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Two Self-Dual Lattices of Signed Integer Partitions

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Abstract: In this paper we study two self-dual lattices of signed integer partitions, $D(m,n)$ and $E(m,n)$, which can be considered also sub-lattices of the lattice $L(m,2n)$, where $L(m,n)$ is the lattice of all the usual integer partitions with at most m parts and maximum part not exceeding n. We also introduce the concepts of k-covering poset for the signed partitions and we show that $D(m,n)$ is 1-covering and $E(m,n)$ is 2-covering. We study $D(m,n)$ and $E(m,n)$ as two discrete dynamical models with some evolution rules. In particular, the 1-covering lattices are exactly the lattices definable with one outside addition rule and one outside deletion rule. The 2-covering lattices have further need of another inside-switch rule.

Keywords: Graded lattices, Integer partitions, Young Diagrams, Discrete Dynamical Models.

1 Introduction

A wide range of papers have studied integer partition posets considered as discrete dynamical systems. Under this approach, the study of the poset is carried out locally, by means of certain rules of evolution. A rule of evolution is a way to deterministically generate a given partition of the poset from another partition on the same poset, and this generation can be accomplished if the partition on which the rule acts has certain characteristics. If the rules of evolution have been determined correctly, it follows that a partition w of the poset covers another partition w' if and only if w' is obtained from w with some evolution rules.

In this paper, we at first generalize to the case of the integer signed partitions the usual order of the classical Young lattice. The concept of signed partition has been recently introduced by Andrews in [9] : a *signed partition* is a finite sequence of integers $a_k,\dots,a_1,a_{-1},\dots,a_{-l}$ such that $a_k \geq \cdots \geq a_1 > 0 > a_{-1} \geq \cdots \geq a_{-l}$. We remark that when Andrews introduced the modern concept of signed partition, he was expanding on apparent acceptance of the possibility of negative parts in a partition in Euler's original work on the subject. Also in [22] have been studied several combinatorial and arithmetical properties of the signed partitions. However in both [9] and [22] the signed partitions are not studied from an order point of view and at present the unique studies in this direction are only partially outlined in [11], [12], [13], [19], [20].

The partial order $\sqsubseteq$ that we obtain on the set P_* of all the signed partitions is a lattice that contains the Young lattice $\mathbb{Y}$ as a sublattice. The importance of the lattice P_* is related to the fact that each finite, or also infinite, sublattice T of $\mathbb{Y}$ can be translated into a corresponding isomorphic sublattice T_* of P_*. Then, in T_*, it is more easy to see eventual symmetric or self-duality properties of T. The process of transition from T to T_* can be considered analogous to a process of reduction to a canonical form of a geometrical equation.

Moreover, it is easy to show that several sublattices of P_* (see for example a class of sublattices defined in [11] and also the sublattices studied in this paper) have an involution structure. An *involution poset* is a partially ordered set $(X,\leq)$ together with an idempotent anti-automorphism $c: X \to X$ that makes X self-dual. The involution posets generalize the Boolean algebras and they are studied in several papers, see for example [1], [2], [3], [14], [18]. Moreover, the involution structures are closely connected with the well-developed theories of Effect Algebra and Quantum Logic, as mentioned, for example in the book [28].

If $m \in \mathbb{Z}$, we can consider the subset $Par(m)$ of P_* whose elements are all the signed integer partitions

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having sum m. Then $Par(m)$ is an infinite subset of P_* that we can think of as a horizontal axis. We introduce the concept of *k-covering sublattice* of P_* so that, in particular, a 1-covering sublattice L is one that intersects each horizontal axis $Par(m)$, when the integer m runs between the sums made respectively on the summands of the minimum signed partition and the maximum signed partition of L.

In the lattice $(P_*, \sqsubseteq)$ we define three evolution rules and we study two finite sublattices of P_*, denoted respectively by $D(m,n)$ and $E(m,n)$, which are "definable" by means of a subset of such rules. The lattice $D(m,n)$ can be considered as a natural extension of the lattice $L(m,n)$ introduced by Stanley in his classical paper [29]. It is defined as the set of all the signed integer partitions with at most m parts and whose sum of the maximum positive part with the minimum negative part does not exceed n. $E(m,n)$ is the sublattice of all the signed partitions of $D(m,n)$ having exactly m non-zero parts.

In particular, we show that $D(m,n)$ is 1-covering and we compute its cardinality. Next, we define a sublattice $E(m,n)$ of $D(m,n)$ and we show that it is 2-covering but it is not 1-covering. Both these lattices have an involution map which makes them self-dual. The 1-covering lattices are exactly the lattices definable with one outside addition rule and one outside deletion rule. The 2-covering lattices have further need of another inside-switch rule.

The way to study a lattice of classical partitions as a discrete dynamical model having some particular evolution rules begins implicitly in [16], where Brylawski proposed a dynamical approach to study the lattice $L_B(m)$ of all the partitions of a fixed positive integer m with the dominance order. The Brylawski model has two evolution rules, and it can be described as follows.

If m is a non negative integer, a configuration of the model is represented by an ordered partition of m, i.e. a decreasing sequence $a = (a_1,...,a_m)$ of non negative integers having sum m, and each positive part is interpreted as a column of movable blocks whose movement respects the following rules:

Rule 1 (vertical rule): *one block can move from a column to the next column if the difference of height of these two columns is greater than or equal to 2.*

Rule 2 (horizontal rule): *If a column containing $p+1$ blocks, is followed by a sequence of columns containing p blocks and next by one column containing $p-1$ blocks, then one block of the first column can slip to the last column.*

In the scope of the discrete dynamical systems (see [4], [5], [6], [7], [8], [10], [21], [27]), the Brylawski lattice can be interpreted then as the model $L_B(m)$, where the movement of a movable block respects the previous Rules 1 and 2. In this paper we study the lattices $D(m,n)$ and $E(m,n)$ as two discrete dynamical models having respectively two and three evolution rules.

We conclude this introduction recalling that there are several recent studies concerning discrete dynamical models that use integer partitions as their configurations: see for example [15], [17], [23], [24], [25].

2 The Signed Partitions Lattice

We begin with the concept of signed partition introduced in [9] and studied in [22] from an arithmetical point of view.

Definition 2.1 Let t and s be two non-negative integers. A signed partition (briefly an s-partition) w with signature (t,s) is a finite sequence of integers $a_t,\ldots,a_1,b_1,\ldots,b_s$, called *parts of* w, such that $a_t \geq \cdots \geq a_1 > 0 > b_1 \geq \cdots \geq b_s$. We write w in the form $w = a_t \ldots a_1 | b_1 \ldots b_s$.

An s-*partition* w is an s-partition having signature (t,s), for some non-negative integers t and s. If $t = s = 0$ we also formally consider the empty signed partition, which we denote by $(|)$. We call $a_t,\ldots,a_1$ the *positive parts of* w and $b_1,\ldots,b_s$ the *negative parts of* w.

In all the numerical examples and also in the graphical representation of the Hasse diagrams, we omit the minus sign for all the parts $b_1,\ldots,b_s$. This means, for example, that we shall write $w = 44|113$ instead of $w = 44|(-1)(-1)(-3)$.

If $w = a_t \ldots a_1 | b_1 \ldots b_s$, we set $w_+ := a_t \ldots a_1|$ and $w_- := |b_1 \ldots b_s$. If m is an integer such that $m = a_q + \cdots + a_1 + b_1 + \cdots + b_p$, we say that w is an *s-partition of the integer* m and we shall write $w \vdash m$.

We shall denote by P_* the set of all the s-partitions. We consider now the distributive lattice $\mathbb{Y} \times \mathbb{Y}^*$, where $\mathbb{Y}$ is the Young lattice and $\mathbb{Y}^*$ its dual. We write the partitions of $\mathbb{Y}$ in decreasing form and the partitions of $\mathbb{Y}^*$ in increasing form. We also denote by $\leqslant$ the partial order on $\mathbb{Y} \times \mathbb{Y}^*$. Since the map $\phi : P_* \to \mathbb{Y} \times \mathbb{Y}^*$ such that

$$\phi(a_t \ldots a_1 | b_1 \ldots b_s) := ((a_t,\ldots,a_1),(-b_1,\ldots,-b_s))$$

and

$$\phi((|)) := (0,0)$$

is bijective, we can consider on P_* the induced order $\sqsubseteq$ from $\leqslant$.

Therefore, if $w, w' \in P_*$, we define:

$$w \sqsubseteq w' :\Longleftrightarrow \phi(w) \leqslant \phi(w') \tag{1}$$

For example, $322|1133 \sqsubseteq 33311|2$ because $((3,2,2),(1,1,3,3)) \leqslant ((3,3,3,1,1),(2))$ in $\mathbb{Y} \times \mathbb{Y}^*$.

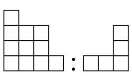
We call $(P_*, \sqsubseteq)$ the *signed partitions poset*. In the sequel, when we must compare two signed partitions w and w' with respect to the partial order $\sqsubseteq$ it will be convenient to adjoin a sufficient number of 0's in w_+ or in w'_+ in order to make them of the same length, and analogously for w_- and w'_-. For example, if we must compare the previous s-partitions $w = 322|1133$ and $w' = 33311|2$, we write $w = 32200|1133$ and $w' = 33311|0002$. In this way, the relation $322|1133 \sqsubseteq 33311|2$ follows from the comparisons of the components: $3 \le 3, 2 \le 3, 2 \le 3, 0 \le 1, 0 \le 1, -1 \le 0, -1 \le 0, -3 \le 0, -3 \le -2$.

We call the process of adding 0s in order to make w_+, w'_+ and w_-, w'_- of the same length *uniformization* of w and w'. Obviously a similar uniformization process can be carried out also when we have a finite set of s-partitions. Therefore, if U is a finite subset of P_*, in the sequel we implicitly make a *uniformization* of U, i.e. we add a necessary quantity of 0s in all the s-partitions $w \in U$ in order to make all their non-negative parts and all their non-positive parts of the same length. In particular, if after the uniformization of U, the resulting non-negative parts of all the s-partitions in U have length q and the resulting non-positive parts of all the s-partitions in U have length p, we say that U is made (q, p)*-uniform*.

After the uniformization, to describe a generic s-partition w we shall use the notation $w = a_q \dots a_1|b_1 \dots b_p$, or $w = (a_q, \dots, a_1|b_1, \dots, b_p)$, where $a_q \ge \dots \ge a_{q-t+1} \ge 0 = a_{q-t} = \dots = a_1 = b_1 = \dots = b_{p-s} = 0 > b_{p-s+1} \ge \dots \ge b_p$, and we set $|w|_{\ge} := q$, $|w|_{\le} := p$, $|w|_{>} := t$, $|w|_{<} := s$, $||w|| := |w|_{>} + |w|_{<}$, $M^+(w) := |a_q|$, $M^-(w) := |b_p|$. Sometimes, if it is not necessary to distinguish which parts of w are non-negative integers and which are non-positive integers, we simply write $w = l_1 \dots l_n$, where $n = q + p$, without specifying the sign of the l_i's. If $w, w' \in P_*$, we write $w \sqsubset w'$ (or $w' \sqsupset w$) if $w \sqsubseteq w'$ and $w \ne w'$.

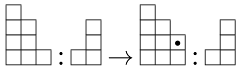
If $w = l_1 \dots l_n \in P_*$ and $1 \le k \le n$ we set $A_k(w) := l_1 \dots l_{k-1}(l_k + 1)l_{k+1} \dots l_n$ if this is an element of P_*. If $1 \le k < s \le n$ we set $A_{ks}(w) := l_1 \dots l_{k-1}(l_k + 1)l_{k+1} \dots \dots l_{s-1}(l_s + 1)l_{s+1} \dots l_n$ if this is an element of P_*.

If $w = a_q \dots a_1|b_1 \dots b_p \in P_*$, we identify w with an ordered pair $D = D_1 : D_2$ of Young diagrams, where D_1 is the Young diagram of the positive parts of w, built with decreasing columns rather than with decreasing rows, and D_2 is the Young diagram of the absolute values of the negative parts of w, built with increasing columns. We call $D = D_1 : D_2$ the *signed Young diagram* (briefly sgYD) of w. For example, if $w = 43310000|00000113 \in P_*$, then we identify w with the following sgYD:

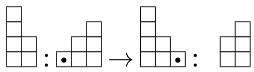


In this context, we adopt the terminology concerning the Sand Piles Models in order to describe some finite sublattices of P_* as discrete dynamic models. Our aim is to prove some properties of these sublattices using some evolution rules for discrete dynamical models. Let $w = a_q \dots a_1|b_1 \dots b_p$ be a fixed element in P_*. We use the letter $i \in \{q, \dots, 1\}$ to identify some non-negative part a_i of w and the letter $j \in \{1, \dots, p\}$ to identify some non-positive part b_j of w. We call a_i the i^{th}*-plus pile* of w and b_j the j^{th}*-minus pile* of w. We call *block* a stacked square in the sgYD of w. We always assume conventionally that $a_0 = b_0 := 0$.
We define the following evolution rules:

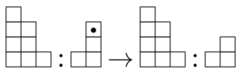
Outside Addition (R_1): If $i \in \{q, \dots, 1\}$ then $w \mapsto A_i(w)$, i.e. we add a block on the i^{th}-plus pile of w.



Switch Right-Left (R_2): If $i \in \{q, \dots, 1\}$ and $j \in \{1, \dots, p\}$ then $w \mapsto A_{ij}(w)$, i.e. at the same time we delete a block from the j^{th}-minus pile of w and we add a block on the i^{th}-plus pile of w.



Outside Deletion (R_3): If $j \in \{1, \dots, p\}$ then $w \mapsto A_j(w)$, i.e. we delete a block from the j^{th}-minus pile of w.



If $w' \in P_*$ is obtained from w with the application of the rule R_k, for some $k \in \{1, 2, 3\}$, we write $w' = R_k(w)$. Now let X be a generic subposet of P_* and $K \subseteq \{R_1, R_2, R_3\}$. We say that X is *definable with* K if K is the smallest subset of $\{R_1, R_2, R_3\}$ such that: whenever $w, w' \in X$, it results that w' covers w in X if and only if $w' = R_k(w)$ for some $R_k \in K$. In our work, a relevant role is played by two classes of sub-posets of $(P_*, \sqsubseteq)$ that we have called k-covering posets and k-stable posets, respectively. Therefore in this section we define such posets and we prove some useful properties for the 1-covering posets. Let $w = l_1 \dots l_n$ and $w' = l'_1 \dots l'_n$ be two distinct s-partitions in P_* and let h be an integer such that $1 \le h \le n$. We write $w' \downarrow^h w$ if w' and w differ in exactly h places $i_1, \dots, i_h$ and $l'_{i_1} = l_{i_1} + 1 \dots, l'_{i_h} = l_{i_h} + 1$.

Definition 2.2. Let k be a fixed positive integer. If $(U, \sqsubseteq)$ is a sub-poset of $(P_*, \sqsubseteq)$, we say that U is k*-covering* if :
(i) whenever $w, w' \in U$ and w' covers w, then $w' \downarrow^h w$ for some integer $1 \le h \le k$;
(ii) there are at least two partitions $w, w' \in U$ such that w' covers w and $w' \downarrow^k w$.

Remark 2.3. (i) If U contains at least two comparable distinct elements, then:
U is 1-covering $\Longleftrightarrow$ (whenever w and w' are two distinct s-partitions in U, w' covers w iff $w' \downarrow^1 w$) $\Longleftrightarrow$ U is definable with $\{R_1, R_3\}$.
(ii) P_* is 1-covering.

Definition 2.4. If $(U,\sqsubseteq)$ is a sub-poset of $(P_*,\sqsubseteq)$, we say that U is k-stable if $||w||=k$ for all $w\in U$.

We recall now some basic facts concerning the graded posets (see for example [30], cap.3). A finite poset X having a minimum $\hat{0}$ is said *graded* of *rank* l if all the maximal chains in X have the length l, in this case the non negative integer l is called *rank* of X and we denote it by $rank(X)$. If X is a graded poset of rank l, then it can easily proved that there exists a unique function $\rho : X\to\mathbb{N}$ (called *rank function* of X) such that $\rho(\hat{0})=0$ and $\rho(y)=\rho(x)+1$ if $x,y\in X$ and y covers x. If k is an integer such that $0\le k\le rank(X)$, the subset $\{x\in X : \rho(x)=k\}$ is denoted by $N_k(X)$ and it is called the *k-level set* of X. It is easily seen that the family $\{N_0(X),N_1(X),\dots,N_l(X)\}$ is a set partition of X. Finally, a finite distributive lattice is a graded poset. In our context, each sub-lattice of $(P_*,\sqsubseteq)$ is distributive, therefore each finite sub-lattice of P^* is a graded poset.
We define now the function $\vartheta : P_*\to\mathbb{Z}$ such that

$$\vartheta(a_q\dots a_1|b_1\dots b_p):=a_q+\cdots a_1+b_1\cdots+b_p,\ \vartheta((|)):=0 \tag{2}$$

Proposition 2.5. Let $(U,\sqsubseteq)$ be a finite sub-lattice with minimum $\hat{0}$ of $(P_*,\sqsubseteq)$ and let $\rho : U\to\mathbb{N}$ such that $\rho(w):=\vartheta(w)-\vartheta(\hat{0})$ for each $w\in U$. Then U is 1-covering if and only if ρ is the rank function of U. Moreover, in this case, $N_k(U)=\{w\in U : w\vdash k+\vartheta(\hat{0})\}$ for $k=0,1,\dots,rank(U)$.

Proof. Straightforward. □

3 Two natural extensions of the lattice $L(m,n)$ and some related sublattices

Let m and n be two fixed non-negative integers. In this section we introduce and study two sublattices of P_* which are both natural extensions of the classical lattice $L(m,n)$ of all the usual integer partitions with at most m parts and maximum part not exceeding n (see [29]). We set

$$C(m,n):=\{w\in P_* : ||w||\le m, M^+(w)\le n, M^-(w)\le n\} \tag{3}$$

and

$$D(m,n):=\{w\in P_* : ||w||\le m, M^+(w)+M^-(w)\le n\} \tag{4}$$

In the sequel we make (m,m)-uniform both $C(m,n)$ and $D(m,n)$. It is clear that $C(m,n)$ and $D(m,n)$ are both distributive sublattices of P_*, and $D(m,n)$ is obviously a sublattice of $C(m,n)$. Now we recall the concept of involution poset. An *involution poset* (IP) is a poset $(X,\le,c)$ with a unary operation $c : x\in X\mapsto x^c\in X$, such that:

$I1)(x^c)^c=x$, for all $x\in X$;
$I2)$if $x,y\in X$ and if $x\le y$, then $y^c\le x^c$.

The map c is called *complementation* of X and x^c the *complement* of x. Let us observe that if X is an involution poset, by $I1)$ follows that c is bijective and by $I1)$ and $I2)$ it holds that if $x,y\in X$ are such that $x<y$, then $y^c<x^c$. If $(X,\le,c)$ is an involution poset and if $Z\subseteq X$, we will set $Z^c=\{z^c : z\in Z\}$. We note that if X is an involution poset then X is a self-dual poset because from $I1)$ and $I2)$ it follows that if $x,y\in X$ we have that $x\le y$ iff $y^c\le x^c$ and this is equivalent to saying that the complementation is an isomorphism between X and its dual poset X^*. If $w = a_m\dots a_1|b_1\dots b_m \in C(m,n)$, we set $w^T=(-b_m)\dots(-b_1)|(-a_1)\dots(-a_m)$. It is easy to verify then that the map $w\mapsto w^T$ is an involution both on $C(m,n)$ and $D(m,n)$. Hence $C(m,n)$ and $D(m,n)$ are both involution posets and therefore ,in particular, they are both self-dual posets.

Proposition 3.1 The lattice $C(m,n)$ is isomorphic to the lattice $L(m,2n)$.

Proof. We write a generic element of $L(m,2n)$ as

$$w=r_m\dots r_{m-k+1}r_{m-k}\dots r_{m-s+1}r_{m-s}\dots r_1,$$

where $2n\ge r_i>n$ for $i=m,\dots,m-k+1$, $r_i=n$ for $i=m-k,\dots,m-s+1$ and $n>r_i\ge 0$ for $i=m-s,\dots,1$. Now we consider the application $\phi : L(m,2n)\to C(m,n)$ defined as $\phi(w) := (r_m-n)\dots(r_{m-k+1}-n)0_{m-k}\dots0_{m-s+1}0_{m-s}\dots0_1|0_1\dots0_s(r_{m-s}-n)\dots(r_1-n)$, where 0_i means 0 in the $i-$th place. It is easy to prove that ϕ is surjective: let $v=t_mt_{m-1}\dots t_1|u_1u_2\dots u_m = t_m\dots t_{m-k+1}0_{m-k}\dots0_1|0_1\dots0_su_{s+1}\dots u_m$ be an element of $C(m,n)$ with $n\ge t_i\ge 1$ for $i=m,\dots,m-k+1$ and $-1\ge u_j\ge -n$ for $j=s+1,\dots,m$. Since $v\in C(m,n)$ implies $k+(m-s)\le m$, if we take $w=(n+t_m)(n+t_{m-1})\dots(n+t_{m-k+1})(n)_{m-k}\dots(n)_s(n+u_{s+1})\dots(n+u_m)$, where $(n)_i$ means n in the $i-$th place, then $w\in L(m,2n)$ and $\phi(w)=v$, i.e. the application ϕ is surjective. It is immediate to prove that ϕ is injective and that ϕ and its inverse ϕ^{-1} are order preserving. □

Corollary 3.2. $C(m,n)$ is 1-covering and $|C(m,n)|=\binom{m+2n}{m}$.

Proof. $L(m,n)$ is 1-covering. Hence $C(m,n)$ is also 1-covering, since the isomorphism ϕ in the proof of the previous proposition is 1-covering preserving. The other part of the thesis follows from the well known formula $|L(m,n)|=\binom{m+n}{m}$. □
The formula for the cardinality of $D(m,n)$ is not much more complicated.

Proposition 3.3. $|D(m,n)| = 1$ if $n=0$ and $\binom{m+n-1}{n-1}\left(\frac{m+mn+n}{n}\right)$ otherwise.

Proof. (We cordially thank an anonymous referee for much simplifying the proof of this proposition and that of the analogous proposition in Section 4.)

The elements in $D(m,n)$ with no negative parts are simply normal partitions with at most m parts of size at most n, known to be counted by $\binom{n+m}{m}$. Those with largest negative part of size $-k$, by addition of k to all parts, become normal partitions with at most $m-1$ parts of size at most n. Since $0 < k \leq n$, we have $|D(m,n)| = \binom{n+m}{m} + n\binom{n+m-1}{m-1} = \binom{m+n-1}{n-1}\left(\frac{m+mn+n}{n}\right)$ as claimed. □

We remark that these values, as a table, are sequence A103450 in the OEIS [26]. They are symmetric under the exchange of n and m, as can also be seen visually from the operation of several of the forms of conjugation of signed partitions given in [22].

The next result shows that the lattice $D(m,n)$ is definable only with the rules R_1 and R_3.

Theorem 3.4. $D(m,n)$ is 1-covering.

Proof. Let $w = (a_m,\dots,a_1|b_1,\dots,b_m) = l_1 \dots l_{2m}$ a generic element of $D(m,n)$. In order to simplify the proof it will be convenient to state explicitly all the subcases when the outside addition and the outside deletion are applicable in the case of $D(m,n)$. They are the following:
D_1: If $a_m = 0$, $|b_m| < n$ and $||w|| < m$, then we apply R_1 on the m^{th}-plus pile of w, that is $a_m = 0 \mapsto 1$.
D_2: If $a_m > 0$ and $a_m + |b_m| < n$, then we apply R_1 on the m^{th}-plus pile of w, that is $a_m \mapsto a_m + 1$.
D_3: If there is $i \in \{m-1,\dots,1\}$ such that $a_{i+1} > a_i = 0$ and $||w|| < m$, then we apply R_1 on the i^{th}-plus pile of w, that is $a_i = 0 \mapsto 1$.
D_4: If there is $i \in \{m-1,\dots,1\}$ such that $a_{i+1} > a_i > 0$, then we apply R_1 on the i^{th}-plus pile of w, that is $a_i \mapsto a_i + 1$.
D_5: If there is $j \in \{1,\dots,m\}$ such that $b_{j-1} > b_j$, then we apply R_3 on the j^{th}-minus pile of w, that is $b_j \mapsto b_j + 1$.
We will write $w \xrightarrow{k} w'$ (or $w' = w \xrightarrow{k}$) to denote that w' is a signed partition obtained from w applying D_k, for $k = 1\dots5$. We also set

$$\nabla(w) = \{w' : w \xrightarrow{k} w', k = 1\dots5\}.$$

It is clear that the thesis is equivalent to proving that $\nabla(w) = \{w' \in D(m,n) : w' \text{ covers } w\}$. Let us note at first that the inclusion $\nabla(w) \subseteq \{w' \in D(m,n) : w' \text{ covers } w\}$ is obvious. Now, it is easy to see that the reverse inclusion $\nabla(w) \supseteq \{w' \in D(m,n) : w' \text{ covers } w\}$ is a consequence of the following statement: if

$$w'' = (a''_m,\dots,a''_1|b''_1,\dots,b''_m) = l''_1 \dots, l''_{2m}$$

is an element in $D(m,n)$ such that $w'' \sqsupset w$, then there exists $w' = (a'_m,\dots,a'_1|b'_1,\dots,b'_m) \in D(m,n)$ such that $w \xrightarrow{k} w'$ for some $k = 1,\dots,5$ and $w'' \sqsupseteq w'$.
We prove then the previous statement. Since $w'' \sqsupset w$, there exists k such that $l''_k > l_k$. We distinguish several cases and conventionally we shall assume that $b_0 = b''_0 := 0$.

(A) $a''_m > a_m$.

(A1) $a_m > 0$ and $a_m + |b_m| < n$. In this case we take $w' = w \xrightarrow{2}$, with $a'_m = a_m + 1 \leq a''_m$. Thus $w'' \sqsupseteq w'$.
(A2) $a_m = 0$, $a_m + |b_m| < n$ and $||w|| < m$. In this case we take $w' = w \xrightarrow{1}$, i.e. $w' = (1,0,\dots,0|b_1,\dots,b_m)$ and we get $a'_m = 1 \leq a''_m$. Then $w'' \sqsupseteq w'$.
(A3) $a_m = 0$ and $||w|| = m$. Then $w = (0,0,\dots,0|b_1,\dots,b_m)$, with $0 > b_1$ from $||w|| = m$. Note that $b''_1 = 0$. Otherwise $0 > b''_1 \geq \dots \geq b''_m$ and $a''_m > a_m = 0$, we would get $||w''|| > m$: a contradiction, since $w'' \in D(m,n)$. Therefore we can write $b''_1 = 0 > b_1$. So that if we take $w' = w \xrightarrow{5}$, with $b'_1 = b_1 + 1$, it follows $w'' \sqsupseteq w'$, from $b'_1 = b_1 + 1 \leq b''_1 = 0$.
(A4) $a_m = 0$, $a_m + |b_m| = n$ and $||w|| < m$. Then there is $j \in \{2,\dots,m\}$ such that

$$w = (0,0,\dots,0|0,0,\dots,0,b_j,b_{j+1},\dots,b_{m-1},-n)$$

Since $w'' \sqsupset w$, we have

$$w'' = (a''_m,\dots,a''_1|0,0,\dots,0,b''_j,\dots,b''_{m-1},b''_m).$$

The condition $a''_m > 0 = a_m$ implies $|b''_m| \leq n-1$, i.e. $b''_m \geq -(n-1)$. Now we take $k \in \{2,\dots,m\}$ minimal such that $b_k = -n$. Hence

$$w = (0,0,\dots,0|0,0,\dots,0,b_j,\dots b_{k-1},-n,\dots,-n)$$

From the minimality of k, we obtain $b_{k-1} > -n = b_k$. In the place k of w'', we must have $b''_k > -n$ from $b''_m > -n$. Therefore if we take $w' = w \xrightarrow{5}$, with $b'_k = b_k + 1 = -n+1$, it holds that $b''_k \geq b'_k$, thus $w'' \sqsupseteq w'$.
(A5) $a_m > 0$, $a_m + |b_m| = n$ and $b_m = 0$. Then $a_m = n$ and $w = (n,a_{m-1},\dots,a_1|0,0,\dots,0)$. This case is impossible, because $w'' \in D(m,n)$ and $a''_m > a_m = n$.
(A6) $a_m > 0$, $a_m + |b_m| = n$ and $|b_m| > 0$. As consequence we get $b''_m > b_m$, otherwise $n \geq a''_m + |b''_m| = a''_m + |b_m| > a_m + |b_m| = n$. Now, if $b_{m-1} > b_m$, we choose $w' = w \xrightarrow{5}$ with $b'_m = b_m + 1$. Otherwise, if $b_{m-1} = b_m$ necessarily $b''_{m-1} > b_{m-1}$, because the condition $b''_{m-1} = b_{m-1}$ implies $b_m < b''_m \leq b''_{m-1} = b_{m-1}$ and this is a contradiction. Now, if $b_{m-2} > b_{m-1}$, we choose $w' = w \xrightarrow{5}$ with $b'_{m-1} = b_{m-1} + 1$. Continuing in the same way, since $a_m > 0$, there is $j \geq 2$ such that $b_m = b_{m-1} = \dots = b_j < 0$, $b_{j-1} = \dots = b_1 = 0$ and $b''_j > b_j$, $b''_{j+1} > b_{j+1},\dots,b''_m > b_m$, and we take $w' = w \xrightarrow{5}$ with $b'_j = b_j + 1$.

(B) $a''_m = a_m,\dots,a''_{i+1} = a_{i+1}, a''_i > a_i$ for some $m > i \geq 1$.

(B1) $a_{i+1} = a_i$. It is impossible because $a_{i+1} = a''_{i+1} \geq a''_i > a_i$.
(B2) $a_{i+1} > a_i > 0$. In this case we take $w' = w \xrightarrow{4}$ with $a'_i = a_i + 1$.
(B3) $a_{i+1} > a_i = 0$ and $||w|| < m$. In such case, we take

$w' = w \xrightarrow{3}$ with $a'_i = a_i + 1$.
(B4) $a_{i+1} > a_i = 0$ and $||w|| = m$. From $||w|| = m$ and $a_i = 0$, there exists at least a $k \geq 1$ such that $b_k < 0$. Let j be the minimum such that $b_j < 0$. Then $b''_j = 0$, otherwise $||w''|| > m$ because $a''_i > a_i = 0$. In such case, we take $w' = w \xrightarrow{5}$, with $b'_j = b_j + 1$ if $j > 1$.

(C) $0 \geq b''_j > b_j$ for some $j \in \{1, \ldots, m\}$.
In this case we can assume that j is minimal, so that $b''_1 = b_1, \ldots, b''_{j-1} = b_{j-1}$ and $b''_j > b_j$. Now, if $b_{j-1} = b_j$, then $b''_{j-1} \geq b''_j > b_j = b_{j-1}$, that is a contradiction. Hence it must be necessarily $b_{j-1} > b_j$ and we can take $w' = w \xrightarrow{5}$, with $b'_j = b_j + 1$. $\square$

From the above theorem, we get the following information about the structure of $D(m,n)$.

Corollary 3.5.
(i) The rank function of $D(m,n)$ is $\rho : D(m,n) \to \mathbb{N}_0$ such that

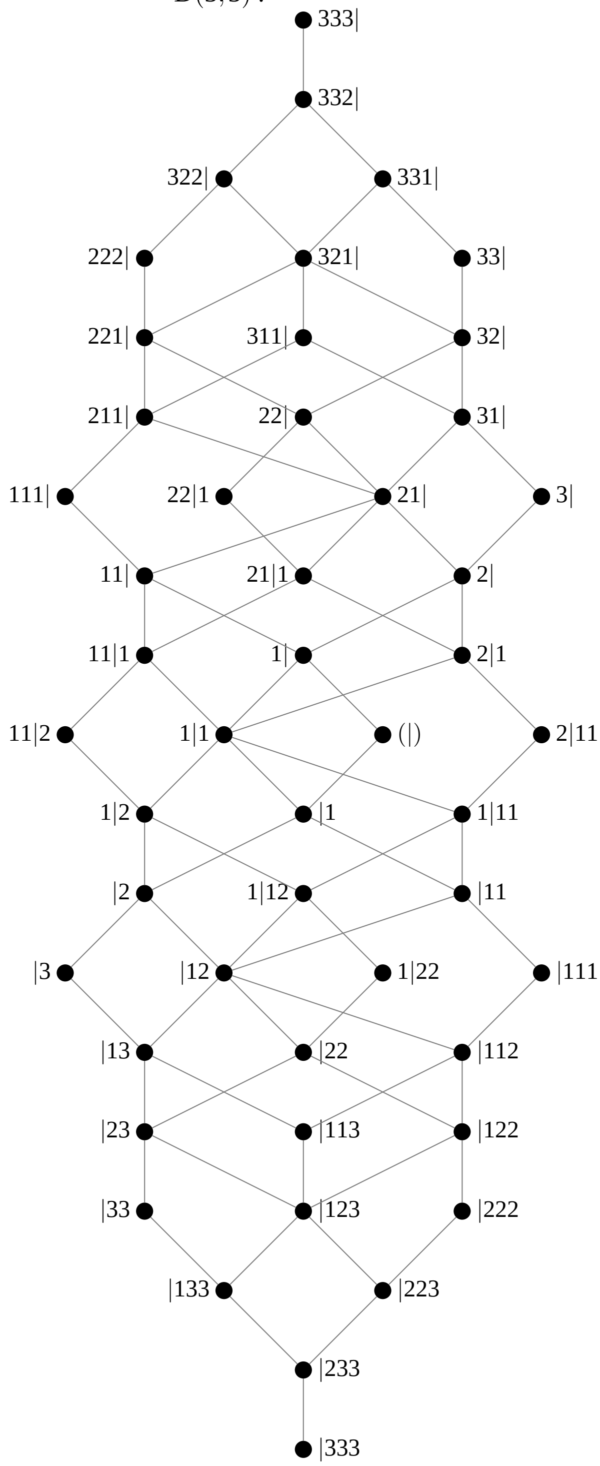
$$\rho((a_n \cdots a_1 | b_1 \cdots b_n)) = a_n + \cdots a_1 + b_1 \cdots + b_n + mn \quad (5)$$

(ii) $N_k(D(m,n)) = \{w \in D(m,n) : w \vdash k - mn\}$.
(iii) The rank of $D(m,n)$ is $2mn$.

Proof. The minimum $\hat{0}$ in $D(m,n)$ is the s-partition $(0, \ldots, 0| -n, \ldots, -n)$, with $-n$ that appears m times. By (5) we have $\rho(\hat{0}) = 0$. Hence (i) and (ii) are direct consequences of the Proposition 2.5 and of the Theorem 3.4. Finally, the maximum $\hat{1}$ in $D(m,n)$ is the s-partition $(n, \ldots, n|0, \ldots, 0)$, where $-n$ appears m times. By (5) we have $\rho(\hat{1}) = 2mn$. Hence (iii) follows by (i). $\square$

Below we draw the Hasse diagram of the lattice $D(3,3)$ (we omit the zeroes in each signed partition):

$D(3,3)$:



4 The Lattice $E(m,n)$

Let m and n be two fixed non-negative integers. We set

$$E(m,n) := \{w \in P_* : ||w|| = m, M^+(w) + M^-(w) \leq n\} \quad (6)$$

Then $E(m,n)$ is an m-stable sublattice of $D(m,n)$ and it is also an involution poset with respect to the restriction of the involution map of $C(m,n)$. Hence, in particular, also $E(m,n)$ is self-dual.

In this section we show that the lattice $E(m,n)$ is 2-covering for $n>1$, and we compute its rank function.

As for $D(m,n)$ and $C(m,n)$, we consider $E(m,n)$ as an (m,m)-uniform sublattice of P_*. Like $D(m,n)$, we can enumerate the s-partitions in $E(m,n)$:

Proposition 4.1. $|E(m,n)|=1$ for $m=0$, 0 for $n=0, m>0$ and $\binom{m+n-2}{n-1}\left(\frac{m+mn+n-1}{m}\right)$ otherwise.

Proof. The proof is similar to that of Proposition 3.3. The border cases are trivial. For nonzero n,m, when there are no negative parts, the partitions are normal partitions of exactly m parts of size at most n, which are counted by $\binom{n+m-1}{m}$. When the largest negative part is of size $-k$, we remove the largest negative part, shorten the partition's profile by removing the last vertical step before the zero line, and add k to all parts, producing a one-to-one bijection between the former partitions and normal partitions with at most $m-1$ parts of size at most $n-1$. This gives us $|E(m,n)| = \binom{n-1+m}{m} + n\binom{n+m-2}{m-1} = \binom{m+n-2}{n-1}\left(\frac{m+mn+n-1}{m}\right)$ as claimed. $\square$

In order to describe also $E(m,n)$ as a discrete dynamical model, we give now the following evolution rules. Let $w=a_m\dots a_1|b_1\dots b_m$ be a fixed element in $E(m,n)$. Also in this case we conventionally assume $b_0:=0$. Let us note that if $n=1$ then $E(m,1)$ is the pair of elements $\{(1,\dots,1,1|0,\dots,0),(0,\dots,0|-1,-1,\dots,-1)\}$.

Theorem 4.2. $E(m,n)$ is definable with $\{R_1,R_2,R_3\}$.

Proof. Let $w=(a_m,\dots,a_1|b_1,\dots,b_m)=l_1\dots l_{2m}$ a generic element of $E(m,n)$. As in the proof of the Theorem 3.4., all the subcases when the rules R_1, R_2 and R_3 are applicable in the case of $E(m,n)$ are the following:
E_1: If $a_m>0$ and $a_m+|b_m|<n$, then we apply R_1 on the m^{th}-plus pile of w, that is $a_m\mapsto a_m+1$.
E_2: If $a_m=0$, $|b_m|<n$ and there is $j\in\{1,\dots,m\}$ such that $b_j=-1$ and $b_{j-1}=0$, then the unique block in the j^{th}- minus pile of w must be shifted in the m^{th}-plus pile of w with the rule R_2, i.e. $a_m\mapsto 1$ and $b_j\mapsto 0$.
E_3: If there is $i\in\{m-1,\dots,1\}$ such that $a_{i+1}>a_i>0$, then we apply R_1 on the i^{th}-plus pile of w, that is $a_i\mapsto a_i'=a_i+1$.
E_4: If there is $i\in\{m-1,\dots,1\}$ such that $a_{i+1}>0$ and $a_i=0$ and there is $j\in\{1,\dots,m\}$ such that $b_j=-1$ and $b_{j-1}=0$ then the unique block in the j^{th}- minus pile of w must be shifted in the i^{th}-plus pile of w with the rule R_2, i.e. $a_i\mapsto 1$ and $b_j\mapsto 0$.
E_5: If there is $j\in\{1,\dots,m\}$ such that $b_{j-1}>b_j$ and $b_j<-1$, then we apply R_3 on the j^{th}-minus pile of w, that is $b_j\mapsto b_j+1$.

We will use now the same notation, $w\xrightarrow{k}w'$ (or $w'=w\xrightarrow{k}$), to denote that w' is a signed partition obtained from w applying the E_k, for some $k=1,\dots,5$. As in the proof of the Theorem 3.4., we are reduced to proving that if $w''=(a''_m,\dots,a''_1|b''_1,\dots,b''_m)=l''_1\dots,l''_{2m}$ is an element in $E(m,n)$ such that $w''\sqsupset w$, then there exists $w'=(a'_m,\dots,a'_1|b'_1,\dots,b'_m)\in E(m,n)$ such that $w\xrightarrow{k}w'$ for some $k=1,\dots,5$ and $w''\sqsupseteq w'$.
We prove then the previous statement also in the case of $E(m,n)$. Since $w''\sqsupseteq w'$, there exists s such that $l''_s>l_s$. Also in this proof we distinguish several cases and conventionally we shall assume that $b_0=b''_0:=0$.

(A) $a''_m>a_m$.

(A1) $a_m>0$ and $a_m+|b_m|<n$. In this case we take $w'=w\xrightarrow{a}$, with $a'_m=a_m+1\le a''_m$. Thus $w''\sqsupseteq w'$.
(A2) $a_m>0$ and $a_m+|b_m|=n$. We must have $0\ge b''_m>b_m$ because $a''_m+|b''_m|\le n$ and $a''_m>a_m$. In this case we take $w'=w\xrightarrow{e}$, with $b'_m=b_m+1\le b''_m$.
(A3) $a_m=0$. In this case $w=(0,\dots,0|b_1,\dots,b_m)$ with $b_1<0$ and $b''_1=0$ because $||w''||=||w||=m$. If $b_1=-1$ we take $w'=w\xrightarrow{b}$, with $j=1$, while if $b_1\le-2$ we $w'=w\xrightarrow{e}$ with $j=1$.

(B) $a''_m=a_m=0,\dots,a''_{i+1}=a_{i+1}, a''_i>a_i$ for some $i\in\{m-1,\dots,1\}$. It is then immediate that it must be $a_{i+1}>a_i$. Now, if $a_i>0$ we take $w'=w\xrightarrow{c}$ with $a'_i=a_i+1$. Otherwise, if $a_i=0$ we have $a''_m=a_m>0,\dots,a''_{i+1}=a_{i+1}>0, a''_i>a_i=0$. As $||w''||=||w||=m$, there is $j\in\{1,\dots,m\}$ such that $b_j<0=b_{j-1}$ and $b''_j=0$. If $b_j\le-2$, we take $w'=w\xrightarrow{e}$ with $b'_j=b_j+1$. If $b_j=-1$, we take $w'=w\xrightarrow{d}$ with $b'_j=b_j+1=0$ and $a'_i=a_i+1=1$.

(C) $a''_m=a_m=0,\dots,a''_1=a_1, b''_1=b_1,\dots,b''_{j-1}=b_{j-1}, b''_j>b_j$. We note that if $b_j=-1$ then $b''_j=0$, and this is impossible because $||w''||=||w||=m$. Then $0>b''_j>b_j$ and therefore we take $w'=w\xrightarrow{e}$ with $b'_j=b_j+1$. $\square$

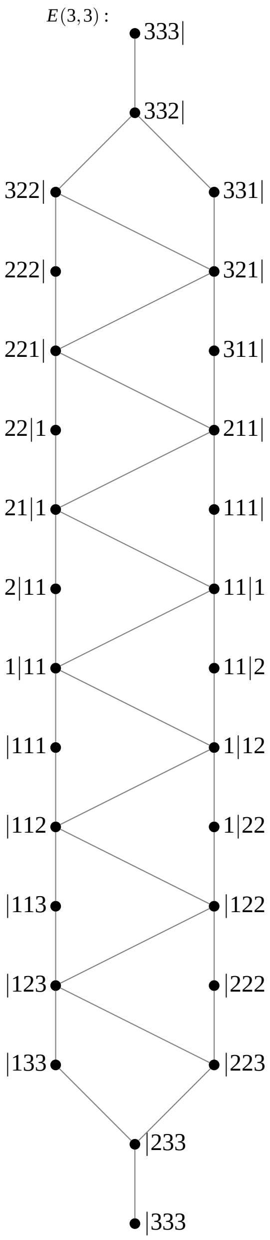
Corollary 4.3.
(i) If $n\ge2$ then $E(m,n)$ is 2-covering.
(ii) The rank function ν of $E(m,n)$ is $\nu(w)=\rho(w)-|w|_>$, where ρ is the rank function of $D(m,n)$
(iii) The rank of $E(m,n)$ is $m(2n-1)$.

Proof. (i) From the Theorem 4.2 it results that if $w,w'\in E(m,n)$, then w' covers w if and only if $w'=w\xrightarrow{k}$, for some $k=a,\dots,e$. Since each Rule $a,\dots,e$ adjoins one block to some part of w or shifts one block from the negative parts into the positive parts, if w' covers w then $w'\downarrow^1 w$ or $w'\downarrow^2 w$. Moreover, if we take $w=(0,\dots,0|-1,\dots,-1)$ and $w'=(1,0,\dots,0|0,-1,\dots,-1)$, then w' covers w in $E(m,n)$ and $w'\downarrow^2 w$.
(ii) Let ρ the rank function of $D(m,n)$. Let $w\in E(m,n)$ and let $w\sqsupset w_t\sqsupset\dots\sqsupset w_1\sqsupset\hat{0}$ be any saturated chain in

$E(m,n)$ from $\hat{0}$ to w. Let us assume that in this chain w is obtained from $\hat{0}$ with k applications of E_2 and E_4, for some integer $k \geq 0$. To each step $l \in \{1,\ldots,t\}$ where we apply E_2 or E_4, there is the following situation: $w_l \sqsupset u_l \sqsupset w_{l-1}$, for exactly one only element $u_l \in D(m,n) \setminus E(m,n)$. This means that $\rho(w) = (t+1)+k$, i.e. $\nu(w) = \rho(w) - k$. The integer k is also the difference between the number of positive parts of w and the number of positive parts of $\hat{0}$, i.e. exactly $|w|_>$, since $\hat{0} = (0,\ldots,0|-n,\ldots,-n)$. Hence the thesis follows.

(iii) The maximum of $E(m,n)$ is $\hat{1} = (n,\ldots,n|0,\ldots,0)$, hence the thesis follows from the previous (ii) and from (ii) in Corollary 3.5. □

Below we draw the Hasse diagram of the lattice $E(3,3)$:



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