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Implementation of a Reducing Algorithm for Differential-Algebraic Systems in Maple

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Abstract: This paper discusses the implementation of a reduction algorithm for differential-algebraic systems in Maple. Using the proposed Maple package, the given system of differential-algebraic equations can be transformed into another simpler system having the same properties. Sample computations are presented to illustrate the proposed algorithm. This algorithm will be helpful to implement this in commercial software packages such as Mathematica, Matlab, Singular, SCIlab etc.

Keywords: Differential-algebraic systems, Reduction algorithms, Elementary row-column operations.

1 Introduction

A first-order matrix differential system can be represented as

$$\mathscr{A}(z)Du(z)+\mathscr{B}(z)u(z)=f(z), \quad (1)$$

where z is a complex variable, $\mathscr{A}(z),\mathscr{B}(z)$ are $m\times n$ matrices of analytic functions, $f(z)$ is an m-dimensional vector of analytic functions, $u(z)$ is an n-dimensional unknown vector to be determined and $D=\frac{d}{dz}$ is a differential operator. In operator notations, the first-order matrix differential system (1) can be represented by an equation of the form

$$Lu=f,$$

where $L=\mathscr{A}D+\mathscr{B}$ is a matrix differential operator. If $m=n$ and $\mathscr{A}$ is regular (i.e., $\det(\mathscr{A})\neq 0$), then the system (1) is called a *system of linear ordinary differential equations* or linear differential system (LDS) [5–10, 12, 14–16]. If $\mathscr{A}\equiv 0$, then the system (1) becomes a *purely algebraic system* and there are several methods available in the literature to find all possible solutions. If $m\neq n$ or $\mathscr{A}$ is singular matrix, then the system (1) turns out to be a *system of differential-algebraic equations* or simply, differential-algebraic system (DAS). Differential-algebraic system is a composed system of ordinary differential equations coupled with purely algebraic equations, hence DAS differ from LDS in many aspects [1–4, 11]. This paper mainly focused on DAS with some necessary conditions. In the literature, there are many algorithms available for solving DAS as well as BVPs with maple implementations, for example see, [1, 2, 4–10, 12–18].

The aim of this paper is to discuss about the implementation of a reduction algorithm. More details about the proposed reduction algorithm for DAS are available in [12] and a part of this work has been presented in [19, 20]. In the proposed algorithm, the given DAS reduces into another simpler and equivalent system where we can easily apply the classical theory of differential equations.

2 Maple Package, `DAS_Reduction`, for System of DAEs

Now the Maple implementation of the proposed algorithm is presented by creating different data types. The implemented Maple package, `DAS_Reduction`, of the proposed algorithm is provided with Maple worksheet at www.srinivasaraothota.webs.com/research. Using the Maple package, one can obtain the two unimodular matrices S,T and the reduced DAS of the given system. In Maple implementation, x is complex variable and $D=\frac{d}{dz}$ is the differential operator.

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2.1 Pseudo-Code [12]

- Input: Coefficient Matrices A, B of a given DAS.
- Output: Two unimodular Matrices S, T and the reduced DAS.

$A, B \leftarrow coefficient\ matricies$
$n \leftarrow size\ of\ A$
$LNS\_A \leftarrow left\ null\ space\ of\ A$
$Sa \leftarrow identity\ matrix\ with\ LNS\_A$
$Aa \leftarrow Sa.A,\ Ba \leftarrow Sa.B$
$LNS\_Ba \leftarrow left\ null\ space\ of\ Ba$
$Sb \leftarrow identity\ matrix\ with\ LNS\_Ba$
$A1 \leftarrow Sb.Aa,\ B1 \leftarrow Sb.Ba,\ S1 \leftarrow Sb.Sa$
$RNS\_A1 \leftarrow right\ null\ space\ of\ A1$
$Ta \leftarrow identity\ matrix\ with\ RNS\_A1$
$A1a \leftarrow A1.Ta,\ B1a \leftarrow B1.Ta$
$RNS\_B1a \leftarrow right\ null\ space\ of\ B1a$
$Tb \leftarrow identity\ matrix\ with\ LNS\_B1a$
$A2 \leftarrow A1a.Tb,\ B2 \leftarrow B1a.Tb,\ T1 \leftarrow Ta.Tb$
$P \leftarrow left\ elementary\ matrix$
$Q \leftarrow right\ elementary\ matrix$
$S \leftarrow P.S1,\ T \leftarrow T1.Q$

2.2 Maple Code

```
with(MatrixPolynomialAlgebra):
with(LinearAlgebra):

DAS_Reduction:=proc(A,B)
local n, L, r, k, Id, transpose_A,
NS_A, Id_part1, NS_part1, Sa, La, Aa,
transpose_Ba, NS_Ba, Id_part2, NS_part2,
Sb, L1, A1, B1, S1, NS_A1,   Id_part3,
NS_part3, Ta, L1a, A1a, B1a, NS_B1a,
Id_part4, NS_part4, Tb, L2, A2, B2,
T1, P_A, Q_A, S, T, Ba, L_reduced;

n := LinearAlgebra:-RowDimension(A);
L := A*delta+B;
n := LinearAlgebra:-RowDimension(A);
r := MTM:-rank(A);
k := MTM:-rank(B);

Id := LinearAlgebra:-IdentityMatrix(n);
transpose_A:=LinearAlgebra:-
     Transpose(A);

NS_A:=LinearAlgebra:-
    NullSpace(transpose_A);
Id_part1:=LinearAlgebra:-Transpose
   (Matrix (LinearAlgebra: Transpose~
   (convert~([seq(Id[i],
   i=1..n-nops(NS_A))], Matrix))));
NS_part1:=LinearAlgebra:-Transpose
   (Matrix(convert~([seq(NS_A[i],
   i=1..nops(NS_A))],Matrix)));
Sa:=convert(linalg:-blockmatrix
    (2,1,[Id_part1,NS_part1]), Matrix);
La := simplify~(Sa.L);
Aa:=seq(Coeff(La, delta,i),i=0..1)[2];
Ba:=seq(Coeff(La, delta,i),i=0..1)[1];
transpose_Ba:=LinearAlgebra:-
    Transpose(Ba);
NS_Ba:=LinearAlgebra:-NullSpace
     (transpose_Ba);
Id_part2:=LinearAlgebra:-Transpose
   (Matrix(`~`[LinearAlgebra:
   Transpose](`~`[convert]
   ([seq(Id[i],i=nops(NS_Ba)..n-1)],
   Matrix))));
NS_part2:=LinearAlgebra:-Transpose
   (Matrix(`~`[convert]([seq
   (NS_Ba[i],i=1..nops(NS_Ba))],
   Matrix)));
Sb:=convert(linalg:-blockmatrix
     (2,1,[NS_part2,Id_part2]), Matrix);
L1 := simplify~(Sb.La);
A1:=seq(Coeff(L1,delta, i),i=0..1)[2];
B1:=seq(Coeff(L1, delta,i),i=0..1)[1];
S1 := simplify~(Sb.Sa);
NS_A1 := simplify~(LinearAlgebra:-
   NullSpace(A1));
Id_part3:=Matrix(`~`[LinearAlgebra:-
   Transpose](`~`[convert]
   ([seq(Id[i],i=1..n-nops(NS_A1))],
   Matrix)));
NS_part3:=Matrix(`~`[convert]
   ([seq(NS_A1[i],i=1..nops(NS_A1))],
   Matrix));
Ta:=convert(linalg:-blockmatrix
   (1,2,[Id_part3,NS_part3]), Matrix);
L1a := simplify~(L1.Ta);
A1a:=seq(Coeff(L1a,delta,i),i=0..1)[2];
B1a:=seq(Coeff(L1a,delta,i),i=0..1)[1];
NS_B1a := simplify~(LinearAlgebra:-
   NullSpace(B1a));
Id_part4:=Matrix(`~`[LinearAlgebra:-
   Transpose](`~`[convert] ([seq(Id[i],
   i = nops(NS_B1a) .. n-1)], Matrix)));
NS_part4:=Matrix(`~`[convert]
   ([seq(NS_B1a[i],i=1..nops(NS_B1a))],
   Matrix));
Tb:=convert(linalg:-blockmatrix
   (1,2,[NS_part4,Id_part4]), Matrix);

L2 := simplify~(L1a.Tb);
A2:=seq(Coeff(L2,delta,i),i=0..1)[2];
B2:=seq(Coeff(L2,delta,i),i=0..1)[1];
T1 := simplify~(Ta.Tb);

P_A:=convert(linalg:-submatrix
   (Student[LinearAlgebra]:-
   ReducedRowEchelonForm
   (<A2|Id>), 1 .. n, n+1 .. 2*n),
```

```
   Matrix);
Q_A:=LinearAlgebra:-Transpose(convert
    (linalg:-submatrix(Student
    [LinearAlgebra]:-
    ReducedRowEchelonForm
    (<LinearAlgebra:-
    Transpose(A2)|Id>),
    1..n,n+1..2*n),Matrix));
S := simplify~(P_A.S1);
T := simplify~(T1.Q_A);
L_reduced:=simplify~(S.L.T);
return S,T,L_reduced;
end proc:
```

3 Sample Computations

Example 1 *Consider a matrix differential operator of DAS*

$$L = \mathscr{A}D + \mathscr{B}$$

$$= \begin{pmatrix} D+z & 0 & D-1 & (2-z)D+1 & D \\ 0 & zD+2 & D+1 & 0 & 1 \\ 2D+z & zD-2 & 3D-2 & (4-2z)D+1 & 2D-1 \\ D+\frac{z}{2} & -2zD+z & -D+\frac{1}{2} & (2-z)D+\frac{1}{2} & D+1 \\ 3D+2z & zD+2 & 4D-1 & (6-3z)D+2 & 3D+1 \end{pmatrix},$$

where $\mathscr{A} = \begin{pmatrix} 1 & 0 & 1 & 2-z & 1 \\ 0 & z & 1 & 0 & 0 \\ 2 & z & 3 & 4-2z & 2 \\ 1 & -2z & -1 & 2-z & 1 \\ 3 & z & 4 & 6-3z & 3 \end{pmatrix}$, *and*

$$\mathscr{B} = \begin{pmatrix} z & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ z & -2 & -2 & 1 & -1 \\ \frac{z}{2} & 2 & \frac{1}{2} & \frac{1}{2} & 1 \\ 2z & 2 & -1 & 2 & 1 \end{pmatrix}.$$

Applying the proposed algorithm, one can obtain the reduced matrix operator as follows

$$\widetilde{L} = \begin{pmatrix} -\frac{1}{12}(3z-5+\frac{1}{z})D & -\frac{1}{6}(z-3-\frac{1}{z})D & 0 & 0 & 0 \\ \frac{1}{12}(1+3z)D & \frac{1}{6}(z-1)D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -z^2+2z-1 \\ 0 & 0 & 0 & \frac{z}{2}+3 & -\frac{1}{2}z^2+z-\frac{1}{2} \end{pmatrix}.$$

Now we apply the Maple implementation to the given system of DAS (1).

```
> A := Matrix([[1,0,1,2-x,1],[0,x,1,0,0],
     [2,x,3,4-2*x,2],[1,-2*x,-1,2-x,1],
     [3,x,4,6-3*x,3]]);

> B := Matrix([[x,0,-1,1,0],[0,2,1,0,1],
     [x,-2,-2,1,-1],[(1/2)*x,2,1/2,1/2,1],
     [2*x,2,-1,2,1]]);
```

$$A := \begin{bmatrix} 1 & 0 & 1 & 2-x & 1 \\ 0 & x & 1 & 0 & 0 \\ 2 & x & 3 & 4-2x & 2 \\ 1 & -2x & -1 & 2-x & 1 \\ 3 & x & 4 & 6-3x & 3 \end{bmatrix}$$

$$B := \begin{bmatrix} x & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ x & -2 & -2 & 1 & -1 \\ \frac{x}{2} & 2 & \frac{1}{2} & \frac{1}{2} & 1 \\ 2x & 2 & -1 & 2 & 1 \end{bmatrix}$$

```
> S, T, Lreduced := DAS_Reduction(A,B);
```

$$\begin{bmatrix} \frac{1}{9}\frac{6x^2-11x+1}{x} & \frac{1}{9}\frac{3x^2-7x-1}{x} & 0 & \frac{2}{9}\frac{x+1}{x} & -\frac{1}{9}\frac{3x^2-5x+1}{x} \\ -\frac{1}{9}-\frac{2}{3}x & \frac{1}{9}-\frac{1}{3}x & 0 & -\frac{2}{9} & \frac{1}{9}+\frac{1}{3}x \\ \frac{4}{3} & \frac{5}{3} & 1 & \frac{2}{3} & -\frac{4}{3} \\ -3 & -1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4}\frac{2x-3}{(x^2-3x+2)(x-1)} & \frac{1}{2}\frac{2x-5}{x^3-4x^2+5x-2} & -\frac{x^2-2x-2}{x^3-4x^2+5x-2} & -1 & x-2 \\ \frac{1}{8(x-2)} & -\frac{1}{4(x-2)} & \frac{1}{x-2} & 0 & 0 \\ -\frac{1}{4(x-2)} & \frac{1}{2(x-2)} & -\frac{x}{x-2} & 0 & 0 \\ -\frac{1}{4}\frac{3x^2-5x+1}{(x^2-3x+2)(x-1)} & -\frac{1}{2}\frac{x^2-3x-1}{x^3-4x^2+5x-2} & -\frac{3x}{x^3-4x^2+5x-2} & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{12}\left(3x-5+\frac{1}{x}\right)\delta & -\frac{1}{6}\left(x-3-\frac{1}{x}\right)\delta & 0 & 0 & 0 \\ \frac{1}{12}(1+3x)\delta & \frac{1}{6}(x-1)\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & -x^2+2x-1 \\ 0 & 0 & 0 & \frac{1}{2}x+3 & -\frac{1}{2}x^2+x-\frac{1}{2} \end{bmatrix}$$

From Maple implementation, we have two unimodular matrices $S, T \in GL_n(\mathbb{K}[[z]])$,

$$S = \begin{pmatrix} \frac{1}{9}\frac{6z^2-11z+1}{z} & \frac{1}{9}\frac{3z^2-7z-1}{z} & 0 & \frac{2}{9}\frac{z+1}{z} & -\frac{1}{9}\frac{3z^2-5z+1}{z} \\ -\frac{1}{9}-\frac{2}{3}z & \frac{1}{9}-\frac{1}{3}z & 0 & -\frac{2}{9} & \frac{1}{9}+\frac{1}{3}x \\ \frac{4}{3} & \frac{5}{3} & 1 & \frac{2}{3} & -\frac{4}{3} \\ -3 & -1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} \frac{1}{4}\frac{2z-3}{(z^2-3z+2)(z-1)} & \frac{1}{2}\frac{2z-5}{z^3-4z^2+5z-2} & -\frac{z^2-2z-2}{z^3-4z^2+5z-2} & -1 & z-2 \\ \frac{1}{8(z-2)} & -\frac{1}{4(z-2)} & \frac{1}{z-2} & 0 & 0 \\ -\frac{1}{4(z-2)} & \frac{1}{2(z-2)} & -\frac{z}{z-2} & 0 & 0 \\ -\frac{1}{4}\frac{3z^2-5z+1}{(z^2-3z+2)(z-1)} & -\frac{1}{2}\frac{z^2-3z-1}{z^3-4z^2+5z-2} & -\frac{3z}{z^3-4z^2+5z-2} & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and the reduced matrix differential operator is

$$\widetilde{L} = \begin{pmatrix} -\frac{1}{12}\left(3z-5+\frac{1}{z}\right)D & -\frac{1}{6}\left(z-3-\frac{1}{z}\right)D & 0 & 0 & 0 \\ \frac{1}{12}(1+3z)D & \frac{1}{6}(z-1)D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -z^2+2z-1 \\ 0 & 0 & 0 & \frac{1}{2}z+3 & -\frac{1}{2}z^2+z-\frac{1}{2} \end{pmatrix}$$

4 Conclusion

In this paper, we discussed the Maple implementation of a reduction algorithm for differential-algebraic systems, `DAS_Reduction`. Sample computations are presented to illustrate the proposed algorithm. This algorithm will be helpful to implement this in commercial software packages such as Mathematica, Matlab, Singular, SCIlab etc. Some numerical examples are presented to demonstrate the proposed algorithm. This package is provided at https://srinivasaraothota.webs.com/research with example worksheets.

Competing interests

The authors declare that they have no competing interests.

References

[1] C. El. Bachaa: Methodes Algebriques pour la Resolution d' Equations Differentielles Matricielles d'Ordre Arbitraire, PhD Thesis, Universite de Limoges, 2011.

[2] M. Barkatou, C. El. Bacha, E. Pugel: Simultaneously row- and column-reduced higher-order linear differential systems, In: Proceedings of ISSAC'10. ACM, Munich, Germany (2010), 45-52.

[3] B. Beckermann, H. Cheng, G. Labahn: Fraction-free row reduction of matrices of ore polynomials, Journal of Symbolic Computation, 41 (5) (2006), 513-543.

[4] W. A. Harris, Y. Sibuya, L. Weinberg: A reduction algorithm for linear differential systems, Funkcialaj Ekvacioj, 11 (1968), 59-67.

[5] S. Thota: On A Symbolic Method for Fully Inhomogeneous Boundary Value Problems, Kyungpook Mathematical Journal, 59(1) (2019), 13-22.

[6] S. Thota: On a New Symbolic Method for Initial Value Problems for Systems of Higher-order Linear Differential Equations, International Journal of Mathematical Models and Methods in Applied Sciences, 12 (2018), 194-202.

[7] S. Thota: Initial value problems for system of differential-algebraic equations in Maple, BMC Research Notes, (2018) 11:651.

[8] S. Thota, S. D. Kumar: Maple Implementation of Symbolic Methods for Initial Value Problems, Research for Resurgence, Vol-1, No. 1(2018), 21-39. ISBN: 978-93-83861-12-5.

[9] S. Thota, S. D. Kumar: Solving system of higher-order linear differential equations on the level of operators, International journal of pure and applied mathematics, 106 (1) (2016), 11-21.

[10] S. Thota, S. D. Kumar: Symbolic algorithm for a system of differential-algebraic equations, Kyungpook Mathematical Journal, 56(4) (2016), 1141-1160.

[11] S. Schulz: Four lectures on differential algebraic equations, Technical Report 497, The University of Auckland (2003).

[12] S. Thota, S. D. Kumar: A New Reduction Algorithm for Differential-Algebraic Systems with Power Series Coefficients, Information Sciences Letters, 10(1) (2021), 59-66.

[13] S. Thota, P. Shanmugasundaram: On A Symbolic Method for Neutral Functional-Differential Equations with Proportional Delays, Cogent Mathematics & Statistics (2020) 7:1, 1813961.

[14] S. Thota: On Solving System of Linear Differential-Algebraic Equations using Reduction Algorithm, International Journal of Mathematics and Mathematical Sciences, Vol. 2020, Article ID 6671926, 10 pages.

[15] S. Thota: On Maple Implementation of A Reduction Algorithm for Differential-Algebraic Systems, International Conference on Mathematical Advances & Applications 2020, November 23-24, 2020, Jointly Organized by REd Talks Daily International, IIM Australia, IIST Malaysia, DRPF Macedonia, IMRF Thailand.

[16] S. Thota, S. D. Kumar: Maple Implementation of a Reduction Algorithm for Differential-Algebraic Systems with Variable Coefficients, International Conference on Recent Trends in Engineering & Technology and Quest for Sustainable Development, May 13-14, 2019, Institute of Technology, Ambo University, Ethiopia.

[17] S. Thota: Maple Implementation of A Symbolic Method for Fully Inhomogeneous Boundary Value Problems, 2021 International Conference on Mathematics and Computers in Science and Engineering (MACISE 2021), January 18-20, 2021, Madrid, Spain.

[18] S. Thota: Maple Implementation of A Symbolic Method for Fully Inhomogeneous Boundary Value Problems, Int. J. of Applied Mathematics, Computational Science and Systems Engineering, 3 (2021), 1-5.

[19] S. Thota, S. D. Kumar: Maple Implementation of A Reduction Algorithm for Differential-Algebraic Systems with Variable Coefficients. International Conference on Recent Trends in Engineering & Technology and the Quest for Sustainable Development (ICRTETQSD 2019), at Institute of Technology, Ambo University, Ethiopia, May 13-14, 2019.

[20] S. Thota: Maple Implementation for Reducing Differential-Algebraic Systems, Conference on Mathematics for the Advancement of Science, Technology and Education, at Ethiopian Mathematics Professionals Association (EMPA), Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University, Ethiopia, March 5-6, 2021.