Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/030402

Derivation of the Fractional Dodson Equation and Beyond: Transient Diffusion With a Non-Singular Memory and Exponentially Fading-Out Diffusivity*

*Jordan Hristov**

Department of Chemical Engineering, University of Chemical Technology and Metallurgy,Sofia, Bulgaria.

Received: 11 Jul. 2016, Revised: 24 Nov. 2016, Accepted: 29 Nov. 2016
Published online: 1 Oct. 2017

Abstract: Starting from the Cattaneo constitutive relation with exponential kernel applied to mass diffusion the derivation of a new form the diffusion equation with a relaxation term expressed through the Caputo-Fabrizio time-fractional operator (derivative) has been developed. The developed equation reduces to the fractional Dodson equation for large relaxation times corresponding to low fractional order of the Caputo-Fabrizio derivative. The approach separates large time effects resulting in the classical Dodson equation with exponentially decaying in time diffusivity and the short time relaxation process modeled by Caputo-Fabrizio time fractional derivative. The solution developed allows seeing a new physical background of the Caputo-Fabrizio time-fractional operator (derivative) and to demonstrate a new interpretation of the Dodson equation incorporating fading memory effects. Moreover a new model with two memories corresponding to large and short time relation effects has been conceived. Defining the diffusion process parameters then the fractional order of the Caputo-Fabrizio time fractional derivative can be determined in a straightforward manner as a function of the Deborah number calculated as a ratio of the relaxation time to the characteristic diffusion time of the process.

Keywords: Dodson equation, new derivation, two-memory model, Caputo-Fabrizio time-fractional derivative, Deborah number.

1 Introduction

1.1 Dodson Diffusion Equation: Physical Background and Original Derivation

The Dodson diffusion equation (1) was derived as result of diffusion of species in minerals [1,2] in an analysis of the cooling history in geochronological systems where the diffusion coefficient depends on temperature in accordance with the Arrhenius equation (2) , namely

$$\frac{\partial C(x,t)}{\partial t} = D(T)\frac{\partial^2 C(x,t)}{\partial x^2}, \tag{1}$$

$$D = D_0 exp\left(-\frac{E}{RT}\right). \tag{2}$$

In general, the solid diffusion process are thermally activated [3,4]. The diffusion coefficient represents the diffusion process at infinitely high temperatures. In (2) the absolute temperature is T , R is the universal gas constant and E is the activation energy of the diffusion process. In accordance to Dodson [1] due to the very strong dependence of the diffusion coefficient, the transitional temperature range can be expected to be reasonably short. Moreover, Dodson used the fact that over a limited range of temperature the cooling history in geochronological systems can be conveniently be approximated

* This article is dedicated to 50th anniversary of the classical Caputo derivative and to the 2nd anniversary of the seminal article of Caputo and Fabrizio published in *Progress in Fractional Differentiation and Applications*, No 1, 2015

* Corresponding author e-mail: jordan.hristov@mail.bg

by a linear increase in time [1,2,3,4,5,6]. The exponential decrease in diffusion coefficient is conventionally described in terms of time constant which is the time taken for $D(t)$ to reduce by a factor of e. Then, following Dodson [1] we have

$$D = D_0 exp\left(-\frac{E}{RT_0} - \frac{t}{\tau}\right) = D(0)e^{-\frac{t}{\tau}}, \tag{3}$$

where D_0 and T_0 are the values of the diffusion coefficient and the temperature at $t = 0$. From the definition (3), which actually is a definition of τ, that is

$$\frac{d}{dt}\left(\frac{E}{RT}\right) = \frac{1}{\tau} = \frac{-RT^2}{E\frac{dT}{dt}}. \tag{4}$$

Avoiding the long and cumbersome expressions of the Dodsons analysis [1,2] and related studies [3,4,5,6,7] the diffusion model (1) was expressed as

$$\frac{\partial C(x,t)}{\partial t} = D(0)e^{\frac{-t}{\tau}}\frac{\partial^2 c(x,t)}{\partial x^2}, \tag{5}$$

or equivalently

$$\frac{\partial C(x,t)}{\partial t} = D(0)e^{-\beta t}\frac{\partial^2 c(x,t)}{\partial x^2}. \tag{6}$$

1.2 Dodson Diffusion Equation: Original Solution Approach

The original Dodson approach (ref.[1]- Appendix A) is to solve (5) in its dimensionless form

$$\frac{\partial}{\partial \theta}\left(\frac{C}{C_0}\right) = -\frac{\tau D(0)}{a^2}e^{-\theta}\frac{\partial^2}{\partial x^2}\left(\frac{C}{C_0}\right), \tag{7}$$

with initial concentration C_0 and dimensionless time $\theta = t/\tau$, and a length scale a of the area where the diffusion takes place. Dodson used a new variable $q = \left(1 - e^{\lambda\tau\theta} - \frac{C}{C_0}\right)$, where λ depends on the geometry of the system, and denoting $M = \tau D(0)/a^2$ transformed (7) to

$$\frac{\partial q}{\partial \theta} = Me^{-\theta}\frac{\partial^2 q}{\partial x^2}, \tag{8}$$

with a second change of variables as $q_s = 1 - e^{-\lambda\tau\theta}$ and initial conditions $q = 0$ at $t = 0$. Then, Dodson used a common approach in solving diffusion equation with a time-dependent coefficient (see the book of Crank [8] by introducing a variable u by a integral transform, namely $u = \int_0^t D(z)dz$ [8], which allowed to reduce (8) to

$$\frac{\partial q}{\partial u} = \frac{\partial^2 q}{\partial x^2}, q_s = 1 - \left(1 - \frac{u}{M}\right)^{\lambda\tau}. \tag{9}$$

In this context, for $\theta = 0$ we get $u = 0$, but at $\theta \longrightarrow \infty$, and consequently $u \longrightarrow M$. The solution of (9) was developed on basis of the result of [9] (see page 104, eq.3). We will avoid the repeating of the cumbersome expressions of this solution, due the practical inconvenience of the result (see (10) as example) to use in the post-solution analysis as well as, because this solution is out of the scope of the present article. For a plane sheet it is, for instance

$$q = 2\sum_{i=1}^{\infty}(-1)^{i+1}\frac{cos(i-1/2)\pi x}{(i-1/2)\pi}\left(1 - \frac{\Gamma(\lambda\tau+1)}{[(i-1/2)^2\pi^2 M]^{\lambda\tau}}\right). \tag{10}$$

To complete this section, in chapter 7 of the book of Crank [8], the fist example (example 7.1, page. 104) is the case of the Dodson equation, briefly mentioned that the transforms of variables used by Dodson leads to (9).

1.3 Dodson Diffusion Equation: Integral-Balance Solution

Here the finite penetration depth concept $\delta(t)$ is assumed and corresponding to finite flux speed (6). The finite diffusion speed is the basic concept of the method of Goodman [10,11,12,13] which, in fact is a simple mass balance over the diffusion layer with a depth δ (with boundary conditions $C(\delta,t)=\partial C(\delta,t)/\partial x=0$), namely

$$\int_0^\delta \frac{\partial C(x,t)}{\partial t}dx=\int_0^\delta D(t)\frac{\partial^2 C(c,t)}{\partial x^2}dx \Longrightarrow \frac{d}{dt}\int_0^\delta C(x,t)dx=-D(t)\frac{\partial C(0,t)}{\partial x}, D(t)=D_0e^{-\beta t}, \beta=1/\tau. \tag{11}$$

The last version of (11) comes from application of the Leibniz rule. The diffusion layer depth $\delta(t)$ should be determined through the solution. Assuming an approximate profile as a function of x/δ we may apply the boundary conditions at the front δ [10,11,12,13].

The transformation $u=1-e^{-\beta t}$ leads to $\partial C/\partial t=(\partial C/\partial u)(\partial u/\partial t)\left(\beta e^{-\beta t}\right)$ which allows expressing (6) in the form

$$\frac{\partial C(x,t)}{\partial u}=\frac{D_0}{\beta}\frac{\partial^2 C(x,u(t))}{\partial x^2}. \tag{12}$$

Then the integral-balance equation [10,11,12,13,14] is

$$\frac{d}{dt}\int_0^\delta C(x,u(t))dx=-\frac{D_0}{\beta}\frac{\partial C(0,u(t))}{\partial x}. \tag{13}$$

The approximate profile is assumed as a parabolic one $C_a(x,u(t))=C_s(1-x/\delta)^n$ [12,13], where the exponent n is unspecified. For the sake simplicity of the present analysis we assume the Dirichlet problem ($C_s=1$). Thus, replacement of $C(x,t)$ by C_a in (13) yields [15]

$$\frac{1}{n+1}\frac{d\delta}{dt}=D_0\frac{n}{\delta} \Longrightarrow \delta^2=\frac{D_0}{\beta}u\left[2n(n+1)\right]. \tag{14}$$

Now, turning to the original variable, the depth of the diffusion layer $\delta(t)$ is

$$\delta=\sqrt{\frac{D_0}{\beta}}\sqrt{\left(1-e^{-\beta t}\right)\left[2n(n+1)\right]}. \tag{15}$$

The ratio (D_0/β) has a dimension of $[m^2]$. However, the function $\left(1-e^{-\beta t}\right)$ is dimensionless and it is growing in time saturating rapidly to 1 when $1/e^{\beta t}\longrightarrow 0$ becomes negligible, i.e. $t\longrightarrow\infty$. In this moment the diffusion layer depth attains it maximum $\delta\equiv\sqrt{D_0/\beta}=const.$, because the diffusion process stops. At this moment, we stop the analysis of the integer -order model of Dodson and will focus the attention on an attempt to derive this equation from an alternative point of view using basic constitutive equations relating the mass flux to the memory integral of the concentration gradient. The results just commented will allow us to define properly the relaxation (damping function).

1.4 Some Critical Remarks

The Dodson equation excludes the case when relaxation process does not exist. Precisely, if $\tau=0(\beta\rightarrow\infty)$, then the diffusion coefficient should attain its maximum value D_0 , but in the form $D_0exp(-\tau/t)=D_0exp(-\beta t)$ for $\tau\rightarrow 0$ we have $D(t)\rightarrow 0$, so there is no diffusion at the beginning when the conditions are the extremely favorable for the diffusion process to take place. The problem just raised will be commented further in this work when the fractional version of the Dodson equation will be developed.

1.5 Aim

The present article focuses on a principle problem regarding the derivation of the Dodson equation (6) starting from basic constitutive equation in contrast to the approach used in the original work [1], where the exponential function and the relaxation time to the diffusion coefficient are added *ad hoc* to the diffusion equation. The first and the most important issue in the following analysis is the problem that the model of Dodson is truly parabolic equation with infinite unphysical speed of the flux and no relaxation is taken into account. For the sake of clarity, do not misunderstand this standpoint with the fact that original Dodsons equation defines the relation time . The second problem at issue is to develop a new form of the Dodson equation in terms of time-fractional derivatives of Caputo-Fabrizio type, which are more general than the original one and its development is based on basic constitutive equations; and reducing to the integer order model when the relaxation of the flux is not taken into account. The study results in two models:

Single-memory model. This is a model with a single relaxation time which can be easily derived from the flux constitutive equation of Cattaneo [16]. The model simply demonstrates how the original Dodson equation could be developed starting from a basic flux-gradient relationship and applying the mass balance law.

Two-memories model. This is a model based on a definition of a new composite relaxation function accounting simultaneously *long-time* and *short-time* relaxation processes.

The article is organized as follows. Section 2 provides the necessary mathematical background regarding the Cattaneo constitutive equation (section 2.1.) and how on its base its is possible to derive the Caputo-Fabrizio time-fractional derivative. The basic properties of the Caputo-Fabrizio time-fractional derivative are presented in section 2.2. Section 3 presents the derivations of the two models with memory which are generalization of the Dodson equation. Especially section 3.3. demonstrates how the fractional order should be related to the Deborah number. Section 4 summarizes the results, compares them to the diffusion equation of Caputo and Fabrizio (see (64)) and previously developed models with different constitutive equations about the flux relaxation.

2 Preliminaries

2.1 The Cattaneo Constitutive Diffusion Equation and the Outcomes

Diffusion phenomena of mass, are generally described as a consequence of the mass conservation law by the relationship

$$\frac{\partial C}{\partial t} = -\frac{\partial j}{\partial x}. \tag{16}$$

The assumption that the mass flux $j(x,t)$ is proportional to the concentration gradient $j(x,t) = -D_0 \partial C(x,t)/\partial x$ in fact is a definition of the diffusivity D_0. Then, applying (16) we get the ordinary diffusion equation (the Fick law) (17),

$$\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial x^2}. \tag{17}$$

The principle drawback of the model (17) is the infinite speed of propagation of the flux which is unphysical.

A relaxation function related to a finite speed of diffusion (heat conduction) in solids was conceived by Cattaneo [16] as a generalization of the Fourier law by a linear superposition of the heat flux and its time derivative related to its history [17,18]. Hence, the flux obeys the constitutive equation [16] involving a memory integral.

$$j(x,t) = -\int_{-\infty}^{t} R(x,t) \nabla C(x,t-s) ds. \tag{18}$$

Setting the lower terminal of the memory integral in (18) at zero we get a more convenient, from engineering point view, expression of the constitutive equation, namely

$$j(x,t) = -\int_{0}^{t} R(x,t) \nabla C(x,t-s) ds. \tag{19}$$

If $R(x,t)$ is assumed as the Dirac delta $\delta_D(s)$ function such that $\int_0^t \delta_D(s) ds = 1$ this immediately leads to the classical Fick (Fourier) equation (17) since *there is no damping effect in the flux propagation*. However, for a homogeneous medium $R(x,t)$ depends only the time and can be represented by a stretched-exponential kernel [16] where the relaxation time τ is finite, i.e.$\tau = const$. Then, the mass balance (16) results in the Cattaneo equations [16]

$$\frac{\partial C(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(-\frac{D_0}{\tau}\int_0^t exp\left(-\frac{t-s}{\tau}\right)\frac{\partial C(x,t)}{\partial x} ds\right). \tag{20}$$

For $\tau \longrightarrow 0$ the limit of the Cattaneo equation (20) reduces to the Fick law. If a first order approximation, with respect to τ [19] is developed (21) we get a first order differential equation (22)

$$j(x,t+\tau) = -D_0 \frac{\partial C(x,t)}{\partial c}, j(x,t+\tau) \approx j(x,t) + \tau \frac{\partial j(x,t)}{\partial x}, \tag{21}$$

$$\frac{1}{\tau}j(x,t)+\frac{\partial j(x,t)}{\partial t}=-\frac{D_0}{\tau}\frac{\partial C(x,t)}{\partial x}. \tag{22}$$

The integration of (22) leads to (20), which can be presented in two equivalent forms

$$\frac{\partial C(x,t)}{\partial t}=\frac{D_0}{\tau}\int_0^t exp\left(-\frac{t-s}{\tau}\right)\frac{\partial^2 C(x,s)}{\partial x^2}ds, \tag{23}$$

$$\frac{\partial C(x,t)}{\partial t}=\beta D_0\int_0^t exp(-\beta(t-s))\frac{\partial^2 C(x,s)}{\partial x^2}ds, \beta=\frac{1}{\tau}. \tag{24}$$

At this end, we have to stress the attention on the memory integrals in (23)and (24) in order to relate them to time-fractional derivatives with non-singular kernels, precisely the time-fractional Caputo-Fabrizio (operator) derivative presented next.

2.2 Time-Fractional (Operator) Derivative of Caputo-Fabrizio

Caputo and Fabrizio [20] suggested a time-fractional (operator) derivative with an exponential kernel defined as

$$_{CF}D_t^{\alpha}=\frac{M(\alpha)}{1-\alpha}\int_0^t exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)\frac{df(t)}{dt}ds=\frac{1}{1-\alpha}\int_0^t exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)\frac{df(t)}{dt}ds. \tag{25}$$

With the operator (25) if we have constant ($f(t)=C=const.$) then as in the classical Caputo derivative [21] we have $_{CF}D_t^{\alpha}f(t)=0$. In [3] an alternative definition of time-fractional (operator) derivative (25) was suggested, namely

$$_{CF}D_t^{\alpha}=\frac{\alpha}{(1-\alpha)^2}\int_{a_0}^t \left(f(t)-f_{a_0}(s)\right)exp\left(-\frac{\alpha}{1-\alpha}\right)ds, t>0. \tag{26}$$

The Laplace transform of $_{CF}D_t^{\alpha}f(t)$ with $a_0=0$ given with p variable is [20]

$$L_T\left[_{CF}D_t^{\alpha}f(t)\right]=\frac{pL_T\left[f(t)-f(0)\right]}{p+\alpha(1-p)}. \tag{27}$$

Both the applications and the properties of the Caputo-Fabrizio time-fractional operator 25 are intensively investigated and for about two years after [20,22] numerous articles have been published, among them: mass-spring damped systems [23], fractional electric circuit [23,24], the Keller-Segel model [25], groundwater flow [26], mechanics and heat transfer of non-Newtonian fluids [27], long wave equations [28] pure mathematical studies [29,30,31] and provoking new computational techniques [32,33], and innovations in creation of fractional derivatives with Mittag-Leffler kernels [34,35].

In most of the articles published in the last 2 years after [20,22], the common approach is: *a just simple change (replacement) of the integer-order time-derivative in the existing models by a time-fractional counterparts* [23,25,26,28,33]. This is a *formalistic fractionalization*, since the appearance of time-fractional derivative *should come from constitutive laws related to real physical processes with relaxations*. In cases of time-fractional derivatives of Riemann-Liouville or Caputo type the formalistic fractionalization is assumed as a rule in purely mathematical articles and this approach was seriously criticized in [37] (Chapter 7). Now, turning on the Caputo-Fabrizio time-fractional derivative, it was demonstrated in [38] that starting from the Cattaneo constitutive equation [16] and using Jeffrey's kernel [39] the Caputo-Fabrizio time-fractional derivative appears naturally in the transient heat conduction (diffusion) equation, but this does not affect the integer-order time derivative. The same approach was used to develop a model with space-memory in the steady state heat conduction [40]. The analysis and the consequent derivation of the diffusion equations in this article are not based on the formalistic fractionalization approach and begin from constitutive equations related to the flux relaxation.

3 Towards the Dodson Equation with Non-Singular Memory

3.1 A Model with a Single Memory

Now, we define the Cattaneo equation as a diffusional flux constitutive equation. For the sake of simplicity, let us consider a virgin medium subjected to a mass load at $x = 0$, that is the following initial and boundary conditions take place

$$C(x,0) = C(0,0) = C(\infty,t) = C_x(x,0) = C_{xx}(x,0) = 0, C(0,t) = 0. \tag{28}$$

Now we focus on eq.(24) and denote $F(x,t) = \partial^2 C(x,t)/\partial x^2$ for the sake of simplicity in calculations. Then, from (28) we have $F(x,0) = \partial^2 C(x,0)/\partial x^2 = 0$.

Integrating by parts of the diffusion term of eq. (24) we get (in details)

$$\beta \int_0^t e^{-\beta(t-s)} F(x,s) ds = \left[e^{-\beta(t-s)} F(x,t) \right]_{s=0}^{s=t} + \beta \int_0^t e^{-\beta(t-s)} \left[F(x,t) - F(x,s) \right] ds. \tag{29}$$

Finally ,

$$\beta \int_0^t e^{-\beta(t-s)} F(x,s) ds = \left(1 - e^{-\beta t} \right) F(x,t) + \beta \int_0^t e^{-\beta(t-s)} \left[F(x,t) - F(x,s) \right] ds. \tag{30}$$

It noteworthy that if the lower terminal of the memory integral is $-\infty$, as in the original Cattaneo concept (see eq.(4), then the first term in (12) is $\left[e^{-\beta t} F(x,t) \right]_{s=0}^{s=t} = 0$ and the exponential terms of (30) will be lost. Hence, one again, it is more realistic to use the second form of the Cattaneo constitutive equation presented by eq. (23) or eq.(24). In terms of the original variable $C(x,t)$ we may present (30) as

$$\beta \int_0^t e^{-\beta(t-s)} \frac{\partial^2 C(x,s)}{\partial x^2} ds = \left(1 - e^{-\beta(t)} \right) \frac{\partial^2 C(x,t)}{\partial x^2} + \beta \int_0^t e^{-\beta(t-s)} \left(\frac{\partial^2 C(x,t)}{\partial x^2} - \frac{\partial^2 C(x,s)}{\partial x^2} \right) ds. \tag{31}$$

The second term in the right-hand side of (31) matches the definition of the Caputo-Fabrizio fractional derivative presented by eq.(26). As it was demonstrated in [38] that this term can be considered as a *pro-Caputo* (non-normalized) derivative denoted as $_{pc}D_t^{\beta}$ with a lower terminal at 0. In terms of $C(x,t)$ we may express two equivalent forms of $_{PC}D_t^{\beta}$, following [20] , namely

$$_{PC}D_t^{\beta} \left(\frac{\partial^2 C(x,t)}{\partial x^2} \right) = \beta \int_0^t e^{-\beta(t-s)} \left(\frac{\partial^2 C(x,t)}{\partial x^2} - \frac{\partial^2 C(x,s)}{\partial x^2} \right) ds, \tag{32}$$

$$_{PC}D_t^{\beta} \left(\frac{\partial^2 C(x,t)}{\partial x^2} \right) = \beta \int_0^t e^{-\beta(t-s)} \frac{d}{dt} \left(\frac{\partial^2 C(x,s)}{\partial x^2} \right) ds. \tag{33}$$

Since the rate constant $\beta \in (0,\infty)$ controls the exponential kernel, then $_{PC}D_t^{\beta}$ can be arranged in the form defined by (25) with a fractional order α . From this concept it follows that for $\alpha \in [0,1] \Longrightarrow 1/\beta \in [0,\infty]$. Consequently, the following relationships are valid [20,22]

$$\frac{1}{\beta} = \frac{1-\alpha}{\alpha} \in [0,\infty], \alpha = \frac{1}{1+1/\beta} \in [0,1], \frac{\alpha}{(1-\alpha)^2} = \frac{\beta}{(1-\alpha)}. \tag{34}$$

From the definition (25) [20,22] we get

$$\partial_{CF} D_t^{\alpha} \left(\frac{\partial^2 C(c,t)}{\partial x^2} \right) = \frac{N(\sigma)}{\sigma} {}_{PC}D_t \left(\frac{\partial^2 C(x,t)}{\partial x^2} \right) = \beta \frac{M(\alpha)}{(1-\alpha)} \int_0^t e^{-\beta(t-s)} \frac{d}{dt} \left(\frac{\partial^2 C(x,s)}{\partial x^2} \right) ds, \tag{35}$$

or equivalently

$$_{CF}D_t^{\alpha} \left(\frac{\partial^2 C(c,t)}{\partial x^2} \right) = \frac{N(\sigma)}{\sigma} {}_{PC}D_t \left(\frac{\partial^2 C(x,t)}{\partial x^2} \right) = \frac{\alpha}{1-\alpha} \frac{M(\alpha)}{(1-\alpha)} \int_0^t e^{-\beta(t-s)} \frac{d}{dt} \left(\frac{\partial^2 C(x,s)}{\partial x^2} \right) ds. \tag{36}$$

with $((1-\alpha)/\alpha)N(\sigma) = M(\alpha)/(1-\alpha)$ and $\sigma = 1/\beta$; $N(\sigma)$ and $M(\alpha)$ are normalizing functions [20,22]. Consequently, we get

$$_{CF}D_t^{\alpha} \left(\frac{\partial^2 C(c,t)}{\partial x^2} \right) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t e^{-\beta(t-s)} \frac{d}{dt} \left(\frac{\partial^2 C(x,s)}{\partial x^2} \right) ds. \tag{37}$$

By help of the definition (26) (see the normalizing function $\alpha/(1-\alpha)^2$), considering $M(\alpha)=1$ as in [1,2] and mainly using the results (31) and (36) we may write the diffusion term of (24) in the form

$$\beta\int_0^t e^{-\beta(t-s)}\frac{\partial C(x,s)}{\partial x^2}ds = D_0\left(1-e^{-\beta t}\right)\left(\frac{\partial^2 C(x,t)}{\partial x^2}\right)+D_0(1-\alpha)\frac{1}{1-\alpha}\int_t^t e^{-\frac{-\alpha}{1-\alpha}(t-s)}\frac{d}{dt}\left(\frac{\partial^2 C(x,s)}{\partial x^2}\right)ds. \tag{38}$$

Finally, the new form of equation (24) is

$$\frac{\partial C(x,t)}{\partial t} = D_0\left(1-e^{-\beta t}\right)\frac{\partial^2 C(x,t)}{\partial x^2}+D_0(1-\alpha)_{CF}D_t^{\alpha}\left(\frac{\partial^2 C(x,t)}{\partial x^2}\right). \tag{39}$$

For $t=0$ when practically no relaxation exists ($\tau\approx 0\Rightarrow\beta\to\infty$) we get $D_0(1-e^{-\beta t})\approx D_0$ and the diffusion coefficient has a maximal value D_0 . Further, $D_0(1-e^{-\beta t})$ can be re-arranged as $D_0e^{-\beta t}\left(\frac{1-e^{-\beta t}}{e^{-\beta t}}\right)$. However, if $\beta t<<1$ we may approximate the exponential term as a series $1-\beta t+((\beta t)^2/2)+O((\beta t)^3)$. Using only the first two terms we have $e^{-\beta t}\approx 1-\beta t$ and consequently the term $\left(\frac{1-e^{-\beta t}}{e^{-\beta t}}\right)$ approximates as $\beta t/(1+\beta t)\approx O(1)$. Hence, with the assumption $\beta t=t/\tau<<1$ we get

$$\frac{\partial C(x,t)}{\partial t}\approx D_0e^{-\beta t}\frac{\partial^2 C(x,t)}{\partial x^2}+D_0(1-\alpha)_{CF}D_t^{\alpha}\left(\frac{\partial^2 C(x,t)}{\partial x^2}\right). \tag{40}$$

The first term in the right-hand side of (40) matches the diffusion term of the Dodson equation. For $\alpha=1$, formally we get the Dodson equation (6). However, this statement should be regarded in view of the fact that when $\beta\to\infty$ we have $\alpha\to 1$ and $\tau\to 0$. Therefore, from the results developed to this point the Dodson equation is an approximation corresponding to situations when $\beta t=t/\tau<<1$ and $\alpha\to 1$. Decreasing in α, that physically means increasing in the damping effect to the mass flux propagation, the weight of the last term in (39) increases, but the approximation which allowed to obtain (40) is not valid yet.

3.2 A Model with Two Memories

3.2.1 Conjecture

Here we conceive a diffusional flux equation $j_a=\int_0^t R_a(t,\tau_0,\tau_s)\frac{\partial C(x,s)}{\partial x}ds$ with a composite memory kernel presented by the following constitutive relationship

$$R_a(t)=e^{-\beta_0(t-s)}\left(1-e^{-\beta_s(t-s)}\right)=e^{-\beta_0(t-s)}-e^{-(\beta_0+\beta_s)(t-s)}. \tag{41}$$

$R_a(t)$ is a product of a *large-time exponential kernel* $e^{-\beta_0(t-s)}$ and a *short-time fading function* $\left(1-e^{-\beta_s(t-s)}\right)$.

The constitutive equation (41) suggests that $\tau_0>>\tau_s$ and consequently we have $\beta_0<<\beta_s$. Thus β_0 corresponds to large-time relaxation processes, while β_s accounts the short time relaxation mechanism. Moreover, the constitutive relation $\beta=\beta_0+\beta_s$ means that the large-time and short-time relaxations occur simultaneously and overlap. Hence, $\beta=1/\tau=1/\tau_0+1/\tau_s$ and therefore $\tau=\tau_0\tau_s/(\tau_0+\tau_s)$. When the time-scale of the diffusion process is order of magnitude of τ_s, taking into account that $\tau_s<<\tau_0$, the approximation is $lim_{t\longrightarrow\tau_s}\tau=\tau_s$ and $\beta\approx\beta_s$, that is $\beta=\beta_0+\beta_s\approx\beta_s$ when only short-time relaxation has to be accounted for. Alternatively, when the time-scale of the process is comparable to τ_0 then $lim_{t\longrightarrow\tau_0}\tau=\tau_0$ and $\beta=\beta_0$. In other words, for short times the term $\left(1-e^{-\beta_s(t-s)}\right)$ dominates since $e^{-\beta_0(t-s)}$ has little effect due to the fact that $\beta_0<<\beta_s$. For large times we have $\left(1-e^{-\beta_s(t-s)}\right)\longrightarrow 1$ and only $e^{-\beta_0(t-s)}$ remains as a memory function.

3.2.2 Approximation of the Memory Integral

Now, following the conjecture, the approximation of the memory integral with the composite damping function is

$$(\beta_0+\beta_s)\int_0^t\left(e^{-\beta_0(t-s)}-e^{-(\beta_0+\beta_s)(t-s)}\right)F(x,s)ds\approx\beta_0\int_0^t e^{-\beta_0(t-s)}F(x,s)ds-\beta_s\int_0^t e^{-\beta_s(t-s)}F(x,s)ds. \tag{42}$$

where $F(x,t) = \partial^2 C(x,t)/\partial x^2$

Therefore, we have two distinguished memory integrals. Further, we will repeat the technique of integration by parts in the right-hand side of (42) as it was done to the single-memory model, but now, to each memory integral separately. Now, recall that from the conjecture $\beta_s >> \beta_0$ and consequently the factor of the first term in the right-hand side of (41) can be approximated as $e^{-\beta_0 t} \approx e^{-\beta_0 t} - e^{-\beta_s t}$ because $e^{-\beta_0}t >> e^{-\beta_s t}$. After these final adjustments and approximations the new time-fractional equation with two memories, and skipping the huge expressions (that could be easily performed by the readers) we get

$$\frac{\partial C}{\partial t} = D_0 \left[P(x,t,\alpha) - Q(x,t,\alpha)\right]. \tag{43}$$

Where the diffusion term in (43) has two components

$$P(x,t,\alpha) = \left(1 - e^{-\beta t}\right) \frac{\partial^2 C(x,t)}{\partial x^2} + (1-\alpha_0)_{CF} D_t^{\alpha_0} \left[\frac{\partial^2 C(x,t)}{\partial x^2}\right], \tag{44}$$

$$Q(x,t,\alpha) = \left(1 - e^{-\beta t}\right) \frac{\partial^2 C(x,t)}{\partial x^2} + (1-\alpha_s)_{CF} D_t^{\alpha_s} \left[\frac{\partial^2 C(x,t)}{\partial x^2}\right]. \tag{45}$$

Hence, the complete form of (43) is

$$\frac{\partial C(x,t)}{\partial t} = D_0 \left[\left(1 - e^{-\beta_0 t}\right) + (1-\alpha_0)_{CF} D_t^{\alpha_0}\right] \frac{\partial^2 C(x,t)}{\partial x^2} - D_0 \left[\left(1 - e^{-\beta_s t}\right) + (1-\alpha_s)_{CF} D_t^{\alpha_s}\right] \frac{\partial^2 C(x,t)}{\partial x^2}. \tag{46}$$

For $\beta_0 t << \beta_s t$ and certainly, when $\beta_0 t << 1$ the first term in (46) reduces to $D_0 e^{-\beta_0 t}$ as it was demonstrated with (40).

When the short time relaxation (damping effect) is neglected, that is when $\alpha_s = 1$, we get the model with a single memory. Besides, when the relaxations in the mass flux are generally neglected, that is when $\alpha_0 = \alpha_s = 1$, we obtain the classical integer-order Dodson equation (6) with fading diffusion coefficient. This does not contradict the fact that the term $e^{-\beta_0 t}$ remains and this point will be especially discussed when the values of the fractional orders have to be specified.

The negative sign of the short-time memory term simply means that short-time relations effects, if they exist, accelerate the total diffusion process. To be exact, let see the construction of the two-memory relaxation kernel and its logical origin. The short time relaxation is modeled by the diminishing function

$$1 - e^{-\frac{(t-s)}{\tau_s}} = 1 - e^{-\beta_s t}. \tag{47}$$

In fact, this is the fading time-dependent function of the penetration depth $\delta(t)$ (see eq. (15) and the related comments). The function (47) appears in the non-linear transform $u = \int_0^t e^{-\beta t} dt = \left(1 - e^{-\beta t}\right)/\beta$ used by Dodson [1] (see also the book of Crank [8]). It rapidly grows from zero to unity for $\beta t \approx 10$ (see Fig. 1).

The expanded expression of $R_a(t)$ can be approximated as (see eq.(41)

$$Ra(t) = e^{-\beta_0(t-s)} - e^{-(\beta_0+\beta_s)(t-s)} \approx e^{-\beta_0(t-s)} - e^{-\beta_s(t-s)}. \tag{48}$$

Hence, we have a counter-current action of the relaxation kernels, which means that the short-time kernel reduces the damping effect of the effect of large-time kernel, thus accelerating the diffusion process; the same as it was commented about eq. (41).

3.3 Fractional Order: How to Define It?

The definition of the fractional order α if of primary importance since the models developed should be practically implemented or at least to be used in numerical simulations. Hence, the reasonable question is: How to calculate the fractional order α if the process parameter such as D_0 and the length scale a are known? We will start the answer with the single-memory model as an instructive example.

The definition of the stretched exponential $exp\left[-\beta(t-s)\right]$ shows directly that the dimensions of β is $[1/s]$. However, while the fractional order α is dimensionless, the rate constant $\beta(\alpha) = \alpha/(1-\alpha)$ has a dimensions of $[1/s]$, or more precisely the ratio $(1-\alpha)/\alpha$ has a dimension of time. Now, the question is how this conflict could be avoided? To overcome the problem we define a time scale that can be defined by the initial conditions of the diffusion process. With

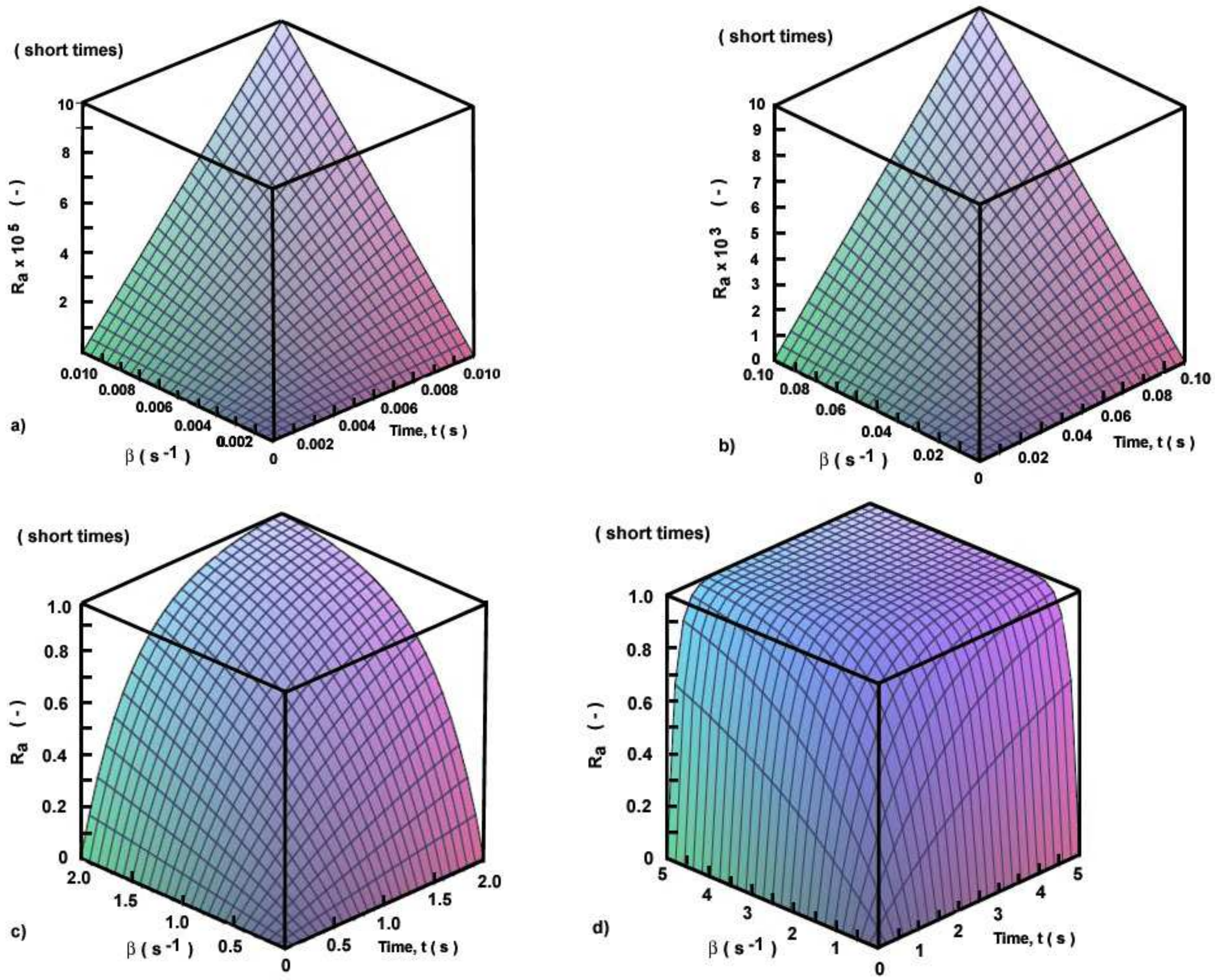

Fig. 1: Short time relaxation kernel $1-e^{-\beta t}$ as a function of the rate constant β and the time t .

D_0 and the length scale a (we use the same notation as Dodson [1]) of the area where the diffusion takes place the characteristic diffusion time is $t_D = a^2/D_0$ and consequently the time t can be scaled as $\overline{t} = t/t_D = D_0 t/a^2$. In fact the dimensionless time t/t_D is the Fourier number $Fo = D_0 t/a^2$ defined through the initial diffusivity D_0. Now we turn on the stretched exponential which can be rescaled as

$$exp\left[-\beta_0(t-s)\right] = exp\left[-\frac{(t-s)}{\tau_0}\right] = exp\left[-\left(\frac{t_D}{\tau_0}\right)(\overline{t}-\overline{s})\right]. \tag{49}$$

Hence, from the definition of the fractional order α_0 we have

$$\frac{\alpha_0}{1-\alpha_0} = \frac{t_D}{\tau_0} = \frac{1}{{}_0De} = \beta_0 t_D \Longrightarrow \alpha_0 = \frac{1}{1+t_D/\tau_0}, \tag{50}$$

or equivalently

$$\alpha_0 = \frac{1}{1+(\tau_0 D_0/a^2)} = \frac{1}{1+(D_0/\beta_0 a^2)} = \frac{1}{1+{}_0De} < 1. \tag{51}$$

The ratio $\tau_0/t_D = \tau_0 D_0/a^2 = {}_0De$ is the Deborah number for the macroscopic (large-time) diffusion relaxation process defined by analogy with the non-Fickian diffusion in complex systems [41,42] . Hence, with known values of a, τ_0, and

D_0 we will be able to define $_0De$, and then to calculate α_0 . When relaxation does not exist, that is for $\tau_0 = 0$ we have $_0De = 0$ and $\alpha_0 = 1$.

Similarly, for the short-time relaxation function with known $\tau_s = 1/\beta_s$ we may calculate the short-time Deborah number $sDe = \tau_s/t_D$ and consequently $\alpha_s = 1/(1 +_s De)$.

The plots in Fig.2 demonstrate the functional relationship $\alpha = \alpha(De)$. It is obvious that the lovely value of $\alpha = 0.5$, commonly used in numerical simulations, corresponds to $De = 1$, that is when the relaxation time equals the characteristic diffusional time of the systems, i.e. when $\tau = a^2/D_0$. The relationships (50) and (51) are quite informative from physical point of view and may be constructive in interpretations of the phenomena behind the model. In addition, similar approach relating the process parameters to the fractional order in case of steady-state heat diffusion (with a spatial memory of Caputo-Fabrizio type) was developed in [40].

Further, we may present the short-time memory kernel as a function of the fractional order α and the Fourier number. This new presentation indicates that for $\alpha \to 1$ and short times, i.e. low Fourier numbers $R_{a(short\ times)} = 1 - e^{-\beta_s(t-s)} = 1 - e^{(-\alpha/(1-\alpha))Fo}$ rapidly grows to 1 (see Fig.3a). The decrease in the value of α hinders the increase of $R_{a(short\ times)}$ at very short times. With increase in Fo the damping effect of α is stronger for $\alpha < 0.5$ (see Fig.3b and Fig.3c).

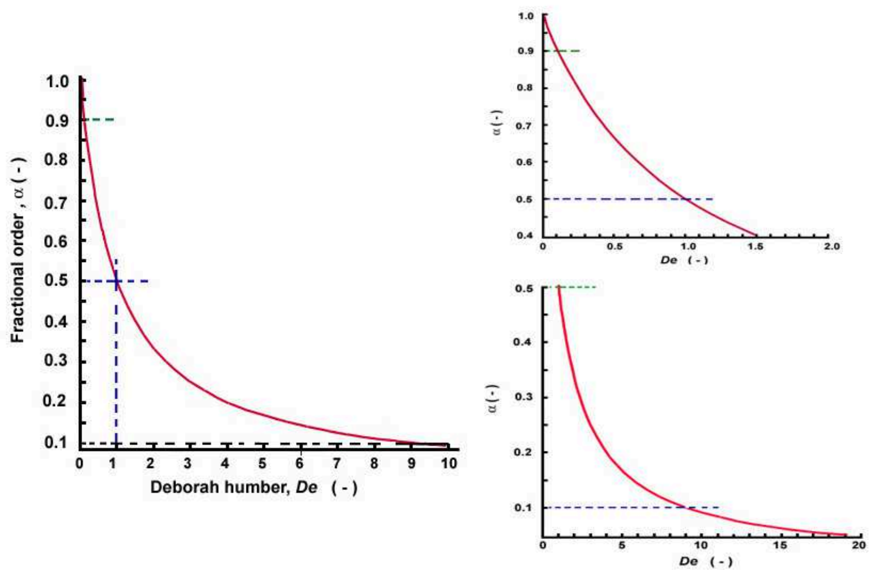

Fig. 2: Functional relationship $\alpha = \alpha(De)$ defining the fraction order as a function of the Deborah number.

3.4 Complete Fractional Expressions of the Dodson Equation with Memory

At this point we focus the attention on the single time-relaxing term of integer order, that is the decaying diffusion coefficient $D_0 e^{-\beta t}$. Following the technology applied to the exponential memory kernel we may present the diffusion coefficient as (taking into account the relationships (50))

$$D_0 exp(-\beta_0 t) = D_0 exp\left(-\beta_0 t_D \left(\frac{t}{t_D}\right)\right) = D_0 exp\left(-\frac{\alpha_0}{1-\alpha_0} Fo\right). \quad (52)$$

From the relations

$$\beta_0 t_D = \frac{1}{_0De} = \frac{t_D}{\tau_0} = \frac{\alpha_0}{1-\alpha_0}, \frac{\tau_0}{t_D} = \frac{1-\alpha_0}{\alpha_0}, \quad (53)$$

it follows directly that if no relaxation exists, that is for $\tau_0 = 0$ we get $\alpha_0 = 1$ and consequently $D = D_0$. However, considers this comment with caution when interpret the physics behind the model because it affects the diffusion coefficient of the original Dodsons equation, especially when it is expressed through the fractional order α, as it s

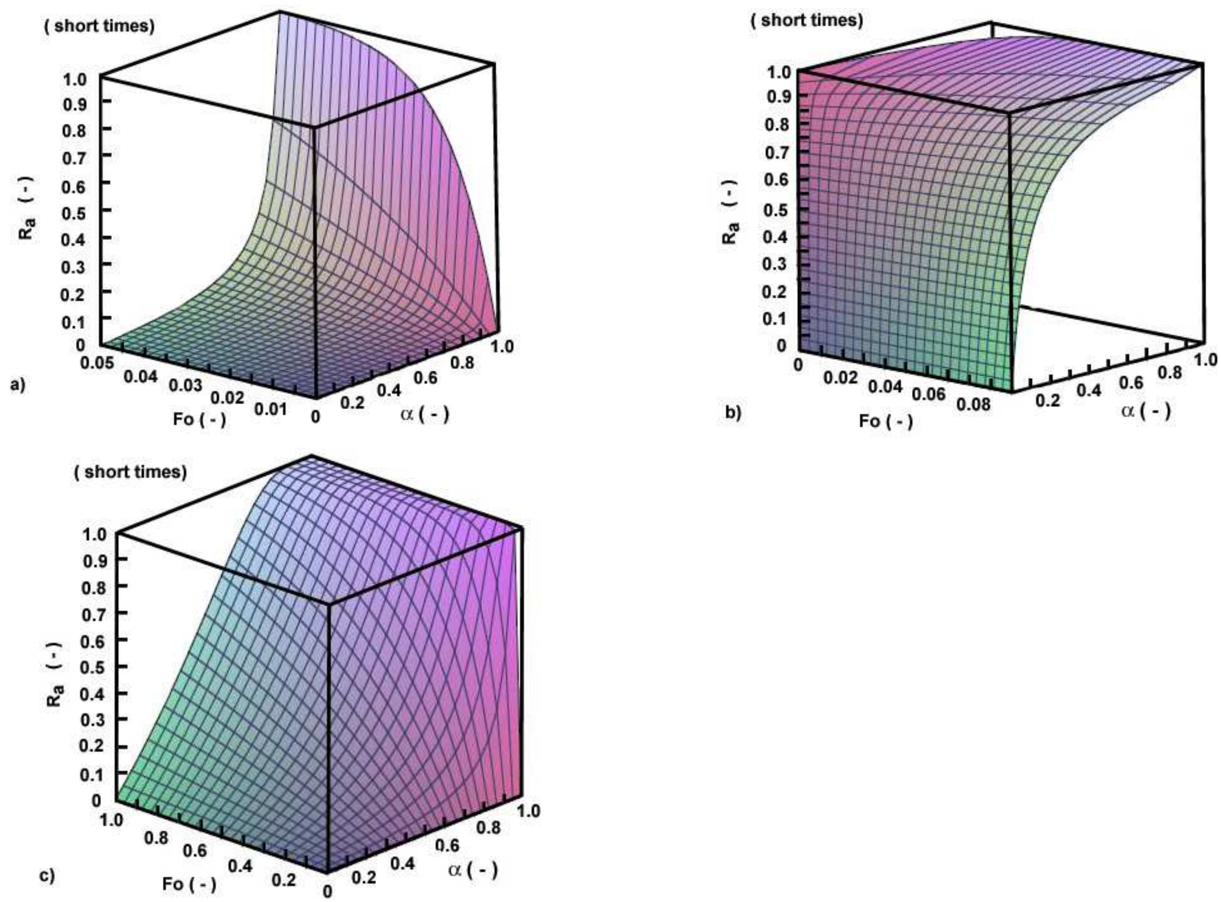

Fig. 3: Short time relaxation kernel $1-exp\left(-\frac{\alpha}{1-\alpha}Fo\right)$ as a function of the fractional order α and the Fourier number Fo

demonstrated next. Now, we may express two extended Dodson equations (39) and (46) in complete fractional forms, namely

Single-memory Model

$$\frac{\partial C(x,t)}{\partial t}=\left(1-e^{-\frac{\alpha_0}{1-\alpha_0}Fo}\right)\left(\frac{\partial^2 C(x,t)}{\partial x^2}\right)+D_0(1-\alpha_0)_{CF}D_t^{\alpha_0}\left(\frac{\partial^2 C(x,t)}{\partial x^2}\right). \tag{54}$$

Two-memory Model

$$\frac{\partial C(x,t)}{\partial t}=D_0\left[\left(1-e^{-\frac{\alpha_0}{1-\alpha_0}Fo}\right)+(1-\alpha_0)_{CF}D_t^{\alpha_0}\right]\frac{\partial^2 C(x,t)}{\partial x^2}-D_0\left[\left(1-e^{-\frac{\alpha_s}{1-\alpha_s}Fo}\right)+(1-\alpha_s)_{CF}D_t^{\alpha_s}\right]\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{55}$$

When memory effects do not exist, i.e. for $\alpha_0=\alpha_s=1$ both models reduce to an ordinary diffusion equation with a diffusion coefficient D_0

$$\frac{\partial C(x,t)}{\partial t}=D_0\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{56}$$

If the assumption $\beta_0 t<<1$ is applicable, then the reduction of the single-memory model (39) to (40) is a valid operation. This simply means that for $\alpha_0\rightarrow 1$, as well as for $\alpha_s\rightarrow 1$ we get

$$\frac{\partial C(x,t)}{\partial t}\approx D_0 e^{-\beta_0 t}\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{57}$$

The reference sources [1,2,3,4,5,6,7] for systems where the Dodson equation was conceived as a model reveal that the order of magnitude of D_0 is $10^{-20} m^2/s$. Moreover Dodson commented that *the relaxation time τ can be million of years* (Sic!) [1]. This makes $\beta = 1/\tau$ extremely small value allowing the approximation $\beta_0 t << 1$ to be accepted as a reasonable step. Now, using (52) we may express (57) as

$$\frac{\partial C(x,t)}{\partial t} \approx D_\alpha \left(\frac{\partial^2 C(x,t)}{\partial x^2} \right), D_\alpha = D_0 e^{-\frac{\alpha_0}{1-\alpha_0} Fo}. \tag{58}$$

Regarding equation (57), if we forget for a while about the idea to use memory integral, it may be considered as a version of the integer-order Dodson equation (1) but now controlled by a single parameter $\alpha \in [0,1] \Rightarrow \beta \in [0,\infty]$, which can be defined in a way demonstrated above. The variations of $D_\alpha / D_0 = exp(-\frac{-\alpha}{1-\alpha} Fo)$ with α and Fo are shown in Fig.4.

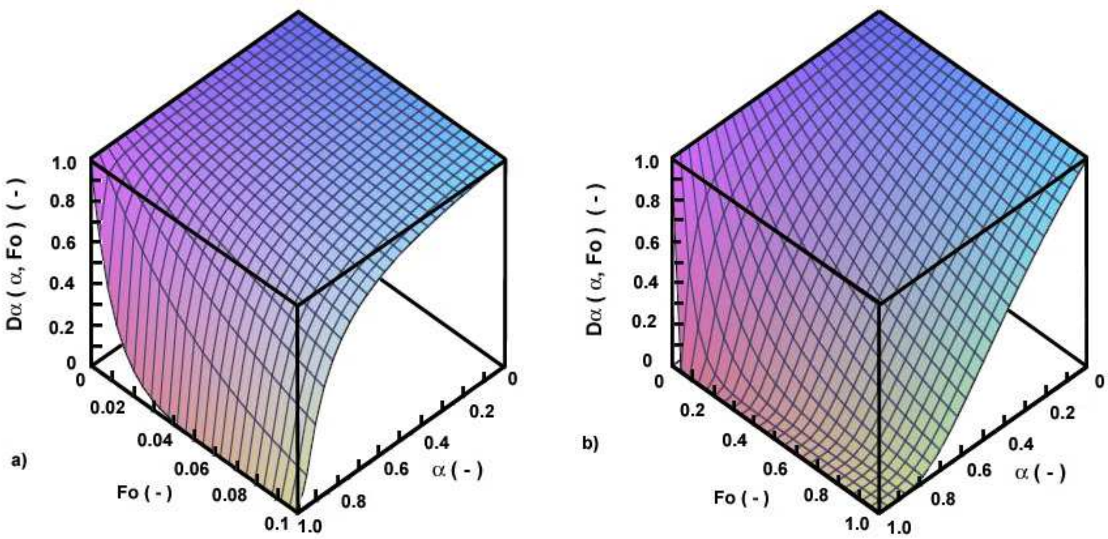

Fig. 4: Dimensionless exponential diffusion coefficient of Dodson as a function of the fractional order α *and the Fourier number Fo*.

4 Comments of the Results and Some Ideas Beyond

4.1 What Really We Derived Starting from the Constitutive Equation of Cattaneo? A Comparative Analysis.

Now, we have to stress the attention on the single-kernel model presented in two equivalent forms: (24) and (54). Actually, we derived in straightforward manner the complete diffusion equation (Fourier or Fick) presented through the Caputo-Fabrizio time-fractional operator (derivative). However, there are three alternative forms of the diffusion equation in terms of the Caputo-Fabrizio operator which are *strongly dependent on the kernel in the initially assumed constitutive equations*. We will comment them in order to demonstrate how the different initial approaches are resulting in different forms of expressions as well as to project the results of this study on the area of existing ones in the literature.

In the second article of Caputo and Fabrizio [22], just a year ago, the associate fractional integral to the derivative (25) was defined as (section 7 of [22])

$$_0I^\alpha f(t) = \frac{1}{\alpha} \int_0^t f(s) exp\left(-\frac{1-\alpha}{\alpha}(t-s) \right) ds, \qquad \alpha \in [0,1]. \tag{59}$$

It is noteworthy that the fractional factor (the temporal rate constant depending on α) in the exponential kernel $(1-\alpha)/\alpha$ is reciprocal to the factor used in the kernel of the derivative $\alpha/(1-\alpha)$ (25). For $\alpha = 0$ this definition provides directly the function $f(t)$ as well as it follows that

$$\frac{d}{dt}\left({}_0I^{\alpha} f(t)\right) = \frac{1}{\alpha} f(t) - \frac{1-\alpha}{\alpha}\,{}_0\left(I^{\alpha} f(t)\right). \tag{60}$$

Caputo and Fabrizio suggested the following constitutive equation for the flux (in terms used here) [22]

$$j(t) = -D_0\left({}_0I^{\alpha} f(t)\left[\frac{\partial C(x,t)}{\partial x}\right]\right) = -D_0\frac{1}{\alpha}\int_0^t \frac{\partial C(x,s)}{\partial x} exp\left(-\frac{1-\alpha}{\alpha}(t-s)\right) ds. \tag{61}$$

Applying the rule (60) to the constitutive equation (61) we get (eq.(33) in [22])

$$\frac{d}{dt} j(t) = -\frac{D_0}{\alpha}\frac{\partial C(x,t)}{\partial x} - \frac{1-\alpha}{\alpha} f(t). \tag{62}$$

Equation (62) coincides with the Cattaneo-Maxwell equation (63)

$$\frac{\alpha}{1-\alpha}\frac{d}{dt} j(t) = -j(t) - \frac{D_0}{1-\alpha}. \tag{63}$$

This equation reduces (for $\alpha = 0$) to the Fourier (Fick) law $j(t) = -D_0\frac{\partial C(x,t)}{\partial x}$.

Recall, that if the constitutive equation is defined as (23) (or (24)) the same result can be derived for $\alpha = 1$, which is in agreement with the definition of the fractional derivative (25). In accordance with the definition (25) for $\alpha = 1$ there is no time delay, i.e. by definition $\tau = 0 \Rightarrow \beta \to \infty \Rightarrow \alpha \to 1$.

Now, applying the mass balance equation (16) we obtain an alternative form of the diffusion equation [22], namely

$$\frac{\partial C(x,t)}{\partial t} = D_0\frac{\partial^2 C(x,t)}{\partial x^2} + \frac{1-\alpha}{\alpha}\left[\frac{1}{\alpha}\int_0^t \frac{\partial C(x,t)}{\partial x} exp\left(-\frac{1-\alpha}{\alpha}(t-s)\right) ds\right]. \tag{64}$$

For $\alpha = 1$ we get the diffusion equation without delay.

The specific feature of (64) is that *the last term is expressed through the associate fractional integral ((61)) instead the Caputo-Fabrizio time-fractional derivative*, as it is in the developed here diffusion model. It easy to check that in (64) the fractional order α is related to the Deborah number by the relation (51) in a manner demonstrated in section 3.3.

It is quite clear that, despite the task to derive the diffusion equation in terms of ${}_{CF}D_t^{\alpha}$, the final form of the equation is strongly dependent on the constitutive equation relating the flux and the gradient. In this context, exploring the idea of heat waves [39], when the relaxation kernel is of Jeffrey type $R_{JP} = k_1\delta_D(s) + (k_2/\tau)exp(-s\tau/)$, where δ_D is Dirac delta function [19,39] (the case of transient heat conduction was at issue, so we preserve the original notations) the constitutive relation about the flux is

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - \frac{k_2}{\tau}\int_{-\infty}^t e^{\left(-\frac{t-s}{\tau}\right)}\frac{\partial T(x,t)}{\partial x} ds. \tag{65}$$

In (65) k_1 and k_2 are *the effective thermal conductivity* and *the elastic thermal conductivity*, respectively. In this case the Fourier law (16) leads to the Jeffrey type integro-differential equation [19]

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{a_2}{\tau}\int_{-\infty}^t e^{\left(-\frac{t-s}{\tau}\right)}\frac{\partial T(x,t)}{\partial x} ds. \tag{66}$$

Here $a_1 = k_1/\rho C_p$ and $a_2 = k_2/\rho C_p$ are *the effective thermal diffusivity* and *the elastic thermal diffusivity*, respectively; ρ is the density while C_p is the heat capacity of the medium.

This equation can be expressed in terms of ${}_{CF}D_t^{\alpha}$ by the relation $\tau \equiv (1-\alpha)/\alpha$ and the analysis performed in [38] resulted in

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\frac{\partial^2 T(x,t)}{\partial x^2}, \qquad t > 0. \tag{67}$$

Hence, the models expressed in terms of ${}_{CF}D_t^{\alpha}$, i.e. (39) and equally (54) and (67), are equivalent to (64) where the relaxation of the flux is represented by the fractional integral ${}_0I^{\alpha}$.

Actually, this work demonstrated the derivation of an alternative form of the diffusion equation in terms of Caputo-Fabrizio time-fractional derivative. For very big relaxation times τ (that is when $\beta << 1$ corresponding to $\alpha \rightarrow 0$) reduces to the fractional version (see (39) and for $\alpha = 1$ to the original integer-order version of the Dodson equation (6).

At the end of this point, the formulated two-memory model is a step beyond the outcome of the task focusing the derivation of the Dodson equation from basic constitutive relationship about the flux relation. This model constitutes a two-kernel composite memory function physically based on the assumption that local disturbances causing short-time transients affect the gross relaxation process. This model reduces simply to the single-kernel memory model when the short-time memory is neglected and further to the original Dodson equation expressed through a fractionalized diffusion coefficient.

4.2 The Task is Completed and What are the Main Outcomes?

Therefore, we derived fractional diffusion equation with exponentially decaying in time diffusivity in terms of the Caputo-Fabrizio time-fractional derivatives straightforwardly starting from the constitutive equation of Cattaneo. Moreover, we demonstrated that the original equation of Dodson is a particular case of the single-memory fractional model.

The developed functional relationship $\alpha = \alpha(De)$ allows calculating the fractional order, *a fact that is essentially missing in the existing publications involving time-fractional Caputo-Fabrizio derivatives*, as well as in the models discussed in preceding section. Moreover, the expression of the original model of Dodson, with diffusion coefficient expressed through the fractional order α and the Fourier number (see eq.(54)) is a step ahead in modeling with this equation, which demonstrates a little progress since the time of it invention.

The formulated two-memory model is a step beyond the outcome of the task focusing the derivation of the Dodson equation from basic constitutive relationship about the flux relaxation. This model constitutes a two-kernel composite memory function physically based on the assumption that local disturbances causing short-time transients affect the gross relaxation process. This model reduces simply to the single-kernel memory model when the short-time memory is neglected and further to the original Dodson equation expressed through a fractionalized diffusion coefficient. To the point where the extended versions of the Dodson equation as a time-fractional equation with Caputo-Fabrizio derivatives was derived the task of this article is completed. Solutions either analytical or numerical as well as tests to real physical situations draw new projects and related ideas beyond the format of this article.

4.3 The Formalistic Fractionalization of the Dodson Equation and what is the Outcome

Finally we have to stress the attention that if we replace directly the time-dependent derivative in (1) by ${}_{CF}D_t^{\alpha}$ we get *via a formalistic fractionalization* the following equation

$$ {}_{CF}D_t^{\alpha}C(x,t) = D_0 e^{-\beta_0 t}\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{68}$$

However, in this case we have no real physically based reasons to express β through the fractional order α since *a constitutive relation about the flux relaxation is missing*. We may suggest only, without a proof, as a conjecture, that (68) may be derived mechanistically if the mass balance equation is expressed as

$$ {}_{CF}D_t^{\alpha}C(x,t) = -\frac{\partial j}{\partial x}. \tag{69}$$

This is a fractional replica of (16) if the diffusion coefficient is constituted *ad hoc* as $D_0 e^{-\beta_0 t}$. However, in this case , as in the classical Fick (Fourier) equation, the flux j should be related to the gradient with a memory integral where the kernel is the Dirac delta function, which physically contradicts the use of the relaxation time τ as a process parameter; because immediately we get a model with unrelated relaxation parameters, i.e. α and τ. However, the analysis of eq.(54) and the principle differences with respect to the models (55), (58) and (69) are beyond the scope of the present work.

5 Conclusion

The present article demonstrated a new derivation of the integer-order Dodsons equation starting from the Cattaneo constitutive relation with exponential kernel. This approach resulted in a fractional order diffusion equation with exponential diffusivity expressed through the fractional order α and the Fourier number. This models simply reduces to the original Dodson equation.

A principle result developed in this work is the development of a straightforward relation of the fractional order and the Deborah number calculated as a ratio of the relaxation time to the characteristic diffusion time of the process.

A new model with two memories corresponding to large and short time relation effects was conceived. It reduces to the single-memory model when the short-time relaxations are neglected.

Therefore the initial task to derive the Dodson equation in a new way resulted in generalized fractional diffusion models. We hope this will be a good contribution to the area when they could be implemented as well as challenging tasks for the modelers interested in applications of the Caputo-Fabrizio time-fractional derivative.

References

[1] M. H. Dodson, Closure temperature in cooling geochronological and petrological systems, *Cont. Mineral. Petrol.* **40**, 259–274 (1973).

[2] M. H. Dodson, Theory of cooling ages,In:E.Jaeger and J.C.Hunziker(Eds),*Lectures in Isotope Geology*, Springer, Berlin Heidelberg, 194–202, 1979.

[3] Y. Zang, Diffusion in mineral and melts: theoretical background, *Rev. Min. Geochem.* **72**, 5–59 (2010).

[4] J. Ganguly and M. Tirone, Diffusion closure temperature and age of a mineral with arbitrary extent of diffusion: theoretical formulation and applications, *Earth. Plant Sci. Lett.* **170**, 131–140 (1999).

[5] Y. Liang, A simple model for closure temperature of a trace element in cooling bi-mineralic systems, *Geoch. Cosmochim. Acta* **165**,35–43 (2015).

[6] B. I. A. McInnes, N. J. Evans, F. Q. Fu and S. Garwin, Application of thermochronology to hydrothermal ore deposits, *Rev. Mineral. Geochem.* **58**, 467–98 (2005).

[7] J. Ganguly, Cation diffusion kinetics in sluminosilicate garnets and geological applications, *Rev. Mineral. Geochem.***72**, 559–601 (2010).

[8] J. Crank, *Mathematics of Diffusion*, 2nd ed., Oxford University Press, UK,1975,

[9] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, London, 1959.

[10] T. R. Goodman, The heat balance integral and its application to problems involving a change of phase, *Transact. ASME* **80**, 335–342 (1958).

[11] T. R. Goodman, Application of integral methods to transient nonlinear heat transfer, in: T. F. Irvine and J. P. Hartnett, (Eds.), *Advances in Heat Transfer*, 1, Academic Press, San Diego, CA,1964, pp. 51122.

[12] J. Hristov, Integral solutions to transient nonlinear heat (mass) diffusion with a power-law diffusivity: a semi-infinite medium with fixed boundary conditions, *Heat Mass Transf.* **52**, 635–655 (2016).

[13] J. Hristov, The heat-balance integral method by a parabolic profile with unspecified exponent: analysis and benchmark exercises, *Thermal Sci.* **13**, 27–48 (2009).

[14] S. L. Mitchell and T. G. Myers, Application of heat balance integral methods to one-dimensional phase change problems, *Int. J. Diff. Eqs.* **2012**, Article ID 187902, doi: 10.1155/2012/187902.

[15] J. Hristov, The non-linear Dodson diffusion equation: approximate solutions and beyond with formalistic fractionalization, *Math. Natur. Sci.* **1**, 1–17 (2017).

[16] C. Cattaneo, On the conduction of heat (In Italian), *Atti Sem. Mat. Fis. Universita Modena* **3**, 83–101 (1948).

[17] S. Carillo, Some remarks on materials with memory: heat conduction and viscoelasticity, *J. Nonlin. Math. Phys.* **12**, 163–178 (2005).

[18] J. A. Ferreira and P. Oliveira, Qualitative analysis of a delayed non-Fickian model, *Applic. Anal.* **87**, 873–886 (2008).

[19] A. Araujo, J. A. Ferreira and P. Oliveira, The effect of memory terms in diffusion phenomena, *J. Comp. Math.* **24**, 91–102 (2000).

[20] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1**, 73–85 (2015).

[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[22] M. Caputo and M. Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, *Progr. Fract. Differ. Appl.* **2**, 1–11 (2016).

[23] J. F. Gomez-Aguilar, H. Yepez-Martinez, C. Calderon-Ramon, I. Cruz-Orduna, R. F. Escobar-Jimenez and V. H. Olivares-Peregrino, Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel, *Entropy* **17**, 6289–6303 (2005).

[24] A. Atangana and J. J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel, *Adv. Mech. Engin.* **7**, 1–7 (2015).

[25] A. Atangana and S. T. A. Badr, Analysis of the KellerSegel model with a fractional derivative without singular kernel,*Entropy* **17**, 4439–453 (2015).

[26] A. Atangana and S. T. A. Badr, New model of groundwater flowing within a confine aquifer: application of Caputo-Fabrizio derivative, *Arab. J. Geosci.*, **9**:8, (2016), doi: 10.1007/s12517-015-2060-8

[27] N. A. Sheikh, F. Ali, M. Saqib, I. Khan and S. A. Alam Jan, A comparative study of Atangana-Baleanu and Caputo-Fabrizio fractional derivatives to the convective flow of a generalized Casson fluid, *Eur. Phys. J. Plus* **132**(54), (2017) doi:10.1140/epjp/i2017-11326-y.

[28] D. Kumar, J. Singh and D. Baleanu, Modified Kawahara equation within a fractional derivative with non-singular kernel, *Thermal Sci.*, (2017), in press: doi: DOI:10.2298/TSCI160826008K.
[29] A. Alsaedi, D. Baleanu, S. Etemad and S. Rezapour, On coupled systems of time-fractional differential problems by using a new fractional derivative, *J. Funct. Spac.* **2016**, Article ID 4626940, doi: 10.1155/2016/4626940.
[30] J. Losada and J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1**, 87–92 (2015).
[31] D. Kumar, J. Singh, D. Baleanu and M. Al Qurashi, Analysis of logistic equation pertaining to a new fractional derivative with non-singular kernel, *Adv. Mech. Engin.* **9**(2,) 1–8 (2017) doi: 10.1177/1687814017690069.
[32] H. Jafari, A. Lia, H. Tejadodi and D. Baleanu, Analysis of Riccati differential equations within a new fractional derivative without singular kernel, *Fundam. Informat.* **151**, 161–171 (2017).
[33] J. F. Gomez-Aguilar, H. Yepez-Martinez, J. Torres-Jimenez, T. Cordova-Fraga, R. F. Escobar-Jimenez and V. H. Olivares-Peregrino, Homotopy perturbation transform method for nonlinear differential equations involving to fractional operator with exponential kernel, *Adv. Differ. Equ.* **2017**:68, doi: 10.1186/s13662-017-1120-7.
[34] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Thermal Sci.* **20**, 763–769 (2016).
[35] D. Baleanu and T. Abdeljawad, Discrete fractional differences with non-singular discrete Mittag-Leffler kernels, *Adv. Differ. Equ.***2016**:232, doi: 10.1186/s13662-016-0949-5.
[36] Il. Koca and A. Atangana, Solutions of Cattaneo-Hristov model of elastic heat diffusion with Caputo-Fabrizio and Atangana-Baleanu fractional derivatives, *Thermal Sci.* **21**, (2017), in press, doi:10.2298/TSCI160209103K.
[37] T. M. Atanackovic, S. Pilipovic, B. Stankovic and D. Zorica,*Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*, ISTE Ltd and John Wiley & Sons, Inc.,ISTE Ltd and John Wiley & Sons, Inc., London, UK, 2014.
[38] J. Hristov, Transient heat diffusion with a non-singular fading memory: from the Cattaneo constitutive equation with Jeffreys kernel to the Caputo-Fabrizio time-fractional derivative, *Thermal Sci.* **20**, 765–770 (2016).
[39] D. D. Joseph and L. Preciozi, Heat waves, *Rev. Mod. Phys.* **61**, 41–73 (1989).
[40] J. Hristov, Steady-state heat conduction in a medium with spatial non-singular fading memory: derivation of Caputo-Fabrizio space-fractional derivative with Jeffrey's kernel and analytical solutions, *Thermal Sci.* **21**, 827–839 (2017).
[41] J. S. Vrentas, C. M. Jarzebski and J. L. Duda, A Deborah number for diffusion in polymer-solvent systems, *AIChE J.* **21**, 894–901 (1975).
[42] J. C. Wu and N. Pepas, Modeling penetrant diffusion in glassy polymers with an integral sorption Deborah number, *J. Pol. Scie.Part B* **31**, 1503–1518 (1993).