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Inference for Linear Exponential Distribution Based on Record Ranked Set Sampling

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Abstract: In this article, we use ranked set sampling (RSS) to develop a Bayesian analysis based on record statistics values. Maximum likelihood estimation (MLE) and Bayes estimators are derived for linear exponential distribution from a simple random sample (SRS) and record ranked set sampling (RRSS) (one- and m-cycle). These estimators are compared via their biases and mean squared error (MSE). This is done with respect to both symmetric and asymmetric loss function. Two numerical examples are used to illustrate these results.

Keywords: Record data, Maximum likelihood estimation, Mean squared error, Symmetric and asymmetric loss functions, Linear exponential distribution

1 Introduction

The idea of RSS has been introduced by McIntyre [1]. RSS is a modification of SRS to provide more structure to the collected sample items. Suppose that $\mathbf{x}_{\text{SRS}} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ be a SRS of size n from a continuous population $F(x)$. Let $X_{i(j:n)} \equiv X_{ij}$, $i = 1, 2, \ldots, n$, be the jth order statistic from the ith the set sample. The process of RSS can be described as follows:

$$\underbrace{\begin{array}{cccc} \underline{X_{11}} & X_{12} & \cdots & X_{1n} \\ X_{21} & \underline{X_{22}} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & \underline{X_{nn}} \end{array}}_{\text{Judgment Rank}} \begin{array}{c} \rightarrow \\ \rightarrow \\ \vdots \\ \rightarrow \end{array} \underbrace{\begin{array}{c} X_{11} \\ X_{22} \\ \vdots \\ X_{nn} \end{array}}_{\text{RSS}}$$

Then, the obtained sample, $X_{\text{RSS}} = (X_{11}, X_{22}, \ldots, X_{nn})$, is called a one-cycle RSS. If this method is repeated m times, a RSS of size mn is obtained. The following are examples of applications for RSS (see Al-saleh et al. [2]): (a) Assume that the length of bacterial cells in a microscope which can be measured with a micrometer is considered, order of length of bacterial cells in the same microscopes may be found easily (b) Suppose that estimation of average milk yield per sheep can be considered, and milk yields of two or three sheep can be easily ranked by the owner of the sheep (c) A medical doctor specialized in chest diseases can rank three or four patients suffering from lung cancer according to their survival time.

RSS has been considered by many authors, Balakrishnan and Li [3] determined the best linear unbiased estimators (BLUEs) based on ORSS and RSS. Helu et al. [4] introduced maximum likelihood estimation using RSS. Kvam and Tiwari [5] derived generalized maximum likelihood estimator based on RSS. Mohie El-Din et al. [6,7] introduced Bayesian estimation and prediction for the pareto distribution using RSS. Sadek et al. [8] studied the estimator of the parameters of exponential distribution based on RSS using asymmetric loss function.

Record values are applied in meteorology, mining and stock market analysis. This type of data has been considered by

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many authors, see for example, Ahmadi and Arghami [9], Ahsanullah [10], Arnold et al. [11] and Panaitescu et al. [12]. A generated record sampling scheme is described as n independent sequences of continuous random variables are deem. Suppose that $R_{\text{RRSS}} = (R_{11}, R_{22}, ..., R_{nn})$ be a one-cycle RRSS of size n is described as (see Salehi and Ahmadi [13]):

$$\begin{array}{ccccccc} \underline{R_{11}} & & & & \rightarrow & R_{11} \\ R_{12} & \underline{R_{22}} & & & \rightarrow & R_{22} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ R_{1n} & R_{2n} & \ldots & \underline{R_{nn}} & \rightarrow & R_{nn} \end{array}$$

where R_{ij} has ith record in jth sequence, $i, j = 1, 2, \ldots, n$, R_{ii} are independent random variables, but not necessarily ordered. If this method is repeated m times, a RRSS of size mn is obtained.

The probability density function (pdf) and the cumulative distribution function (cdf) of the linear exponential distribution are given by

$$f(x;\alpha,\beta) = (\alpha+\beta x)\exp\left(-\alpha x - \frac{\beta}{2}x^2\right), \quad x \geq 0, \quad \alpha, \beta > 0, \tag{1}$$

and

$$F(x;\alpha,\beta) = 1 - \exp\left(-\alpha x - \frac{\beta}{2}x^2\right), \quad x \geq 0 \quad \alpha, \beta > 0, \tag{2}$$

respectively. The linear exponential distribution with parameters α and β will be denoted by LExp(α, β). Its reliability and hazard functions at mission time t are given respectively by

$$R(t;\alpha,\beta) = \exp\left(-\alpha t - \frac{\beta}{2}t^2\right) \quad \text{and} \quad h(t,\theta) = \alpha + \beta t, \quad t > 0, \tag{3}$$

This model has many applications in reliability analysis and applied statistics. It is easily noted that if $\alpha = 0$, the LExp(α, β) reduces to Rayleigh distribution. Carbone et al. [14] used it to study the survival pattern of patients with lymphoma. References and more details on this model may be found in Al-Khedhairi [15], Mahmoud et al. [16], Seo and Yum [17].

***Loss function*:** Asymmetric loss function of the function of parameters $\phi = \phi(\alpha, \beta)$ can be expressed from the assumption that the minimal loss occur at $\phi^* = \phi$ as

$$L(\Delta) \propto \exp(c\Delta) - c\Delta - 1, \quad c \neq 0,$$

where $\Delta = (\phi^* - \phi)$, ϕ^* is an estimator of ϕ. The posterior expectation of the LINEX loss function is given by

$$E[L(\phi^* - \phi)] \propto \exp(c\phi^*) E_\phi[\exp(-c\phi)] - c(\phi^* - E_\phi(\phi)) - 1, \tag{4}$$

where $E_\phi(.)$ is the posterior expectation of the posterior density of ϕ. The linear exponential (LINEX) loss function is obtained by minimizing (4), then

$$\phi^*_{BL} = -\frac{1}{c}\ln\left\{E_\phi[\exp(-c\phi)]\right\}. \tag{5}$$

Recently, asymmetric loss function considered in Bayesian inference such as Al-Aboud [18], Al-Hossain [19], Amin [20], Hassan [21], Howladera and Hossainb [22], Kim and Song [23], Ku and Kaya [24], Kundua and Howladerb [25], Martz and Waller [26], Mohammadi and Pazira [27], Soliman et al. [28,29] and Zellner [30].

In the next section, we present Bayes estimation to estimate the unknown parameters based on one-and m-cycle RRSS and SRS. The Bayes estimates are obtained using both squared error loss function (SEL) and LINEX loss functions. In Section 3, we discuss the MLEs of the unknown parameters. In Section 4, we use illustrative examples based on both real and simulated data sets. A conclusion will be presented in Section 5.

2 Bayes Estimation

In this section, we obtain Bayes estimation based on RRSS and SRS of the unknown parameters of LExp(α, β). This is done with respect to both SEL and LINEX loss functions.

2.1 Bayes estimation based on RRSS

Consider $Y_1, Y_2, \ldots, Y_n$ form one-cycle RRSS, each of them has the LExp(α, β). Then, the density of Y_j is given (Arnold et al. [11])

$$g(y_j) = \frac{1}{(j-1)!}\left[-\log(\overline{F}(y_j))\right]^{j-1} f(y_j).$$

The joint pdf of the RRSS of y_i, is

$$f(\mathbf{y}) = \prod_{j=1}^{n} g(y_j) \propto \prod_{j=1}^{n}\left[-\log(\overline{F}(y_j))\right]^{j-1} f(y_j), \tag{6}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

2.1.1 Bayes estimates based on one-cycle RRSS

From Equations (1) and (2) in (6), the likelihood function is given by

$$L_1(\alpha, \beta; \mathbf{y}) \propto \prod_{j=1}^{n}\left(\alpha + \frac{\beta}{2} y_j\right)^{j-1}(\alpha + \beta y_j)\exp\left(-\alpha\sum_{j=1}^{n} y_j - \frac{\beta}{2}\sum_{j=1}^{n} y_j^2\right), \tag{7}$$

Using the following relations (see Balakrishnan [31] and Kotb and Raqab [32])

$$\prod_{j=1}^{n}\sum_{t=0}^{j-1} h_t(j) = \sum_{t_1=0}^{0}\sum_{t_2=0}^{1}\cdots\sum_{t_n=0}^{n-1}\prod_{j=1}^{n} h_{t_j}(j), \tag{8}$$

and

$$\prod_{j=1}^{n}(\alpha + \beta y_j) = \sum_{\nu=0}^{n}\alpha^{n-\nu}\beta^{\nu}\xi_{\nu}, \tag{9}$$

where

$$\xi_{\nu} = \sum_{b_1=1}^{n-\nu+1} y_{b_1}\sum_{b_2=b_1+1}^{n-\nu+2} y_{b_2}\times\cdots\times\sum_{b_{\nu}=b_{\nu-1}+1}^{n} y_{b_{\nu}},$$

the likelihood function in Equation (7) becomes

$$L_1(\alpha, \beta; \mathbf{y}) \propto \sum_{\underline{t}}\sum_{\nu=0}^{n} c_{\underline{t},\nu}\alpha^{n+V_{\underline{t}}-\nu}\beta^{U_{\underline{t}}+\nu}\exp\left(-\alpha\sum_{j=1}^{n} y_j - \frac{\beta}{2}\sum_{j=1}^{n} y_j^2\right), \tag{10}$$

where $\sum_{\underline{t}} = \sum_{t_1=0}^{0}\sum_{t_2=0}^{1}\cdots\sum_{t_n=0}^{n-1}$, $\underline{t} = (t_1, \cdots, t_n)$, $V_{\underline{t}} = \sum_{j=1}^{n} j - t_j - 1$, $U_{\underline{t}} = \sum_{j=1}^{n} t_j$ and

$$c_{\underline{t},\nu} = \xi_{\nu}\prod_{j=1}^{n}\binom{j-1}{t_j}\left(\frac{y_j}{2}\right)^{t_j}. \tag{11}$$

For the Bayesian estimation setup, we need a suitable prior parameter distribution. Here, we consider the prior density suggested by Al-khedhairi [15] as

$$\pi(\alpha, \beta; \delta) = \eta\rho\exp\{-\alpha\eta - \beta\rho\}, \quad \beta, \alpha > 0, \tag{12}$$

where δ is the vector of prior parameters and η, ρ are known positive constants.

Then, from Equations (10) and (12), the posterior density function of α and β becomes

$$\pi_1^*(\alpha, \beta|\mathbf{y}) = J_{1,1}^{-1}\sum_{\underline{t}}\sum_{\nu=0}^{n} c_{\underline{t},\nu}\alpha^{n+V_{\underline{t}}-\nu}\beta^{U_{\underline{t}}+\nu} \times\exp\left(-\alpha\left(\eta + \sum_{j=1}^{n} y_j\right) - \beta\left(\rho + \sum_{j=1}^{n}\frac{y_j^2}{2}\right)\right), \tag{13}$$

where the normalized constant $J_{\tau_1,\tau_2} = J^{0,0}_{\tau_1,\tau_2}$ and

$$J^{\upsilon_1,\upsilon_2}_{\tau_1,\tau_2} = \sum_{\underline{t}} \sum_{v=0}^{n} c_{\underline{t},v} \frac{\Gamma(n+V_{\underline{t}}-v+\tau_1)}{\left(\upsilon_1+\eta+\sum_{j=1}^{n} y_j\right)^{n+V_{\underline{t}}-v+\tau_1}} \times \frac{\Gamma(U_{\underline{t}}+v+\tau_2)}{\left(\upsilon_2+\rho+\sum_{j=1}^{n}\frac{y_j^2}{2}\right)^{U_{\underline{t}}+v+\tau_2}}.$$

Bayesian estimators of α and β under a SEL function are

$$\tilde{\alpha}_{BS} = E(\alpha|\mathbf{y}) = \int_0^\infty \int_0^\infty \alpha \pi_1^*(\alpha,\beta|\mathbf{y}) \mathrm{d}\alpha \mathrm{d}\beta = \frac{J_{2,1}}{J_{1,1}} \quad \text{and} \quad \tilde{\beta}_{BS} = \frac{J_{1,2}}{J_{1,1}}. \tag{14}$$

From Equation (5), the Bayesian estimates of α and β based on the LINEX loss function are given by

$$\tilde{\alpha}_{BL} = -\frac{1}{c}\ln[E(e^{-c\alpha}|\mathbf{y})] = -\frac{1}{c}\ln\left[\frac{J^{c,0}_{1,1}}{J_{1,1}}\right] \quad \text{and} \quad \tilde{\beta}_{BL} = -\frac{1}{c}\ln\left[\frac{J^{0,c}_{1,1}}{J_{1,1}}\right]. \tag{15}$$

2.1.2 Bayes estimates based on m-cycle RRSS

Suppose that $y_{\ell j}$, $j = 1,2,\ldots,n$, $\ell = 1,2,\ldots,m$ are m cycles of upper record data. From Equation (6), the likelihood function is

$$L_2(\alpha,\beta;\underline{\mathbf{y}}) \propto \prod_{\ell=1}^{m}\prod_{j=1}^{n}\left(\alpha+\frac{\beta}{2}y_{\ell j}\right)^{j-1}(\alpha+\beta y_{\ell j}) \times \exp\left(-\alpha\sum_{\ell=1}^{m}\sum_{j=1}^{n} y_{\ell j} - \frac{\beta}{2}\sum_{\ell=1}^{m}\sum_{j=1}^{n} y_{\ell j}^2\right). \tag{16}$$

Using Equations (8) and (9), the likelihood function in Equation (16) is written as

$$L_2(\alpha,\beta;\underline{\mathbf{y}}) \propto \prod_{\ell=1}^{m}\sum_{\underline{t}_\ell}\sum_{v_\ell=0}^{n} c_{\underline{t}_\ell,v_\ell}\alpha^{nm+V_{\underline{\mathbf{t}}}-v^*}\beta^{U_{\underline{\mathbf{t}}}+v^*} \times \exp\left(-\alpha\sum_{\ell=1}^{m}\sum_{j=1}^{n} y_{\ell j} - \frac{\beta}{2}\sum_{\ell=1}^{m}\sum_{j=1}^{n} y_{\ell j}^2\right), \tag{17}$$

where $\underline{\mathbf{y}} = (y_{11},y_{12},\ldots,y_{1n};y_{21},y_{22},\ldots,y_{2n};\ldots;y_{m1},y_{m2},\ldots,y_{mn})$,

$$\sum_{\underline{t}_\ell} = \sum_{t_{\ell,1}=0}^{0}\sum_{t_{\ell,2}=0}^{1}\cdots\sum_{t_{\ell,n}=0}^{n-1}, \quad V_{\underline{\mathbf{t}}} = \sum_{\ell=1}^{m}\sum_{j=1}^{n} j - t_{\ell,j} - 1, \quad U_{\underline{\mathbf{t}}} = \sum_{\ell=1}^{m}\sum_{j=1}^{n} t_{\ell,j},$$

$$c_{\underline{t}_\ell,v_\ell} = \xi_{v_\ell}\prod_{j=1}^{n}\binom{j-1}{t_{\ell,j}}\left(\frac{y_{\ell j}}{2}\right)^{t_{\ell,j}},$$

and

$$\xi_{v_\ell} = \sum_{b_1=1}^{n-v_\ell+1} y_{\ell b_1}\sum_{b_2=b_1+1}^{n-v_\ell+2} y_{\ell b_2}\times\cdots\times\sum_{b_{v_\ell}=b_{v_\ell-1}+1}^{n} y_{\ell b_{v_\ell}}.$$

Using Equations (12) and (17), the posterior density function is

$$\pi_2^*(\alpha,\beta|\underline{\mathbf{y}}) = K_{1,1}^{-1}\prod_{\ell=1}^{m}\sum_{\underline{t}_\ell}\sum_{v_\ell=0}^{n} c_{\underline{t}_\ell,v_\ell}\alpha^{nm+V_{\underline{\mathbf{t}}}-v^*}\beta^{U_{\underline{\mathbf{t}}}+v^*} \times \exp\left(-\alpha\left(\eta+\sum_{\ell=1}^{m}\sum_{j=1}^{n} y_{\ell j}\right)-\beta\left(\rho+\sum_{\ell=1}^{m}\sum_{j=1}^{n}\frac{y_{\ell j}^2}{2}\right)\right), \tag{18}$$

where the normalized constant $K_{\tau_1,\tau_2} = K^{0,0}_{\tau_1,\tau_2}$ and

$$K^{\upsilon_1,\upsilon_2}_{\tau_1,\tau_2} = \prod_{\ell=1}^{m}\sum_{\underline{t}_\ell}\sum_{\nu_\ell=0}^{n} c_{\underline{t}_\ell,\nu_\ell} \frac{\Gamma(nm+V_{\underline{\mathbf{t}}}-\nu^*+\tau_1)}{\left(\upsilon_1+\eta+\sum_{\ell=1}^{m}\sum_{j=1}^{n} y_{\ell j}\right)^{nm+V_{\underline{\mathbf{t}}}-\nu^*+\tau_1}} \times \frac{\Gamma(U_{\underline{\mathbf{t}}}+\nu^*+\tau_2)}{\left(\upsilon_2+\rho+\sum_{\ell=1}^{m}\sum_{j=1}^{n}\frac{y_{\ell j}^2}{2}\right)^{U_{\underline{\mathbf{t}}}+\nu^*+\tau_2}}.$$

Bayesian estimators of α and β under a SEL function are

$$\tilde{\alpha}_{BS} = \frac{K_{2,1}}{K_{1,1}} \quad \text{and} \quad \tilde{\beta}_{BS} = \frac{K_{1,2}}{K_{1,1}}, \tag{19}$$

respectively, while the Bayesian estimators of α and β based on the LINEX loss function are given by

$$\tilde{\alpha}_{BL} = -\frac{1}{c}\ln\left[\frac{K^{c,0}_{1,1}}{K_{1,1}}\right] \quad \text{and} \quad \tilde{\beta}_{BL} = -\frac{1}{c}\ln\left[\frac{K^{0,c}_{1,1}}{K_{1,1}}\right]. \tag{20}$$

2.2 Bayes estimation based on SRS

Suppose that $x_{\ell i}$, $i = 1,2,\ldots,n$, $\ell = 1,2,\ldots,m$ are m upper record sets, then the joint density is

$$\begin{aligned} f(\underline{\mathbf{x}}|\alpha,\beta) &= \prod_{\ell=1}^{m}\left(f(x_{\ell n})\prod_{i=1}^{n-1}\frac{f(x_{\ell i})}{\overline{F}(x_{\ell i})}\right) \\ &= \left(\prod_{\ell=1}^{m}\prod_{i=1}^{n}(\alpha+\beta x_{\ell i})\right)\exp\left(-\alpha\sum_{\ell=1}^{m} x_{\ell n} - \frac{\beta}{2}\sum_{\ell=1}^{m} x_{\ell n}^2\right). \end{aligned} \tag{21}$$

where $\underline{\mathbf{x}} = (x_{11},\ldots,x_{1n},\ldots,x_{m1},\ldots,x_{mn})$. Using Equation (9), the likelihood function in Equation (21) becomes

$$L_3(\alpha,\beta;\underline{\mathbf{x}}) = \prod_{\ell=1}^{m}\sum_{\kappa_\ell=0}^{n}\zeta_{\kappa_\ell}\alpha^{mn-d_{\underline{\kappa}}}\beta^{d_{\underline{\kappa}}}\exp\left(-\alpha\sum_{\ell=1}^{m}x_{\ell n}-\frac{\beta}{2}\sum_{\ell=1}^{m}x_{\ell n}^2\right), \tag{22}$$

where $d_{\underline{\kappa}} = \sum_{\ell=1}^{m}\kappa_\ell$ and

$$\zeta_{\kappa_\ell} = \sum_{b_1=1}^{n-\kappa_\ell+1} x_{\ell b_1}\sum_{b_2=b_1+1}^{n-\kappa_\ell+2} x_{\ell b_2}\times\cdots\times\sum_{b_{\kappa_\ell}=b_{\kappa_\ell-1}+1}^{n} x_{\ell b_{\kappa_\ell}}.$$

From Equations (12) and (22), the posterior density function is

$$\pi_3^*(\alpha,\beta|\underline{\mathbf{x}}) = A_{1,1}^{-1}\prod_{\ell=1}^{m}\sum_{\kappa_\ell=0}^{n}\zeta_{\kappa_\ell}\alpha^{mn-d_{\underline{\kappa}}}\beta^{d_{\underline{\kappa}}}\exp\left(-\alpha\left(\eta+\sum_{\ell=1}^{m}x_{\ell n}\right)-\beta\left(\rho+\sum_{\ell=1}^{m}\frac{x_{\ell n}^2}{2}\right)\right), \tag{23}$$

where $A_{\tau_1,\tau_2} = A^{0,0}_{\tau_1,\tau_2}$ and

$$A^{\upsilon_1,\upsilon_2}_{\tau_1,\tau_2} = \prod_{\ell=1}^{m}\sum_{\kappa_\ell=0}^{n}\zeta_{\kappa_\ell}\frac{\Gamma(mn-d_{\underline{\kappa}}+\tau_1)}{(\upsilon_1+\eta+\sum_{\ell=1}^{m}x_{\ell n})^{mn-d_{\underline{\kappa}}+\tau_1}}\times\frac{\Gamma(d_{\underline{\kappa}}+\tau_2)}{\left(\upsilon_2+\rho+\sum_{\ell=1}^{m}\frac{x_{\ell n}^2}{2}\right)^{d_{\underline{\kappa}}+\tau_2}}.$$

Bayesian estimators of α and β under a SEL function are

$$\tilde{\alpha}_{BS} = \frac{A_{2,1}}{A_{1,1}} \quad \text{and} \quad \tilde{\beta}_{BS} = \frac{A_{1,2}}{A_{1,1}}. \tag{24}$$

Under the LINEX loss function, the Bayesian estimates of α and β are given, respectively, by

$$\tilde{\alpha}_{BL} = -\frac{1}{c}\ln\left[\frac{A^{c,0}_{1,1}}{A_{1,1}}\right] \quad \text{and} \quad \tilde{\beta}_{BL} = -\frac{1}{c}\ln\left[\frac{A^{0,c}_{1,1}}{A_{1,1}}\right]. \tag{25}$$

3 Maximum likelihood estimation (MLE)

In this section, maximum likelihood estimators for the parameters of LExp(α,β) based on RRSS and SRS are derived.

3.1 MLE based on m-cycle RRSS

Based on Equation (16), we have

$$\ln L\left(\alpha,\beta;\underline{\mathbf{y}}\right) = \sum_{\ell=1}^{m}\sum_{j=1}^{n}\left[-\ln(j-1)!+(j-1)\ln\left(\alpha+\frac{\beta}{2}y_{\ell j}\right)\right. \\ \left. +\ \ln\left(\alpha+\beta y_{\ell j}\right)-\left(\alpha y_{\ell j}+\frac{\beta}{2}y_{\ell j}^{2}\right)\right].$$

Under the assumption that both the parameters α and β are unknown, the $\tilde{\alpha}_{ML}$ and $\tilde{\beta}_{ML}$ can be solved numerically from the following equations

$$\sum_{\ell=1}^{m}\sum_{j=1}^{n}\left[\frac{j-1}{\alpha+\frac{\beta}{2}y_{\ell j}}+\frac{1}{\alpha+\beta y_{\ell j}}\right]=mn\overline{Y}, \tag{26}$$

and

$$\sum_{\ell=1}^{m}\sum_{j=1}^{n}y_{\ell j}\left[\frac{j-1}{2\alpha+\beta y_{\ell j}}+\frac{1}{\alpha+\beta y_{\ell j}}\right]=\frac{mn}{2}\overline{\overline{Y}}. \tag{27}$$

where

$$\overline{Y}=\frac{1}{m}\sum_{\ell=1}^{m}\overline{\mathbf{y}}_{\ell},\quad \overline{\mathbf{y}}_{\ell}=\frac{1}{n}\sum_{j=1}^{n}y_{\ell j}\quad\text{and}\quad \overline{\overline{Y}}=\frac{1}{mn}\sum_{\ell=1}^{m}\sum_{j=1}^{n}y_{\ell j}^{2}.$$

3.2 MLE based on SRS

Based on Equation (21), the log likelihood function is given by

$$\ln L(\alpha,\beta;\underline{\mathbf{x}}) = \sum_{\ell=1}^{m}\sum_{i=1}^{n}\left[\ln\left(\alpha+\frac{\beta}{2}x_{\ell i}\right)-\left(\alpha x_{\ell n}+\frac{\beta}{2}x_{\ell n}^{2}\right)\right].$$

The MLE of α and β can be solved numerically from the following equations

$$\sum_{\ell=1}^{m}\sum_{i=1}^{n}\frac{1}{\alpha+\beta x_{\ell i}}=m\overline{X}\quad\text{and}\quad\sum_{\ell=1}^{m}\sum_{i=1}^{n}\frac{x_{\ell i}}{\alpha+\beta x_{\ell i}}=\frac{m}{2}\overline{\overline{X}}, \tag{28}$$

where

$$\overline{X}=\frac{1}{m}\sum_{\ell=1}^{m}x_{\ell n}\quad\text{and}\quad\overline{\overline{X}}=\frac{1}{m}\sum_{\ell=1}^{m}x_{\ell n}^{2}.$$

4 Numerical Results

We illustrate our previous theoretical results of the inferences discussed, two simulated record sets of sizes $n=3$ and 5 from the LExp(α,β) based on RRSS and SRS are obtained.

1. Choose values of the prior parameters ($\eta=0.5$ and $\rho=0.3$), then generate $\alpha=0.3328$ from Gamma$(1,\eta)$ and $\beta=0.4995$ from Gamma$(1,\rho)$.
2. Based on those generated values of α and β in step (1), using the algorithms given in Aboeleneen [33], we generate n record values from the LExp(α,β) using $X_i=\sqrt{\left(\frac{\alpha}{\beta}\right)^2-\frac{2}{\beta}\ln\left(1-U_i^*\right)}-\frac{\alpha}{\beta}$ where $U_i^*=1-\prod_{j=1}^{i}U_j$ and U_j from $U(0,1)$ for $i=1,\cdots,n$. Then using the procedure of one-cycle ($m=1$) RRSS, samples of size $n=3,5$ can be obtained.
3. To obtain two-cycle RRSS, the previous step is replicated two times, so a sample size of $2n$ is obtained.

4.The different Bayes estimates $(.)_{BS}$ and $(.)_{BL}$ of α and β are computed through Equations (14), (15), (19), (20), (24) and (25), as well as the ML $(.)_{ML}$ of α and β are calculated numerically from Equations (26), (27) and (28).

5.Repeat Steps 1-4 for 1000 runs to obtain MSE and bias of all estimates for SRS and RRSS of one- and two- cycle, respectively. MSE and bias are computed as, $\text{MSE}(\theta) = \frac{1}{1000}\sum_{i=1}^{1000}(\tilde{\theta}_i - \theta)^2$, $\tilde{\theta}_i$ is the estimator of $\theta = (\alpha, \beta)$ for the ith simulated data and $\tilde{\theta}_{\text{Bias}} = (\bar{\tilde{\theta}} - \theta)$, $\bar{\tilde{\theta}}$ is the average of the 1000 estimates of $\tilde{\theta}$. The bias and MSE of all the estimates are in Tables 1 and 2, respectively.

4.1 Applications

The real data set is taken from Hand et al.([34], p. 124) to illustrate Bayesian estimation techniques based on SRS and RRSS for one- and two-cycle. Cox and Lewis [35] introduced 799 recorded waiting times which represent time intervals between successive pulses along a nerve fibre measured in seconds.
To check whether the linear exponential distribution is suitable for this data, Cramér von Mises test is used to test the null hypothesis

$$H_o : F(x) = \text{linear exponential distribution}, \quad H_1 : F(x) \neq \text{linear exponential distribution}.$$

H_o is rejected at a significance level of $\alpha = 0.05$ if $p-$value $< \alpha$. The Cramér von Mises test statistic is 0.2053 with an associated $p-\text{value} = 0.2575 > 0.05$, so linear exponential distribution is fitted to the above real data set. The one- and two- cycle RRSS is in Tables 3 and 4. Then, the Bayes estimates of α and β are in Table 5.

Table 1: Bias based on SRS and RRSS when $\alpha = 0.3328$, $\beta = 0.4995$, $\eta = 0.5$ and $\rho = 0.3$.

			SRS				RRSS			
			$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$	
m	n	*Par.*			$c=-1$	$c=1$			$c=-1$	$c=1$
1	3	α	0.2270	0.2734	0.5369	0.1488	0.1329	0.1136	0.1483	0.0868
		β	0.1793	0.0905	0.3449	0.0420	0.1140	0.0593	0.0831	0.0450
	5	α	0.2301	0.4366	0.6052	0.3188	0.1349	0.2351	0.2834	0.1915
		β	0.0073	0.0596	0.1903	-0.0196	0.0406	0.0107	0.0512	-0.0260
2	3	α	0.2175	0.2451	0.4159	0.1462	0.0993	0.0874	0.0982	0.0781
		β	0.0721	0.0385	0.0573	0.0311	0.0491	0.0355	0.0367	0.0345
	5	α	0.2444	0.4369	0.5761	0.3302	0.1080	0.1653	0.1889	0.1431
		β	-0.0426	-0.0333	0.0149	-0.0740	-0.0004	-0.0249	-0.0142	-0.0356

Table 2: MSE based on SRS and RRSS when $\alpha = 0.3328$, $\beta = 0.4995$, $\eta = 0.5$ and $\rho = 0.3$.

			SRS				RRSS			
			$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$	
m	n	*Par.*			$c=-1$	$c=1$			$c=-1$	$c=1$
1	3	α	0.5102	0.8511	3.7446	0.3422	0.1995	0.2667	0.5247	0.1443
		β	2.4584	0.3689	2.8984	0.0975	0.3824	0.1139	0.3752	0.0454
	5	α	0.3248	0.4647	1.1243	0.2251	0.1236	0.1264	0.1961	0.0798
		β	0.2263	0.1160	0.4200	0.0527	0.0662	0.0259	0.0328	0.0238
2	3	α	0.2270	0.2734	0.5369	0.1488	0.1329	0.1136	0.1483	0.0868
		β	0.1793	0.0905	0.3449	0.0420	0.1140	0.0593	0.0831	0.0450
	5	α	0.2175	0.2451	0.4159	0.1462	0.0993	0.0874	0.0982	0.0781
		β	0.0721	0.0385	0.0573	0.0311	0.0491	0.0355	0.0367	0.0345

Table 3: A record ranked set sample design with sample size $n = 5$ when $m = 1, 2$.

m	*RRSS*				
1	0.2100				
	0.0300	0.0500			
	0.1100	0.5900	0.9400		
	0.7300	0.7400	1.2100	1.3800	
	0.1500	0.2300	0.3100	0.7400	1.1000
2	0.1600				
	0.7800	1.1200			
	0.5000	0.5800	0.8300		
	0.0400	0.1400	0.3400	0.6500	
	0.0300	0.1700	0.3800	0.4000	0.4400

Table 4: The data of RRSS and SRS for one- and two- cycle for $n = 5$.

m	Samples				
			RRSS		
1	0.2100	0.0500	0.9400	1.3800	1.1000
2	0.1600	1.1200	0.8300	0.6500	0.4400
			SRS		
1	0.1500	0.2300	0.3100	0.7400	1.100
2	0.0300	0.1700	0.3800	0.4000	0.4400

Table 5: Bayesian estimates and MLE based on SRS and RRSS for $n = 5$.

		SRS				RRSS			
		$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$	
m	*Par.*			$c = -0.1$	$c = 1$			$c = -0.1$	$c = 1$
1	α	0.7229	1.1288	1.1530	0.9259	3.2213	3.0767	3.1441	2.4845
	β	1.9915	1.5949	1.6553	1.1542	1.5555	1.8444	1.9589	1.1675
2	α	1.9634	2.4742	2.5037	2.1898	3.8464	3.8403	3.8635	3.5966
	β	4.4192	2.8800	3.0494	1.7730	1.6857	1.6854	1.7658	1.1488

5 Conclusion

Based on both SRS and RRSS of the upper record values, Bayesian estimation and MLE are used to estimate the two unknown parameters for the LExp(α, β). The Bayes estimators obtained using both SEL and LINEX functions. Comparisons made between the different estimators based on a simulation study and real record values taken from Hand et al. [34]. We notice from results presented in Tables 1 and 2

1. The MLE and different Bayes estimates based on RRSS have the smallest MSE compared with the MLE and Bayes estimates based on SRS in all cases considered. This demonstrates the efficiency of inference based on RRSS.
2. It is clear that the Bayes estimates based on both SRS and RRSS with two-cycle ($m = 2$) are better than the Bayes estimates with one-cycle ($m = 1$). In general, the better results are obtained using a large number of cycles.
3. The the Bayes estimates relative to the LINEX loss function have the smallest MSE and bias compared with the SEL loss function of Bayes estimates.
4. It is also observed that when the sample size is large ($n = 5$), the MLE and Bayes estimates have a small MSE for RRSS and SRS.
5. The relationships in Equations (8) and (9) are quite useful in getting closed form expressions appeared frequently in the previous sections.

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Conflict of Interest

The author(s) professed that no conflicts of interest for this publication ,research and authorship of this article.

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