \\ & +|B(t)A(t)|^2|20\rangle\langle20| + |B^2(t)|^2|22\rangle\langle22|] \\ & +|\sin(\theta)|^2|00\rangle\langle00| \\ & +\frac{\sin(2\theta)}{2}[B(t)^2|22\rangle\langle00|] \\ & +\frac{\sin(2\theta)}{2}[B^*(t)^2|00\rangle\langle22|]. \end{aligned} \tag{29}$$

Therefore, the concurrence of the two cavities in this case is

$$\begin{aligned}C_{f_1 f_2}(t) = & 2\max\Big\{0, |\frac{\sin(2\theta)}{2}||B(t)|^2 \\ & -|\cos(\theta)|^2|B(t)A(t)|^2\Big\}. \end{aligned} \tag{30}$$

In Fig. 9, we plot the concurrence of two fields as a function of the scaled time and for different values of the Kerr-like medium parameter ($\chi/g = 0$, $\chi/g = 0.4$, and $\chi/g = 0.8$), and we consider the initial state of our system (B). We observe that when $\chi = 0$, the ESB phenomenon occurs when $\theta = \pi/21.5$, $\theta = \pi/17$, and $\theta = \pi/12$ more so than when $\theta = \pi/4$. In Fig. 9b, when we consider the effect of the Kerr-like medium parameter $\chi/g = 0.4$, we find that the concurrence is smaller than that in the previous case, but we observe that the period time of the ESB appearance is somewhat smaller than that in the first case when $\chi/g = 0$ in Fig. 9a. In Fig. 9c, by increasing the value of the Kerr-like medium parameter to $\chi/g = 0.8$, it can be found that the concurrence is smaller than in the previous case, and when $\theta = \pi/21.5$ and $\theta = \pi/17$, the ESB phenomenon disappears. This means that larger values of the Kerr-like medium parameter can be used to prevent ESB.

## 3 Summary

In this paper, we investigated the entanglement dynamics of two two-level atoms interacting with two fields of two spatially separated cavities in the presence of either the Stark shift or of a Kerr-like medium. We considered that the two atoms are initially entangled and that the fields are in vacuum states. The results show that no ESD is observed when we ignore the Stark-shift effect. This effect is absent when the Stark shift is considered. In addition, we show that the Kerr-like medium plays an important role in preventing ESD and in creating ESB and enhances the degree of atom-atom entanglement. A more interesting finding is that both the Kerr-like medium and Stark shift play the same role for some initial state settings.

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