Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/130403

A New Generalized Moment Generating Function of Random Variables

Bruno Monte Castro and Marcelo Bourguignon*

Departament of Statistics, Federal University of Rio Grande do Norte, Natal, Brazil.

Received: 12 Mar. 2019, Revised: 11 May 2019, Accepted: 14 May 2019
Published online: 1 Jul. 2019

Abstract: Many of the important characteristics and features of a distribution are obtained through the ordinary moments and generating function. The main goal of this paper is to address a new approach to compute, without using multiple integrals and derivatives, $\mathbb{E}\left[\frac{(X^a+b)^r}{(X^c+d)^s}\right]$ for a nonnegative random variable, where a,b,c,d are any real number. The proposed approach is discussed in detail and illustrated through a few examples.

Keywords: moment-generating function; inverse moments; probability distributions.

1 Introduction

Many of the important characteristics and features of a probability distribution are obtained through the ordinary moments and generating function. The moment-generating function with a random variable X is defined as $M_X(t) := \mathbb{E}[e^{tX}]$, see Feller [6] and Grimmett and Welsh [7]. The moment-generating function can be used to compute a distribution's moments. The nth moment about 0 is the nth derivative of the moment-generating function, evaluated at 0, i.e.,

$$\mathbb{E}[X^n] = \left.\frac{\mathrm{d}\,\varphi_X(t)}{\mathrm{d}\,t^n}\right|_{t=0}.$$

However, $M_X(t)$ is usually difficult to evaluate and may not exist a closed-form expression for some probability distributions, for instance the Hypergeometric distribution. In this context, Michael et al [1] have used the probability-generating function to find the negative integer moments of $X+A>0$, where X is a random variable and A is a constant. Cressie and Borkent [5] worked on obtaining all ordinary moments from the moment-generating function, and links between factorial moment-generating functions and all factorial moments. If X is integer valued, Chakraborti et al [2] obtained an alternative expression for $\mathbb{E}[X^n]$. Michael et al [1] performed a generalized method of generating the moments of random variables. They presented examples are only in continuous probability distribution.

Let X be a random variable defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$. Let a,b,c,d be any real numbers and let $\varphi_X^{a,b;c,d}(t,u)$ be generalized moment-generating function of X defined as

$$\varphi_X^{a,b;c,d}(t,u) = \mathbb{E}\left[e^{t(X^a+b)+u(X^c+d)}\right]. \tag{1}$$

Using the results of Chao and Strawderman [3] and Cressie et al [4], we have that

$$\mathbb{E}\left[\frac{(X^a+b)^r}{(X^c+d)^s}\right] = \int_{-\infty}^{0}\int_{-\infty}^{u_1}\cdots\int_{-\infty}^{u_{s-1}} \left.\frac{\mathrm{d}^r\varphi_X^{a,b;c,d}(t,u)}{\mathrm{d}t^r}\right|_{t=0} \mathrm{d}u\cdots\mathrm{d}u_2\mathrm{d}u_1.$$

Which not have closed-form expressions for mgf of X^a, $a\in\mathbb{R}$. In this context, the main goal of this paper is to address a new approach to compute $\mathbb{E}\left[\frac{(X^a+b)^r}{(X^c+d)^s}\right]$ without using multiple integrals and derivatives, see Jones [8]. Furthermore, we extend the generalization proposed by Michael et al [1] in order to add inverse moments of random variables.

The rest of the paper is organized as follows. In Section 2, we introduce a generalized method of generating the moments of continuous random variables. In Section 3, the same method is introduced for discrete random variables. Finally, some concluding remarks are made in Section 4.

* Corresponding author e-mail: bruno.monte19@gmail.com

2 Continuous probability distribution

Theorem 1. *Let X be a continuous random variable with probability density function (pdf) $f_X(x)$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let a,b,c,d be any real number, then*

$$\begin{aligned}\varphi_{r,s}(a,b;c,d) &= \mathbb{E}\left[\frac{(X^a+b)^r}{(X^c+d)^s}\right] \\ &= \sum_{i=0}^{r}\binom{r}{i}b^{r-i}\int_{-\infty}^{\infty}\frac{x^{ai}}{(x^c+d)^s}f_X(x)dx.\end{aligned}$$

Proof. Using the binomial series we have

$$(a+b)^n = \sum_{i=0}^{r}\binom{n}{i}a^i b^{n-i}.$$

Then

$$\begin{aligned}\varphi_{r,s}(a,b;c,d) &= \mathbb{E}\left[\sum_{i=0}^{r}\binom{r}{i}\frac{X^{ai}b^{r-i}}{(X^c+d)^s}\right] \\ &= \sum_{i=0}^{r}\binom{r}{i}b^{r-i}\int_{-\infty}^{\infty}\frac{x^{ai}}{(x^c+d)^s}f_X(x)dx.\end{aligned}$$

Corollary 1. *Consider $c,d \in \mathbb{R}$. The variance of the random variable X can be written by*

$$Var(X) = \varphi_{2,0}(1,-\mathbb{E}(X);c,d).$$

The skewness of X is given by

$$A(X) = \frac{\varphi_{3,0}(1,-\mathbb{E}(X);c,d)}{[\varphi_{2,0}(1,-\mathbb{E}(X);c,d)]^{3/2}}.$$

The kurtosis of X is

$$K(X) = \frac{\varphi_{4,0}(1,-\mathbb{E}(X);c,d)}{[\varphi_{2,0}(1,-\mathbb{E}(X);c,d)]^{2}}.$$

Proof. The kth central moment of X is

$$\begin{aligned}\varphi_{k,0}(1,-\mathbb{E}(X);c,d) &= \mathbb{E}\left[\frac{(X-\mathbb{E}(X))^k}{(X^c+d)^0}\right] \\ &= \mathbb{E}\left[(X-\mathbb{E}(X))^k\right].\end{aligned}$$

Therefore

$$\mathrm{Var}(X) = \varphi_{2,0}(1,-\mathbb{E}(X);c,d).$$

The skewness and kurtosis are obtained similarly.

2.1 Gamma probability distribution

A random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, if its density function is given by

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x} \quad x > 0.$$

Then,

$$\begin{aligned}\varphi_{r,s}(a,b;c,d) &= \int_0^\infty\sum_{i=0}^{r}\binom{r}{i}\frac{x^{ai}b^{r-i}}{(x^c+d)^s}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x}\mathrm{d}x \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{r}\binom{r}{i}b^{r-i}\int_0^\infty\frac{x^{\alpha+ai-1}\mathrm{e}^{-\beta x}}{(x^c+d)^s}\mathrm{d}x.\end{aligned}$$

For any $c,d \in \mathbb{R}$, $s=0$ and $r=a=1$, we have that

$$\begin{aligned}\varphi_{1,0}(1,b;c,d) &= \mathbb{E}(X+b) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\left[b\int_0^\infty x^{\alpha-1}\mathrm{e}^{-\beta x}dx + \frac{1}{\beta}\int_0^\infty x^{\alpha}\mathrm{e}^{-\beta x}\mathrm{d}x\right] \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\left[b\frac{\Gamma(\alpha)}{\beta^\alpha} + \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}\right] \\ &= b + \frac{\alpha}{\beta}.\end{aligned}$$

Now, for any $a,b \in \mathbb{R}$, $r=d=0$ and $s=c=1$, we have that

$$\begin{aligned}\varphi_{0,1}(a,b;1,0) &= \mathbb{E}\left(\frac{1}{X}\right) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-2}\mathrm{e}^{-\beta x}\mathrm{d}x \\ &= \frac{\beta}{\alpha-1}, \quad \alpha > 1.\end{aligned}$$

where X^{-1} is an inverse gamma distribution with shape parameters $\alpha > 0$ and $\beta > 0$.

2.2 Beta probability distribution

A random variable X follows a beta distribution with shape parameters $\alpha > 0$ and $\beta > 0$, if its pdf is given by

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1,$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function. Therefore,

$$\begin{aligned}\varphi_{r,s}(a,b;c,d) &= \int_0^1\sum_{i=0}^{r}\binom{r}{i}\frac{x^{ai}b^{r-i}}{(x^c+d)^s}\cdot\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\mathrm{d}x \\ &= \frac{1}{B(\alpha,\beta)}\sum_{i=0}^{r}\binom{r}{i}b^{r-i}\int_0^1\frac{x^{\alpha+ai-1}(1-x)^{\beta-1}}{(x^c+d)^s}\mathrm{d}x.\end{aligned}$$

For any $c,d \in \mathbb{R}$, we have that

$$\begin{aligned}\varphi_{1,0}(1,b;c,d) = \mathbb{E}(X+b) &= \frac{1}{B(\alpha,\beta)}\left[bB(\alpha,\beta)+B(\alpha+1,\beta)\right] \\ &= b + \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = b + \frac{\alpha}{\alpha+\beta}.\end{aligned}$$

The first inverse moment of X is obtained by

$$\begin{aligned}\varphi_{0,1}(a,b;1,0) &= \mathbb{E}(X^{-1}) \\ &= \frac{1}{B(\alpha,\beta)}\left[B(\alpha-1,\beta)\right] \\ &= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \\ &= \frac{\alpha+\beta-1}{\alpha-1}, \quad \alpha > 1.\end{aligned}$$

2.3 F-distribution

A random variable X follows a F-distribution with degrees of freedom $n > 0$ and $m > 0$, if its pdf is given by

$$f_X(x) = \frac{n^{n/2}m^{m/2}x^{n/2-1}}{B\left(\frac{n}{2},\frac{m}{2}\right)(m+nx)^{(n+m)/2}}, \quad x > 0.$$

Thus,

$$\begin{aligned}
&\varphi_{r,s}(a,b;c,d) \\
&= \int_0^\infty \sum_{i=0}^{r}\binom{r}{i}\frac{x^{ai}b^{r-i}}{(x^c+d)^s}\cdot\frac{n^{n/2}m^{m/2}x^{n/2-1}}{B\left(\frac{n}{2},\frac{m}{2}\right)(m+nx)^{(n+m)/2}}\mathrm{d}x \\
&= \frac{n^{n/2}m^{m/2}}{B\left(\frac{n}{2},\frac{m}{2}\right)}\sum_{i=0}^{r}\binom{r}{i}\int_0^\infty\frac{b^{r-i}x^{ai+n/2-1}}{(x^c+d)^s(m+nx)^{(m+n)/2}}\mathrm{d}x.
\end{aligned}$$

The second inverse moment of X is given by

$$\begin{aligned}
\varphi_{0,1}(a,b;2,0) = \mathbb{E}(X^{-2}) &= \frac{n^{n/2}m^{m/2}}{B\left(\frac{n}{2},\frac{m}{2}\right)}\int_0^\infty\frac{x^{n/2-3}}{(m+nx)^{(n+m)/2}}\mathrm{d}x \\
&= \frac{\left(\frac{n}{m}\right)^2\Gamma(\frac{n}{2}-2)\Gamma(\frac{m}{2}+2)}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \\
&= \frac{n^2(m+2)}{m(n-2)(n-4)}, \; n > 4.
\end{aligned}$$

3 Discrete probability distribution

Let X be a random variable with integer values. It is not possible to solve

$$\varphi_{r,s}(a,b;c,d) = \sum_{x\in\mathscr{X}}\sum_{i=0}^{r}\binom{r}{i}b^{r-i}\frac{x^{ai}}{(x^c+d)^s}\mathbb{P}(X=x).$$

where $\mathscr{X}$ is the set of values of X. We extend the result of Michael et al. (2017) and apply to the random variables with integer values. The kth moment of X is

$$\mathbb{E}(X^k) = \sum_{l=0}^{k}\frac{1}{l!}\sum_{m=0}^{l}(-1)^m\binom{l}{m}(l-m)^k\mathbb{E}\left[\frac{X!}{(X-l)!}\right], \; k\in\mathbb{N}. \quad (2)$$

Theorem 2. *Let X be a random variable on a probability space $(\Omega,\mathscr{F},\mathbb{P})$. Let a, $r\in\mathbb{N}$ and $b\in\mathbb{R}$, then*

$$\begin{aligned}
\phi_r(a,b) &= \mathbb{E}[(X^a+b)^r] \\
&= \sum_{i=0}^{r}\binom{r}{i}b^{r-i}\sum_{l=0}^{ai}\frac{1}{l!}\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m)^{ai}\mathbb{E}\left[\frac{X!}{(X-l)!}\right].
\end{aligned}$$

Proof. Using only the the Binomial series the proof follows as in Theorem 1.

3.1 Binomial probability distribution

A random variable X is said to be a Binomial random variable with parameters $n\in\mathbb{N}$ and $p\in(0,1)$ if the probability function is

$$\mathbb{P}(X=x) = \binom{n}{x}p^x(1-p)^{n-x}\underset{\{0,\cdots,n\}}{I}(x), \; n\in\mathbb{N}, \; 0\le p\le 1.$$

The lth factorial moment of X is

$$\mathbb{E}\left[\frac{X!}{(X-l)!}\right] = \frac{n!p^l}{(n-l)!}.$$

Thus, using the theorem 2

$$\phi_r(a,b) = \sum_{i=0}^{r}\binom{r}{i}b^{r-i}\sum_{l=0}^{ai}\binom{n}{l}p^l\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m)^{ai}.$$

The first moment of X considers $r=a=1$, then

$$\begin{aligned}
\phi_1(1,b) &= \mathbb{E}(X+b) \\
&= \sum_{i=0}^{1}\binom{1}{i}b^{1-i}\sum_{l=0}^{i}\binom{n}{l}p^l\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m)^i \\
&= b\sum_{l=0}^{0}\binom{n}{l}p^l\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m)^0 \\
&+ \sum_{l=0}^{1}\binom{n}{l}p^l\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m) \\
&= b+np.
\end{aligned}$$

Taking $b=0$ we have

$$\phi_1(1,0) = \mathbb{E}(X) = np.$$

3.2 Poisson probability distribution

Let X be a Poisson random variable with parameter $\lambda > 0$ and with probability function given by

$$\mathbb{P}(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}\underset{\{0,1,\ldots\}}{I}(x), \; \lambda > 0.$$

The lth factorial moment of X is

$$\mathbb{E}\left[\frac{X!}{(X-l)!}\right] = \lambda^l.$$

Therefore

$$\phi_r(a,b) = \sum_{i=0}^{r}\binom{r}{i}b^{r-i}\sum_{l=0}^{ai}\frac{\lambda^l}{l!}\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m)^{ai}.$$

The first moment of $X+b$ is obtained when $r=a=1$

$$\begin{aligned}
\phi_1(1,b) &= \mathbb{E}(X+b) \\
&= \sum_{i=0}^{1}\binom{1}{i}b^{1-i}\sum_{l=0}^{i}\frac{\lambda^l}{l!}\sum_{m=0}^{l}(-1)^l\binom{l}{m}(l-m)^i \\
&= b+\lambda.
\end{aligned}$$

4 Concluding remarks

In this paper, we have developed a new generalized moment-generating function of random variables. The main contributions of the present paper are: (i) we extend the generalization proposed by Michael et al [1] in order to add inverse moments of random variables, and (ii) we compute $\mathbb{E}\left[(X^a+b)^r/(X^c+d)^s\right]$ without using multiples integrals and derivatives. The approach are discussed in detail and illustrated through a few examples. We hope this generalization may attract applications in statistics.

References

[1] M.C. Michael, O.C. Anene, A.M. Akudo and I.J. Nwabueze, Generalized Moment Generating Functions of Random Variables and Their Probability Density Functions, *American Journal of Applied Mathematics and Statistics*, **5**, 49-53 (2017).

[2] S. Chakraborti, F. Jardim and E. Epprecht, Higher-order moments using the survival function: The alternative expectation formula, *The American Statistician*, **73**(2), 191-194 (2018).

[3] M.T. Chao and W.E. Strawderman, Negative Moments of Positive Random Variables, *Journal of the American Statistical Association*, **67**, 429-431 (1972).

[4] N.A.C. Cressie, A.S. Davids, J.L. Folks and G.E. Policello, The moment-generating function and negative integer moments, *Amer. Statist.*, **35**, 148-150 (1981).

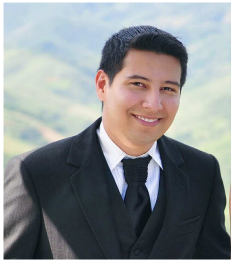
[5] N.A.C. Cressie and M. Borkent, The moment generating function has its moments, *J. Statist. Plann. Inference*, **13**, 337-344 (1986).

[6] W. Feller, *An Introduction to Probability Theory and Applications*, 2nd ed., Vol 2. John Wiley and Sons, Inc. New York, (1966).

[7] G. Grimmett and P. Welsh, *Probability: An Introduction*, Oxford University Press, (1986).

[8] M. C. Jones, Inverse factorial moments, *Statist. Probab. Lett.*, **6**, 37-42 (1987).

Bruno Monte Castro is an assistant professor at the Federal University of Rio Grande do Norte, located in Natal (Brazil). He received the PhD degree in Statistics, at University of São Paulo. His researchs interests are in the areas of probability and statistics, focusing on stochastic process, regression models and statistical model selection.



Marcelo Bourguignon is an assistant professor at the Federal University of Rio Grande do Norte, located in Natal (Brazil). He carried out his doctoral studies (2011-2014) at the Federal University of Pernambuco, Recife, Brazil. His studies were related to fatigue life models and count time series.