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Avoiding Certain Graphs for a Variation of Toughness

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Abstract: For an undirected simple graph G, a variation of toughness is defined as

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)-1}\Big|\omega(G-S)\geq 2\}$$

if G is not complete, and $\tau(G)=\infty$ if G is complete. In this paper, we determine the connected graph families $\mathscr{F}$ such that every large enough connected $\mathscr{F}$-free graph is τ-tough.

Keywords: graph, toughness, variation of toughness, marriage theorem, minimal cut

1 Introduction

We only consider simple undirected graphs in this paper. The notation and terminology used but undefined in this paper can be found in [1]. The notion of *toughness* was first introduced by chvátal in [2]: if G is complete graph, $t(G)=\infty$. If G is not complete,

$$t(G) = \min\{\frac{|S|}{\omega(G-S)}\Big|\omega(G-S)\geq 2\}$$

and where $\omega(G-S)$ is the number of connected components of $G-S$. A variation of toughness is defined as

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)-1}\Big|\omega(G-S)\geq 2\}$$

if G is not complete, and $\tau(G)=\infty$ if G is complete.

Several papers contributed to the properties of $\tau(G)$. Enomoto [3] proved that if $\tau(G)\geq k$, $k|G|$ is even, and $|G|\geq k^2-1$, then G has a k-factor. Zhou [4] presented that a graph has a fractional k-factor if $\tau(G)>k$ where $k=1,2$. Other related research can refer to [5], [6], [7] and [8].

For two given connected graphs G and H, we say G is H-free if G does not contain H as an induced subgraph. Let $\mathscr{F}$ be a family of connected graphs. We say a graph G is $\mathscr{F}$-free if G is H-free for each $H\in\mathscr{F}$. Let G be a connected graph and τ be a positive real number. A graph G is said to be τ-tough if $\tau\cdot(\omega(G-S)-1)\leq |S|$ establishes for every cutset $S\subseteq V(G)$. The $\tau(G)$ is the maximum τ for which G is τ-tough.

In this article, we first raise following problem for τ and then solve the Problem 1.

Problem 1. Let τ be a positive real number. Characterize the connected graph families $\mathscr{F}$ such that every large enough connected $\mathscr{F}$-free graph is τ-tough.

The answer is expressed in the following section. The rest of this paper is organized as follows. In next Section, we present some definitions and show our main result. In Section 3, we give the detail proofs for our main result.

2 Definitions and main result

For two connected graphs H_1 and H_2, the notion $H_1\preceq H_2$ denote that H_1 is an induced subgraph of H_2. If there are two different graphs $H_1,H_2\in\mathscr{F}$ such that $H_1\preceq H_2$, then we say a family of connected graphs $\mathscr{F}$ is redundant. Hence, our problem is restricted to consider only nonredundant families. Let $\mathscr{G}$ be the set of all nonredundant families of connected graphs, and $\mathbf{H}(\tau)$ be the set of families $\mathscr{F}\in\mathscr{G}$ satisfies that all $\mathscr{F}$-free

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connected graphs G with $|V(G)| \geq n_0$ are τ-tough with a constant $n_0 = n_0(\tau, \mathscr{F})$. In this sense, the answer of Problem 1 is reduced to determine all the elements in the set $\mathbf{H}(\tau)$.

For $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{G}$, if for each $H_2 \in \mathscr{F}_2$, there is an $H_1 \in \mathscr{F}_1$ such that $H_1 \preceq H_2$, then we say that $\mathscr{F}_1 \leq \mathscr{F}_2$. Clearly, any $\mathscr{F}_1$-free graph is also $\mathscr{F}_2$-free if $\mathscr{F}_1 \leq \mathscr{F}_2$.

Let Y_m^n be the graph obtained from identifying the center of a $K_{1,n}$ with the first vertex of a path on m vertices. The last vertex of the path is called the tail of the Y_m^n. Let $Z_{m,r}^n$ be the graph yielded by identifying one vertex of a K_r with the tail of a Y_m^n. Let $\mathscr{F}^A(m,l,r) = \{K_{1,l}, P_m, Z_{1,r}^1\}$ and $\mathscr{F}^B(m,l,r) = \{K_{1,l}, Y_{m+2}^n, Z_{1,r}^n, \cdots, Z_{m,r}^n\}$.

Now, we define the following subsets of $\mathscr{G}$.

$$\mathbf{F}^A = \{\mathscr{F} \in \mathscr{G} : \mathscr{F} \leq \mathscr{F}^A(m,l,r) \text{ for some } m \geq 4, l \geq 3 \text{ and } r \geq 3\}.$$

$$\mathbf{F}^B = \{\mathscr{F} \in \mathscr{G} : \mathscr{F} \leq \mathscr{F}^B(m,l,r) \text{ for some } m \geq 1, l \geq n+2 \text{ and } r \geq 3\}.$$

Our main result to be proved in the next section can be stated as follows:

Theorem 1. *Let τ be a positive real number. Then,*
- *If $\tau > 1$, then $\mathbf{H}(\tau) = \mathbf{F}^A$.*
- *If $0 < \tau \leq 1$, then $\mathbf{H}(\tau) = \mathbf{F}^B$, where $n = \lfloor \frac{1}{\tau} \rfloor$.*

Before on the way to proof our main result, we should give some useful definitions.

For $v \in V(G)$, let $N_G^i(v) = \{w \in V(G) : d(v,w) = i\}$. Note that $N_G^0(v) = v$ and $N_G^1(v) = N_G(v)$. We can denote $N^i(v)$ for $N_G^i(v)$ if graph G is obvious from the context. Let l and r be two positive integers. The Ramsey number $R(l,r)$ is the minimum positive integer R such that any graph of order at least R contains either an independent set of cardinality l or a clique of cardinality r.

We denote $v \sim w$ if $vw \in E(G)$ for $v,w \in V(G)$. Let $S \subseteq V(G)$ be a cutset of G and $x \in S$. Let

$$C_S(x) = \{C : C \text{ is a component of } G-S \text{ such that } N(x) \cap V(C) \neq \emptyset\}.$$

Define $C_S(X) = \cup_{x \in X} C_S(x)$ for $X \subseteq S$. We write $C(x)$ instead of $C_S(x)$ if there is no ambiguity about the set S.

A nonempty set $S \subseteq V(G)$ is a τ-tough cut if $(\omega(G - S) - 1) > \frac{|S|}{\tau}$. A τ-tough cut $S \subseteq V(G)$ is a minimal τ-tough cut if for every $S' \subset S$, S' is not a τ-tough cut. Let $S \subseteq V(G)$ be a τ-tough cut, $x \in S$ and $D \subseteq C_S(x)$ be a set of components. A set $A \subseteq V(G)$ is a selection for x from D if $A \subseteq N(x)$ and for every $C \in D$, $|A \cap V(C)| = 1$. A set $A \subseteq V(G)$ is a selection for x if A is selection for x from $C_S(x)$.

The following result is a direct corollary of Hall's marriage theorem. We will use it in the next Section.

Theorem 2. *Let G be a bipartite graph with partite sets X and Y with $X = \{x_1, \cdots, x_k\}$. Suppose that for all $X' \subseteq X$, $|N(X')| \geq n|X'|$. Then there are pairwise disjoint subsets $Y_1, \cdots, Y_k$ of Y such that for all $1 \leq i \leq k$, $Y_i \subseteq N(x_i)$ and $|Y_i| = n$.*

3 Proof of Theorem 1

The process of the proof can be divided into a number of cases.

3.1 Case $\tau > 1$

Theorem 3. *Let $\tau > 1$. Then $\mathbf{F}^A \subseteq \mathbf{H}(\tau)$.*

Proof. Let $\mathscr{F} \in \mathbf{F}^A$. Let $m \geq 4$, $l \geq 3$, and $r \geq 3$ such that $\mathscr{F} \leq \mathscr{F}^A(m,l,r)$. Let G be a connected $\mathscr{F}$-free graph. Suppose that G is not τ-tough. Hence, there exist a cutset $S \subseteq V(G)$ such that $|S| < \tau(\omega(G - S) - 1)$. We may suppose that S is minimal under inclusion.

Claim. There is a vertex $y \in N(S) - S$ such that $|N(y) \cap S| < l\tau$.

Proof. On the contrary, suppose that for all $y \in N(S) - S$, $|N(y) \cap S| \geq l\tau$. Let k be the number of pairs (x, C) with $x \in S$ and $C \in C(x)$. We have,

$$k = \textstyle\sum_{x \in S} |C(x)| \text{ and } k = \sum_{C \in C(S)} |N(C) \cap S|.$$

Then $|C(x)| < l$ for all $x \in S$ since G is $K_{1,l}$-free. We obtain

$$k = \sum_{x \in S} |C(x)| < l|S| < l\tau(\omega(G-S)-1).$$

Let $C \in C(S)$ and $y \in V(C) \cap N(S)$. Then $|N(C) \cap S| \geq |N(y) \cap S| \geq l\tau$. Therefore, $|N(C) \cap S| \geq l\tau$ for each $C \in C(S)$, and

$$k = \sum_{C \in C(S)} |N(C) \cap S| \geq l\tau|C(S)| = l\tau(\omega(G-S)-1),$$

a contradiction. □

Let y_1 be a vertex in $N(S) - S$ as in Claim 3.1 and $x_0 \in S \cap N(y_1)$. Let $C_1 \in C(x_0)$ such that $y_1 \in C_1$. $|C(S)| = \omega(G-S) \geq 2$ since S is a cutset. If $|S| = 1$, then $|C(x_0)| = |C(S)| \geq 2$. Suppose $|S| \geq 2$. If $|C(x_0)| \leq 1$, then $S' = S - \{x_0\}$ is also a cutset with $\omega(G - S') \geq \omega(G - S)$ by connected of G. Thus, $\tau(w(G - S') - 1) \geq \tau(\omega(G - S) - 1) > |S| > |S'|$. This contradicts the minimality of S. In conclusion, $|C(x_0)| \geq 2$.

So, there exist a component $C_2 \in C(x_0)$ with $C_2 \neq C_1$. Let $y_2 \in N(x_0) \cap V(C_2)$. We infer $N^{m-1}(x_0) = \emptyset$ by G is P_m-free. Next, we show that $N^i(x_0)$ is bounded for all $1 \leq i \leq m-2$.

$N(x_0)$ has no independent set of size l because $\{x_0\} \cap N(x_0)$ has no $K_{1,l}$. Since $\{y_1, x_0\} \cap (N(x_0) - N(y_1))$ contains no $Z_{1,r}^1$, $N(x_0) - N(y_1)$ does not contain a clique of size $r - 1$. Thus, $|N(x_0) - N(y_2)|$ does not contain a clique of size $r-1$ and $|(N(x_0) - N(y_1)) \cap (N(x_0) - N(y_2))| < 2R(l,r)$. Let $X = N(x_0) \cap N(y_1) \cap N(y_2)$. Since y_1 and y_2 are not in the same components of $G - S$, X and $X \subseteq S$ have neighbors in more than one component of $G - S$. We yield

$|X| < l\tau$ by the choose of y_1, and deduce that $|N(x_0)| < 2R(l,r) + l\tau$.

For $i \geq 1$, we show that $|N^{i+1}(x_0)| < R(l,r) \cdot |N^i(x_0)|$. Let $x_i \in N^i(x_0)$. It is sufficient to show that $|N(x_i) \cap N^{i+1}(x_0)| < R(l,r)$. Since $\{x_i\} \cup (N(x_i) \cap N^{i+1}(x_0))$ does not contain $K_{1,l}$, $N(x_i) \cap N^{i+1}(x_0)$ has no independent set of size l. Let $x_{i-1} \in N^{i-1}(x_0)$. $x_{i-1} = x_0$ if $i = 1$. At last, $N(x_i) \cap N^{i+1}(x_0)$ does not contain a clique of size r since $\{x_{i-1}, x_i\} \cup (N(x_i) \cap N^{i+1}(x_0))$ does not contain a $Z^1_{1,r}$.

Therefore, we obtain that for all $i \geq 0$

$$|N^i(x_0)| < R(l,r)^{i-1}|N(x_0)| < R(l,r)^{i-1}(2R(l,r) + l\tau).$$

According to $N^{m-1}(x_0) = \emptyset$, we get

$$|V(G)| = \sum_{i=0}^{m-2} |N^i(x_0)| < \sum_{i=0}^{m-2} (2R(l,r) + l\tau)R(l,r)^{i-1}$$
$$= \left(\frac{2R(l,r) + l\tau}{R(l,r)}\right)\left(\frac{R(l,r)^{m-1} - 1}{R(l,r) - 1}\right)$$

□

From the proof of Theorem 3, we lead the following more precise statement.

Theorem 4. *Let $\tau \geq 1$. Then every $\mathscr{F}^A(l,m,r)$-free connected graph G with $|V(G)| \geq n_0 = n_0(l,m,r,t)$ is τ-tough, where $n_0(l,m,r,t) = (\frac{2R(l,r)+l\tau}{R(l,r)})(\frac{R(l,r)^{m-1}-1}{R(l,r)-1})$.*

Theorem 5. *Let $\tau > 1$. Then $\mathbf{H}(\tau) \subseteq \mathbf{F}^A$.*

Proof. Let $\mathscr{F} \in \mathbf{H}(t)$. Then, there exist a positive integer n_0 satisfies that each $\mathscr{F}$-free connected graph of order at least n_0 is τ-tough. Let n_1 be an integer with $n_1 \geq \max(n_0, 3)$.

Consider the family $\mathscr{F}' = \mathscr{F}^A(n_1, n_1, n_1)$. K_{1,n_1} has toughness $\frac{1}{n_1-1} < 1$. P_{n_1} has toughness 1. Z^1_{1,n_1} has toughness 1. Hence, all the graphs in $\mathscr{F}'$ have toughness at most 1 and so none of them is τ-tough. All the graphs in $\mathscr{F}'$ are connected graphs of order at least n_0 by $n_1 \geq n_0$. Thus, no graph of $\mathscr{F}'$ is $\mathscr{F}$-free. i.e., for each graph $H' \in \mathscr{F}'$, there exist a graph $H \in \mathscr{F}$ such that $H \preceq H'$. By definition of $\mathscr{F} \leq \mathscr{F}'$ and $\mathscr{F}' \in \mathbf{F}^A$, we infer $\mathscr{F} \in \mathbf{F}^A$. □

3.2 Case $0 < t \leq 1$

Theorem 6. *Let $0 < \tau \leq 1$. Then $\mathbf{F}^B_n \subseteq \mathbf{H}(\tau)$, where $n = \lfloor \frac{1}{\tau} \rfloor$.*

We split the proof of theorem 6 in several lemmas.

Lemma 1. *Let G be a connected graph, $0 < \tau \leq 1$, and S be a minimal τ-tough cut. Then $|C_S(X)| > \frac{1}{\tau}|X|$ for each nonempty $X \subseteq S$. In particular, $|C_S(x)| > \frac{1}{\tau}$ for any $x \in S$.*

Proof. According to the definition of τ-tough cut, $(\omega(G-S) - 1) > \frac{1}{\tau}|S|$. Let $S' = S - X$. By the minimality of S, $(\omega(G - S') - 1) \leq \frac{1}{\tau}|S'|$. We have $C_S(S) - C_S(X) \subseteq C_{S'}(S')$ and $\omega(G-S) - |C_S(X)| \leq \omega(G-S')$ since each component of $G-S$ not in $C_S(X)$ is a component of $G-S'$. This implies

$$\frac{1}{\tau}|S| - |C_S(X)| < (\omega(G-S) - 1) - |C_S(X)|$$
$$\leq (\omega(G-S) - 1) \leq \frac{1}{\tau}|S'|$$
$$= \frac{1}{\tau}(|S| - |X|).$$

Then, we get $|C_S(X)| > \frac{1}{\tau}|X|$. □

Lemma 2. *Let G be a connected graph, $n \geq 2$, $0 < \tau \leq \frac{1}{n}$, S be a minimal τ-tough cut and $x_0 \in S$. If G is Y^n_m-free for some $m \geq 1$, then $N^{m}(x_0) = \emptyset$, where $m' = 2\max(n, m+1) + m$.*

Proof. Suppose $N^{m'}(x) \neq \emptyset$. Let $P = x_0 \cdot x_{m'}$ be a path satisfies that $x_i \in N^i(x_0)$. Note that P is an induced path. We use the notation $v^{+j} = x_{i+j}$ and $v^{-j} = x_{i-j}$ if $v \in P$ with $v = x_i$. Let $q = \max(n, m+1)$. A subsequence $v_1, \cdots, v_q$ of $x_0, \cdots, x_{m'}$ and sets $A_1, \cdots, A_q$ constructed with the following properties:
(i) $v_i \in S$ for all $1 \leq i \leq q$,
(ii) v_{i+1} is either v_i^{+1} or v_i^{+2} for all $1 \leq i \leq q-1$,
(iii) A_i is a selection for v_i for all $1 \leq i \leq q$, and
(iv) $|A_i - A_{i+1}| \leq n-1$ for all $1 \leq i \leq q-1$.

Choose $v_1 = x_0$ and let A_1 be any selection for x_0. Let $1 \leq i < q$ and suppose $v_1, \cdots, v_i$ and $A_1, \cdots, A_i$ are chosen. We choose v_{i+1} and A_{i+1} in the following way.

By condition (ii), $h \leq 2i - 2 \leq 2q - 4$ if $v_i = x_h$. Hence, $m' = 2q + m > h + m$ and v_i^{+j} exists for all $1 \leq j \leq m$. For $j \geq 3$, the distance between v_i and v_i^{+j} is j, $N(v_i) \cap N(v_i^{+j}) = \emptyset$ and $A_i \cap N(v_i^{+j}) = \emptyset$. Let $Y_1 = A_i \cap N(v_i^{+1})$ and $Y_2 = A_i \cap N(v_i^{+2})$.

Suppose $|Y_2| = 1$ and let $y \in Y_2$. Then, $y \sim v_i$, $y \sim v_i^{+2}$, and $y \not\sim v_i^{+j}$ for all $3 \leq j \leq m-1$. Since vertices of A_i and y are in different components of $G-S$, we have $N(y) \cap A_i = \emptyset$. By Lemma 1, $|A_i| > \frac{1}{\tau} \geq n$ and $|A_i - \{y\}| \geq n$. Note that $A_i - \{y\}$ is an independent set since the vertices of A_i are in different components. But then, $(A_i - \{y\}) \cup \{v_i, y, v_i^{+2}, v_i^{+3}, \cdots, v_i^{+m-1}\}$ contains a Y^n_m, a contradiction.

Suppose $|Y_2| = 0$ and $|Y_1| \leq 1$. We get $(A_i - Y_1) \cap N(v_i^{+1}) = \emptyset$. Also, $|A_i| \geq n+1$ and then $|A_i - Y_1| \geq n$. But $(A_i - Y_1) \cup \{v_i, v_i^{+1}, v_i^{+2}, \cdots, v_i^{+m-1}\}$ contains a Y^n_m, a contradiction. Then, we have that either $|Y_2| \geq 2$ or $|Y_2| = 0$ and $|Y_1| \geq 2$.

If $|Y_2| \geq 2$, then v_i^{+2} has neighbors in at least two components of $G-S$ and $v_i^{+2} \in S$. Choose $v_{i+1} = v_i^{+2}$ and let A_{i+1} be any selection for v_i^{+2} with $Y_2 \subseteq A_{i+1}$. Let $y \in Y_2$. Similarly, since $(A_i - A_{i+1}) \cup \{v_i, y, v_i^{+2}, v_i^{+3}, \cdots, v_i^{+m-1}\}$ does not contain a Y_m^n we have $|A_i - A_{i+1}| \le n-1$. If $|Y_2| = 0$ and $|Y_1| \ge 2$, then $v_i^{+1} \in S$. Choose $v_{i+1} = v_i^{+1}$ and let A_{i+1} be any selection for v_i^{+1} with $Y_1 \subseteq A_{i+1}$. Since $(A_i - A_{i+1}) \cup \{v_i, v_i^{+1}, v_i^{+2}, \cdots, v_i^{+m-1}\}$ does not contain a Y_m^n then $|A_i - A_{i+1}| \le n-1$.

Claim. $|A_q| \le 2(n-1)$.

Proof. For $j \ge 3$, we have $A_q \cap N(v_q^{-j}) = \emptyset$. Suppose that $A_q \cap N(v_q^{-2}) \ne \emptyset$ and let $y \in A_q \cap N(v_q^{-2})$. Since $(A_q - N(v_q^{-2})) \cup \{v_q, y, v_q^{-2}, \cdots, v_q^{-(m-1)}\}$ does not contain a Y_n^m, then $|A_q - N(v_q^{-2})| \le n-1$. Since $(A_q \cap N(v_q^{-2})) \cup \{v_q^{-2}, \cdots, v_q^{-(m-1)}\}$ does not contain a Y_n^m, then $|A_q \cap N(v_q^{-2})| \le n-1$. Then $|A_q| = |A_q - N(v_q^{-2})| + |A_q \cap N(v_q^{-2})| \le (n-1)+(n-1) = 2(n-1)$.

Suppose $A_q \cap N(v_q^{-2}) = \emptyset$. Since $(A_q - N(v_q^{-1})) \cup \{v_q, v_q^{-1}, v_q^{-2}, \cdots, v_q^{-(m-1)}\}$ does not contain a Y_n^m, then $|A_q - N(v_q^{-1})| \le n-1$. Since $(A_q \cap N(v_q^{-1})) \cup \{v_q^{-1}, v_q^{-2}, \cdots, v_q^{-m}\}$ does not contain a Y_n^m, then $|A_q \cap N(v_q^{-1})| \le n-1$. Then $|A_q| = |A_q - N(v_q^{-1})| + |A_q \cap N(v_q^{-1})| \le (n-1)+(n-1) = 2(n-1)$. □

By Lemma 1, we deduce

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_q| &\ge |C_S(v_1) \cup \cdots \cup C_S(v_q)| \\
&= |C_S(v_1, \cdots, v_q)| \\
&> nq.
\end{aligned}
$$

Furthermore, we yield

$$
\begin{aligned}
& |A_1 \cup \cdots \cup A_q| \\
= & |A_1 - \cup_{i=2}^{q} A_i| + |A_2 - \cup_{i=3}^{q} A_i| \\
& + \cdots + |A_{q-1} - A_q| + |A_q| \\
\le & |A_1 - A_2| + |A_2 - A_3| + \cdots + |A_{q-1} - A_q| + |A_q| \\
\le & (n-1)(q-1) + 2(n-1) = (n-1)(q+1).
\end{aligned}
$$

Hence, $(n-1)(q+1) > nq$ and then $q < n-1$, which contradicts $q = \max(n, m+1)$. □

Lemma 3. *Let G be a connected graph, $n \ge 2$, $0 < \tau\frac{1}{n}$, and S be a minimal τ-tough cut. Let $X \subseteq S$ be a clique. If G is $\{K_{1,l}, Z_{1,r}^n\}$-free for some $r \ge 3$ and $l \ge n+2$, then $|X| < l(r-1)$.*

Proof. Let $Y = C(X)$ and $Y_x = C(x)$ for any $x \in X$.

Claim. For each $x \in X$, there exist a set $Y_x \subseteq V(G)$ that is a selection for x from some set $Y_x^{'} \subseteq Y_x$ with $|Y_x| = n$, and so that for all $x_1, x_2 \in X (x_1 \ne x_2)$, $Y_{x_1}^{'} \cap Y_{x_2}^{'} = \emptyset$.

Proof. Let G' be the bipartite graph with vertex set $V(G') = X \cup Y$ and edge set $E(G') = \{(x, C) : x \in X, C \in Y_x\}$. By $X \subseteq S$ and Lemma 1, for all $X' \subseteq X$, we have $|N_{G'}(X')| = |C(X')| > n|X'|$. Applying Theorem 2 to G', for each $x \in X$ there exist a set $Y_x^{'} \subseteq Y_x$ with $|Y_x^{'}| = n$ and for all $x_1, x_2 \in X (x_1 \ne x_2)$, $Y_{x_1}^{'} \cap Y_{x_2}^{'} = \emptyset$. For each $x \in X$, let $Y_x \subseteq V(G)$ be a selection for x from $Y_x^{'}$. Then, the claim holds. □

Let $x \in X$. If $|X - N(Y_x)| \ge r-1$ then $Y_x \cup \{x\} \cup (X - N(Y_x))$ contains a $Z_{1,r}^n$, a contradiction. Then for all $x \in X$, $|X - N(Y_x)| < r-1$. Suppose that $|X| \ge l$. Let $x_1, \cdots, x_l \in X$. If there exist a vertex $x \in X - \cup_{i=1}^{l}(X - N(Y_{x_i}))$, then for all $1 \le i \le l$, we have $N(x) \cap Y_{x_i} \ne \emptyset$. Note that the Y_{x_i}'s are selections from pairwise disjoint $Y_{x_i}^{'}$'s, hence $N(x) \cup \cup_{i=1}^{l} Y_{x_i}$ contains a $K_{1,l}$, a contradiction. Thus $X = \cup_{i=1}^{l}(X - N(Y_{x_i}))$. But $|X| = |\cup_{i=1}^{l}(X - N(Y_{x_i}))| < l(r-1)$. □

Lemma 4. *Let G be a connected graph, $n \ge 2$, $0 < \tau \le \frac{1}{n}$, S be a minimal τ-tough cut and $x_0 \in S$. Let $X \subseteq N(x_0)$ be a clique and $q = r(l+1)$. If G is $Z_{1,r}^n$-free for some $r \ge 3$, then $|X| < q$.*

Proof. Let $X_1 = X - S$ and $X_2 = X \cap S$. We have $|X_2| < l(r-1)$ by Lemma 3. Let Y_0 be a selection for x_0. By Lemma 1, $|Y_0| \ge n+1$. Let Y be any subset of Y_0 with $|Y| = n+1$. Since $X_1 \cap S = \emptyset$, then there exist a component C of $G - S$ with $X_1 \subseteq V(C)$. Let $Y' = Y \cap V(C)$. Then $|Y'| \le 1$ and $|Y - Y'| \ge n$. By $X_1 \subseteq V(C)$, there are no edges between $Y - Y'$ and X_1. We infer $X_1 < r$ since $(Y - Y') \cup \{x_0\} \cup X_1$ does not contain a $Z_{1,r}^n$. Thus, $|X| = |X_1| + |X_2| < r + l(r-1) < r(l+1) = q$. □

Lemma 5. *Let G be a connected graph, $n \ge 2$, $0 < \tau \le \frac{1}{n}$, S be a minimal τ-tough cut, and $x_0 \in S$. Let $x_1 \in N(x_0)$ and $X \subseteq N(x_1) \cap N^2(x_0)$ be a clique. If G is $\{Z_{1,r}^n, Z_{2,r}^n\}$-free for some $r \ge 3$, then $|X| < q$, where $q = r(l+1)$.*

Proof. If $x_1 \in S$, then $|X| < r(l+1)$ by Lemma 4. We suppose that $x_1 \notin S$. Let $X_1 = X - S$ and $X_2 = N \cap S$. We get $|X_2| < l(r-1)$ from Lemma 3. Let Y_0 be a selection for x_0. Then $|Y_0| \ge n+1$ By Lemma 1. Let Y be any subset of Y_0 with $|Y| = n+1$.

Since $X_1 \cap S = \emptyset$, then there exist a component C of $G - S$ satisfies that $X_1 \subseteq V(C)$. Suppose $x_1 \in V(C)$. Let $Y' = Y \cap V(C)$. Then $|Y'| \le 1$ and $|Y - Y'| \ge n$. Moreover, since $x_1 \in V(C)$ and $X_1 \subseteq V(C)$, there are no edges between x_1 and $Y - Y'$, and no edges between X_1 and $Y - Y'$. But by $(Y - Y') \cup \{x_0, x_1\} \cup X_1$ does not contain a $Z_{2,r}^n$, we obtain $|X_1| < r$. Hence, $|X| = |X_1| + |X_2| < r + l(r-1) < r(l+1) = q$. □

Lemma 6. *Let G be a connected graph, $n \ge 2$, $0 < \tau \le \frac{1}{n}$, S be a minimal τ-tough cut, $x_0 \in S$, $i \ge 0$ and $q = r(l+1)$. If G is $\{K_{1,l}, Z_{1,r}^n, \cdots, Z_{i+1,r}^n\}$-free for some $r \ge 3$ and $l \ge n+2$, then $|N^{i+1}(x_0)| < |N^i(x_0)| \cdot R(l,q)$.*

Proof. Let $x_i \in N^i(x_0)$. Note that $x_i = x_0$ if $i = 0$. We infer $N(x_i) \cap N^{i+1}(x_0)$ does not contain an independent set of size at least l since $\{x_i\} \cup (N(x_i) \cap N^{i+1}(x_0))$ does not contain a $K_{1,l}$. Let $X \subseteq N(x_i) \cap N^{i+1}(x_0)$ be a clique. Let $P = x_0 \cdots x_i$ be a path from x_0 to x_i such that for all $0 \le j \le i$, $x_j \in N^j(x_0)$. Note that P is an induced path. Let

$k = \max\{j : 0 \leq j \leq i$ and $x_j \in S\}$. Such an index k exists by $x_0 \in S$. If $k = i$ or $k = i-1$ then the result draws from Lemma 4 and Lemma 5 respectively by taking x_k as the x_0 in the corresponding lemma.

Suppose that $k \leq i-2$. Let Y be a selection for x_k. By Lemma 1, we get $|Y| \geq n+1$. Let P' be the subpath of P going from x_k to x_i. Then P' is a shortest path from x_k to x_i, $|N(Y) \cap P| \subseteq \{x_k, x_{k+1}, x_{k+2}\}$, and $|N(Y) \cap X| = \emptyset$. Let $Y_1 = Y \cap N(x_{k+1})$ and $Y_2 = Y \cap N(x_{k+2})$. We deduce that none of x_{k+1} and x_{k+2} is in S and hence $|Y_1| \leq 1$ and $|Y_2| \leq 1$.

Suppose that $|Y_2| = 1$ and let $y \in Y_2$. We obtain $|X| < r < r(l+1) = q$ since $(Y - \{y\}) \cup \{x_k, y, x_{k+2}, \cdots, x_i\} \cup X$ does not contain a $Z^n_{i-k+1,r}$. Hence, we suppose $|Y_2| = 0$. Since $|Y_1| \leq 1$, then $|Y - Y_1| \geq n$. But then, according to $(Y - Y_1) \cup \{x_k, x_{k+1}, x_{k+2}, \cdots, x_i\} \cup X$ does not contain a $Z^n_{i-k+1,r}$, we yield that $|X| < r < q$. So, we conclude that $|N(x_i) \cap N^{i+1}(x_0)| < R(l,q)$. □

Proof of Theorem 6. Let $\mathscr{F} \in \mathbf{F}^B_n$, $m \geq 1$, $l \geq n+2$, and $r \geq 3$ such that $\mathscr{F} \leq \mathscr{F}^B_n(m,l,r)$. Let G be an $\mathscr{F}$-free connected graph. Suppose that G is not τ-tough. Then, G has a τ-tough cut. We suppose S is a minimal τ-tough cut. Let $x_0 \in S$.

Notice that G is $Z^n_{i,r}$-free for all $i \geq m+1$ since G is Y^n_{m+2}-free. We can infer that G is $Z^n_{i,r}$-free for all $i \geq 1$. Note that $\tau \leq \frac{1}{n}$ by $n = \lfloor \tau \rfloor$. Hence, G satisfies all the conditions of Lemmas 2 and 6.

Let $m' = 2 \cdot \max(n, m+1) + m$. Using Lemma 2, we have $N^{m'}(x_0) = \emptyset$. Thus, it is sufficient to show that $N^i(x_0)$ is bounded for each $1 \leq i \leq m'-1$. Let $q = r(l+1)$. By Lemma 6, $|N^{i+1}(x_0)| < R(l,q) \cdot |N^i(x_0)|$ for all $i \geq 0$. We obtain $|N^i(x_0)| < R(l,q)^{i-1}$ for all $i \geq 1$. Since $N^{m'}(x_0) = \emptyset$, we infer $|N^i(x_0)| < R(l,q)^{m'-2}$ for all $1 \leq i \leq m'-1$. □

Theorem 7. *Let* $0 < \tau \leq 1$. *Then* $\mathbf{H}(\tau) \subseteq F^B_n$, *where* $n = \lfloor \frac{1}{\tau} \rfloor$.

Proof. Let $\mathscr{F} \in \mathbf{H}(\tau)$. Then there exist a positive integer n_0 such that every $\mathscr{F}$-free connected graph of order at least n_0 is τ-tough. Let n_1 be an integer with $n_1 \geq \max(n_0, n+2)$.

Consider the family $\mathscr{F}' = \mathscr{F}_n(n_1, n_1, n_1)$. Note that $\mathscr{F}' \in \mathbf{F}^B_n$. K_{1,n_1} has toughness $\frac{1}{n_1-1}$. $Y^n_{n_1+2}$ has toughness $\frac{1}{n}$. Z^n_{m,n_1} has toughness $\frac{1}{n}$ for all $1 \leq m \leq n_1$. Thus, all the graphs in $\mathscr{F}'$ have toughness at most $\frac{1}{n}$. Since $n = \lfloor \frac{1}{\tau} \rfloor$, then $\tau > \frac{1}{n}$ and so no graph of $\mathscr{F}'$ is τ-tough. Just as in Theorem 5, we obtain $\mathscr{F} \in \mathbf{F}^B_n$. □

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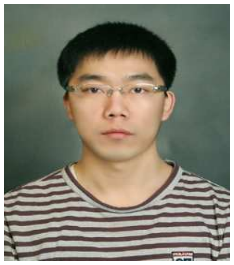

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