Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070436

A Parameter-uniform Method for Two Parameters Singularly Perturbed Boundary Value Problems via Asymptotic Expansion

D. Kumar[1,*], *A. S. Yadaw*[2] *and M. K. Kadalbajoo*[3]

[1]Department of Mathematics, B.I.T.S. Pilani, 333031, India
[2]Department of Pharmacology & Systems Biology Mount Sinai School of Medicine, New York, NY 10029
[3]Department of Mathematics & Statistics, I.I.T. Kanpur, 208016, India

Received: 11 Dec. 2012, Revised: 13 Jan. 2013, Accepted: 4 Feb. 2013
Published online: 1 Jul. 2013

Abstract: An approximate method for two parameters singularly perturbed boundary value problems having boundary layers at both end points is given. The method is motivated by the asymptotic behavior of the solution. In the outer region the solution of the problem is approximated by the zeroth order asymptotic expansion while in the inner region the solution of the problem is obtained by using B-spline collocation method. The method is iterated on the transition point of the boundary layer region. To demonstrate the applicability of the method two test examples are considered.

Keywords: Singular perturbation, two parameters, boundary layer, asymptotic expansion, B-spline collocation method.

1. Introduction

The boundary value problems for ordinary differential equations in which one or more small positive parameters multiplying the derivatives arise in the field of physics and applied mathematics. The problems with one small positive parameter multiplying to highest derivative have considered by many authors [1]. Motivated by asymptotic expansion [2,3] a second order boundary value problem with two small parameters multiplying to the highest and second highest derivative is considered. This type of problems arise in chemical reactor theory, engineering, biology, lubrication theory etc.

Consider the problem

$$Ly(x) \equiv -\epsilon y''(x) - \mu a(x)y'(x) + b(x)y(x) = f(x), \quad (1)$$

$x \in [0,1]$, with

$$y(0) = \alpha, \quad y(1) = \beta, \quad (2)$$

where ϵ and μ are small positive parameters satisfying $\epsilon/\mu^2 \to 0$ as $\mu \to 0$. The functions $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth satisfying $a(x) \geq a^* > 0$, $b(x) \geq b^* > 0$.

In 1967, Malley gave the asymptotic solution of two parameters singularly perturbed boundary value problems and demonstrate the roll of ϵ and μ on the solution. After three decades, various mathematician [4,5,6] gave the numerical solution of two parameters problems. Malley [2,3,7,8,9] examined the nature of asymptotic solution of the continuous problem where the ratio of μ^2 to ϵ was identified as significant. In [10,11], the standard upwind finite difference operator on two different choices of Shishkin mesh was shown to be parameter-uniform of first order. In [12] parameter-uniform methods on a uniform mesh were constructed. Vulanović [13], used the higher order finite difference scheme on a piecewise uniform mesh both of Shishkin and Bakhavalov type for solving quasi-linear boundary value problems with small parameters.

2. Solution of the problem

2.1. Asymptotic solution

Consider the asymptotic expansion of (1) with (2) of the form

$$y(x,\epsilon,\mu) = (y_0 + (\epsilon/\mu)y_1 + (\epsilon/\mu)^2 y_2 + O((\epsilon/\mu)^3)) \quad (3)$$

* Corresponding author e-mail: dkumar2845@gmail.com

$$+(z_0+(\epsilon/\mu)z_1+(\epsilon/\mu)^2z_2+O((\epsilon/\mu)^3)) \quad (4)$$
$$=(y_0+z_0)+(\epsilon/\mu)(y_1+z_1) \quad (5)$$
$$+(\epsilon/\mu)^2(y_2+z_2)+O((\epsilon/\mu)^3) \quad (6)$$
$$=W_0+(\epsilon/\mu)W_1+(\epsilon/\mu)^2W_2+O((\epsilon/\mu)^3), \quad (7)$$

where the zeroth order asymptotic expansion W_0 is given by $W_0(x)=y_0+z_0$, where y_0 is the solution of the reduced problem

$$-\mu a(x)y_0'(x)+b(x)y_0(x)=f(x),\ y_0(1)=\beta, \quad (8)$$

and $y_1(x)$ is the solution given by

$$-\epsilon y_1''(x)-\mu a(x)y_1'(x)+b(x)y_1(x)=\mu y_0''(x), \quad (9)$$
$$y_1(0)=0,\ y_1(1)=-(\epsilon/\mu)^{-1}z_0(1), \quad (10)$$

and z_0 is a layer correction given by

$$-d^2z_0/d\tau^2-a(0)dz_0/d\tau=0,$$
$$z_0(0)=\alpha-y_0(0), z_0(\infty)=0, \quad (11)$$

where $\tau=x/(\epsilon/\mu)$.

The solution of Eq. (8) is given by

$$y_0(x)=\exp\left(\int_x^1\frac{b(s)}{\mu\,a(s)}ds\right) \quad (12)$$
$$\left[\beta-\int_x^1\frac{f(s)}{\mu\,a(s)}\exp\left(-\int_x^1\frac{b(s)}{\mu\,a(s)}d\,s\right)ds\right] \quad (13)$$

and the solution of Eq. (11) is given by

$$z_0(x)=(\alpha-y_0(0))\exp(-a(0)x/(\epsilon/\mu)).$$

Thus the zeroth order asymptotic expansion $W_0(x)$ is obtained.

It is easy to see that the zeroth order asymptotic expansion $W_0(x)$ of the problem (1) with (2) satisfies the following inequality:

Theorem 1. *The zeroth order asymptotic expansion $W_0(x)$ of the solution $y(x)$ of (1) with (2) satisfies*

$$|y(x)-W_0(x)|\le C(\epsilon/\mu),$$

where $\epsilon/\mu^2\to 0$ as $\mu\to 0$.

Also the solution $y(x)$ of (1) with (2) and its derivatives satisfy the following inequality:

Theorem 2. *Suppose $a(x), b(x)$ and $f(x)$ are sufficiently smooth and has derivatives at least of order k then we have*

$$|y^{(i)}(x)|\le C[1+\mu^{-i}\exp(-\nu(1-x)/\mu) \quad (14)$$
$$+\epsilon^{-i}\exp(-\alpha x/(\epsilon/\mu))],\ i=0,1,\ldots,k, \quad (15)$$

where $\frac{b(x)}{a(x)}\le\nu$.

2.2. Discrete solution

We divide the interval $[0,1]$ into three non overlapping subintervals $[0,k(\epsilon/\mu)]$, $(k(\epsilon/\mu),1-k\mu)$ and $[1-k\mu,1]$ where k be any positive integer such that $k\mu\ll 1$. The original problem is then divided into three equivalent problems (two inner regions and one outer region problem). To obtain the boundary conditions at the transition points $k(\epsilon/\mu)$ and $(1-k\mu)$, we use the zeroth order asymptotic expansion W_0. Thus if $W_0(k(\epsilon/\mu))=\gamma_1$ and $W_0(1-k\mu)=\gamma_2$ then the three problems are

P_1:The left inner region problem $(x\in[0,k(\epsilon/\mu)])$

$$-\epsilon y''(x)-\mu a(x)y'(x)+b(x)y(x)=f(x), \quad (16)$$
$$y(0)=\alpha,\ y(k(\epsilon/\mu))=\gamma_1. \quad (17)$$

P_2:The outer region problem $(x\in(k(\epsilon/\mu),1-k\mu))$

$$-\epsilon y''(x)-\mu a(x)y'(x)+b(x)y(x)=f(x), \quad (18)$$
$$y(k(\epsilon/\mu))=\gamma_1,\ y(1-k\mu)=\gamma_2. \quad (19)$$

P_3:The right inner region problem $(x\in[1-k\mu,1])$

$$-\epsilon y''(x)-\mu a(x)y'(x)+b(x)y(x)=f(x), \quad (20)$$
$$y(1-k\mu)=\gamma_2,\ y(1)=\beta. \quad (21)$$

After solving the inner and outer region problems, we combine their solutions to obtain an approximate solution of the problem (1) with (2) in the whole interval $[0,1]$. We change the value of k until the solution profiles do not differ much from iteration to iteration. For this we use the absolute error criteria $|y^{(m+1)}(x)-y^{(m)}(x)|\le\sigma$, where σ is prescribed tolerance error bound. We use B-spline collocation method in the inner region and for P_1 we have $x_0=0$, $x_N=k\epsilon/\mu$, $h_1=k\epsilon/\mu N$, $x_i=x_0+ih_1$, $i=1,2,\ldots,N$ and for P_3 we have $x_0=1-k\mu$, $x_N=1$, $h_2=k\mu/N$, $x_i=x_0+ih_2$, $i=1,2,\ldots,N$.

For $0<l_1<l_2<1$, we define $L_2[l_1,l_2]$ a vector space of all the square integrable function on $[l_1,l_2]$ and let X be the linear subspace of $L_2[l_1,l_2]$. Now define for $i=0,1,2,\ldots,N$

$$B_i(x)=\frac{1}{h^3}\begin{cases}(x-x_{i-2})^3,\ x_{i-2}\le x\le x_{i-1},\\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2\\ -3(x-x_{i-1})^3,\ x_{i-1}\le x\le x_i,\\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2\\ -3(x_{i+1}-x)^3,\ x_i\le x\le x_{i+1},\\ (x_{i+2}-x)^3,\ x_{i+1}\le x\le x_{i+2},\\ 0,\text{otherwise}.\end{cases} \quad (22)$$

Let $\pi=\{x_0,x_1,\ldots,x_N\}$ be the partition of $[l_1,l_2]$. We introduce four additional knots $x_{-2}<x_{-1}<x_0$ and $x_{N+2}>x_{N+1}>x_N$. It is easy to check that each of the function $B_i(x)$ is twice continuously differentiable on the entire real line. Also

$$B_i(x_j)=\begin{cases}4,\text{ if } i=j,\\ 1,\text{ if } i-j=\pm1,\\ 0,\text{ if } i-j=\pm2,\end{cases} \quad (23)$$

and that $B_i(x) = 0$ for $x \geq x_{i+2}$ and $x \leq x_{i-2}$. We can also see that

$$B_i'(x_j) = \begin{cases} 0, & \text{if } i = j, \\ \pm\frac{3}{h}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \tag{24}$$

and

$$B_i''(x_j) = \begin{cases} \frac{-12}{h^2}, & \text{if } i = j, \\ \frac{6}{h^2}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2. \end{cases} \tag{25}$$

Each $B_i(x)$ is also a piecewise cubic at π and $B_i(x) \in X$. Let $\Pi = \{B_{-1}, B_0, B_1, \ldots, B_{N+1}\}$ and let $\Phi_3(\pi)$ is span of Π. Then Π is linearly independent on $[l_1, l_2]$, thus $\Phi_3(\pi)$ is $(N+3)$-dimensional. In fact $\Phi_3(\pi)$ is a subspace of X. Let L be a linear operator whose domain is X and whose range is also in X. Now we define

$$S(x) = \sum_{i=-1}^{N+1} c_i B_i(x), \tag{26}$$

where c_i are unknown real coefficients. Here we have introduced two extra cubic B-splines, B_{-1} and B_{N+1} to satisfy the boundary conditions. Thus

$$LS(x_i) = f(x_i), \quad 0 \leq i \leq N, \tag{27}$$

and

$$S(x_0) = \alpha, \quad S(x_N) = \beta. \tag{28}$$

On solving the collocation equations (27) and putting the values of B-spline functions B_i and of derivatives at mesh points, we obtain a system of $(N+1)$ linear equations in $(N+3)$ unknowns

$$(-6\epsilon + 3\mu a_i h + b_i h^2)c_{i-1} + (12\epsilon + 4b_i h^2)c_i$$
$$+(-6\epsilon - 3\mu a_i h + b_i h^2)c_{i+1} = f_i h^2. \tag{29}$$

The given boundary conditions become

$$c_{-1} + 4c_0 + c_1 = \alpha, \quad c_{N-1} + 4c_N + c_{N+1} = \beta. \tag{30}$$

Thus the Eqs. (29) and (30) lead to an $(N+3) \times (N+3)$ system with $(N+3)$ unknowns $c_{-1}, c_0, \ldots, c_N, c_{N+1}$. Now eliminating c_{-1} from first equation of (29) and from first equation of (30)

$$(36\epsilon - 12\mu a_0 h)c_0 + (-6\mu a_0 h)c_1$$
$$= h^2 f_0 - \alpha(-6\epsilon + 3\mu a_0 h + b_0 h^2). \tag{31}$$

Similarly, eliminating c_{N+1} from the last equation of (29) and from second equation of (30)

$$(6\mu a_N h)c_{N-1} + (36\epsilon + 12\mu a_N h)c_N$$
$$= h^2 f_N - \beta(-6\epsilon - 3\mu a_N h + b_N h^2). \tag{32}$$

Coupling equations (31) and (32) with the second through $(N-1)$st equations of (29). Thus by the elimination of c_{-1} and c_{N+1}, we lead to a system $Tx_N = d_N$ of $(N+1)$ linear equations in $(N+1)$ unknowns, where $x_N = (c_0, c_1, \ldots, c_N)^t$ and $d_N = (h^2 f_0 - \alpha(-6\epsilon + 3\mu a_0 h + b_0 h^2), h^2 f_1, \ldots, h^2 f_{N-1}, h^2 f_N - \beta(-6\epsilon - 3\mu a_N h + b_N h^2))$. The elements $t_{i,j}$ of the tridiagonal matrix T are given by

$$t_{i,j} = \begin{cases} 36\epsilon - 12\mu a_0 h, & i = j = 0, \\ -6\mu a_0 h, & i = 0, j = 1, \\ -6\epsilon + 3\mu a_i h + b_i h^2, & i = j+1, j = 0(1)N-2, \\ 12\epsilon + 4b_i h^2, & i = j = 1(1)N-1, \\ -6\epsilon - 3\mu a_i h + b_i h^2, & i = j-1, j = 2(1)N, \\ 6\mu a_N h, & i = N, j = N-1, \\ 36\epsilon + 12\mu a_N h, & i = j = N, \\ 0, & |i-j| > 1. \end{cases} \tag{33}$$

It is easy to see that the matrix T is strictly diagonally dominant and hence nonsingular. Since T is nonsingular, we can solve the system $Tx_N = d_N$ for $c_0, c_1, \ldots, c_N$ and substitute into the boundary conditions (30) to obtain c_{-1} and c_{N+1}.

Lemma 1. *The B-splines $B_{-1}, B_0, \ldots, B_{N+1}$ defined in equation (22), satisfy the inequality*

$$\sum_{i=-1}^{N+1} |B_i(x)| \leq 10, \; 0 \leq x \leq 1.$$

Proof. We know that $|\sum_{i=-1}^{N+1} B_i(x)| \leq \sum_{i=-1}^{N+1} |B_i(x)|$. At any node x_i, we have

$$\sum_{i=-1}^{N+1} |B_i| = |B_{i-1}| + |B_i| + |B_{i+1}| = 6 < 10.$$

Also we have $|B_i(x)| \leq 4$ and $|B_{i-1}(x)| \leq 4, \forall x_{i-1} \leq x \leq x_i$. Similarly $|B_{i-2}(x)| \leq 1$ and $|B_{i+1}| \leq 1, \forall x_{i-1} \leq x \leq x_i$. Now for any point $x_{i-1} \leq x \leq x_i$ we have $\sum_{i=-1}^{N+1} |B_i(x)| = |B_{i-2}| + |B_{i-1}| + |B_i| + |B_{i+1}| \leq 10$.

Theorem 3. *Let $S(x)$ be the collocation approximation from the space of cubic splines $\Phi_3(\pi)$ to the solution $y(x)$ of the boundary value problem (1)–(2). If $f \in C^2[l_1, l_2]$, then the parameter-uniform error estimate is given by*

$$\|y(x) - S(x)\|_\infty \leq Ch^2,$$

where C is a positive constant independent of ϵ and N.

Proof. To estimate the error $\|y(x) - S(x)\|_\infty$, let Y_n be the unique spline interpolate from $\Phi_3(\pi)$ to the solution $y(x)$ of our boundary value problem (1)–(2). If $f(x) \in C^2[l_1, l_2]$ then $y(x) \in C^4[l_1, l_2]$ and it follows from De Boor-Hall error estimates [14] that

$$\|D^j(y(x) - Y_n)\|_\infty \leq \gamma_j h^{4-j}, \quad j = 0, 1, 2, \tag{34}$$

where h is uniform mesh spacing and $\gamma_j's$ are constants independent of h and N. Let

$$Y_n(x) = \sum_{i=-1}^{N+1} p_i B_i(x). \tag{35}$$

It follows immediately from the estimates (34) that

$$|LS(x_i) - LY_n(x_i)| = |f(x_i) - LY(x_i) \\ +Ly(x_i) - Ly(x_i)| \leq \lambda h^2, \quad (36)$$

where $\lambda = \epsilon\gamma_2 + \mu\gamma_1\|a(x)\|_\infty h + \gamma_0\|b(x)\|_\infty h^2$. Also $LS(x_i) = Ly(x_i) = f(x_i)$. Let $LY_n(x_i) = \hat{f}_n(x_i)$ for $i = 1, 2, \ldots, N$ and $\hat{f}_n = (\hat{f}_n(x_0), \hat{f}_n(x_1), \ldots, \hat{f}_n(x_N))^t$. Now from the system $Tx_N = d_N$ and (34), it is clear that the ith coordinate $[T(x_N - y_N)]_i$ of $T(x_N - y_N)$, where $y_N = (p_0, p_1, \ldots, p_N)^t$, satisfies the inequality

$$|[T(x_N - y_N)]_i| \leq \lambda h^4. \quad (37)$$

Since $(Tx_N)_i = h^2 f(x_i)$ and $(Ty_N)_i = h^2\hat{f}_n(x_i)$ for $i = 0, 1, 2, \ldots, N$. The ith coordinate of $[T(x_N - y_N)]$ is the ith equation

$$(-6\epsilon + 3\mu a_i h + b_i h^2)\delta_{i-1} + (12\epsilon + 4b_i h^2)\delta_i \\ +(-6\epsilon - 3\mu a_i h + b_i h^2)\delta_{i+1} = \xi_i, \ 1 \leq i \leq N-1, \quad (38)$$

where $\delta_i = p_i - q_i, -1 \leq i \leq N+1$ and $\xi_i = h^2(f(x_i) - \hat{f}_n(x_i)), 1 \leq i \leq N-1$. Clearly $|\xi_i| \leq \lambda h^4$. Let $\xi = \max_{1\leq i\leq N-1}|\xi_i|$. Also consider $\delta = (\delta_{-1}, \delta_0, \ldots, \delta_{N+1})^t$, then we define $e_i = |\delta_i|$ and $\tilde{e} = \max_{1\leq i\leq N}|e_i|$. Now equation (38) becomes

$$(12\epsilon + 4b_i h^2)\delta_i\xi_i \leq \xi_i + (6\epsilon - b_i h^2)(\delta_{i-1} + \delta_{i+1}) \\ +3\mu a_i h(\delta_{i+1} - \delta_{i-1}), \ 1 \leq i \leq N-1. \quad (39)$$

Taking absolute values with sufficiently small h, we obtain

$$(12\epsilon + 4b_i h^2)e_i \leq \xi + 2\tilde{e}(6\epsilon + 3\mu a_i h - b_i h^2). \quad (40)$$

Since $0 < a^* \leq a(x)$ and $0 < b^* \leq b(x)$, we get

$$(12\epsilon + 4b^* h^2)e_i \leq \xi + 2\tilde{e}(6\epsilon + 3\mu a^* h - b^* h^2) \quad (41)$$
$$\leq \xi + 2\tilde{e}(6\epsilon + 3\mu a^* h + b^* h^2). \quad (42)$$

In particular

$$(12\epsilon + 4b^* h^2)\tilde{e} \leq \xi + 2\tilde{e}(6\epsilon + 3\mu a^* h + b^* h^2). \quad (43)$$

Solving for $\tilde{e}$, we obtain $(2b^*h^2 - 6\mu a^* h)\tilde{e} \leq \xi \leq \lambda h^4$ or

$$\tilde{e} \leq \frac{\lambda h^3}{2b^*h - 6\mu a^*}. \quad (44)$$

Now to estimate e_{-1}, e_0, e_N and e_{N+1}, we observe that the first and last equation of the the system $T(x_N - y_N) = h^2(\bar{f}_n - \hat{f}_n)$ where $\bar{f}_n = (f_0, f_1, \ldots, f_N)$, gives

$$e_0 \leq \frac{2\lambda b^* h^5}{(36\epsilon - 12a^* h\mu)(2b^*h - 6\mu a^*)}, \quad (45)$$

and

$$e_N \leq \frac{2\lambda b^* h^5}{(36\epsilon + 12a^* h\mu)(2b^*h - 6\mu a^*)}. \quad (46)$$

Now e_{-1} and e_{N+1} can be evaluated using boundary conditions $\delta_{-1} = (4\delta_0 - \delta_1)$ and $\delta_{N+1} = (4\delta_N - \delta_{N-1})$

$$e_{-1} \leq \frac{\lambda h^3}{(2b^*h - 6\mu a^*)}\left(\frac{2b^*h^2 + 9\epsilon - 3a^*h\mu}{9\epsilon + 3a^*h\mu}\right), \quad (47)$$

and

$$e_{N+1} \leq \frac{\lambda h^3}{(2b^*h - 6\mu a^*)}\left(\frac{2b^*h^2 + 9\epsilon - 3a^*h\mu}{9\epsilon + 3a^*h\mu}\right). \quad (48)$$

Using value $\lambda = \epsilon\gamma_2 + \mu\gamma_1\|b(x)\|_\infty h + \gamma_0\|b(x)\|_\infty h^2$ and since there exists a constant C such that

$$e = \max_{-1\leq i\leq N+1}\{e_i\} \leq Ch^2. \quad (49)$$

The above inequality enables us to estimate $\|S(x) - Y_n(x)\|_\infty$, and therefore $\|y(x) - S(x)\|_\infty$. In particular

$$S(x) - Y_n(x) = \sum_{i=-1}^{N+1}(q_i - p_i)B_i(x). \quad (50)$$

Thus

$$|S(x) - Y_n(x)| \leq \max|q_i - p_i|\sum_{i=-1}^{N+1}|B_i(x)|. \quad (51)$$

Combining equations (49), (51) and using Lemma 1 we obtain

$$\|S - Y_n\|_\infty \leq Ch^2.$$

Since $\|y - Y_N\|_\infty \leq \gamma_0 h^4$ and $\|y - S\|_\infty \leq \|y - Y_N\|_\infty + \|Y_N - S\|_\infty$, we obtain

$$\|y - S\|_\infty \leq Ch^2.$$

Combining the results we get the required estimate.

3. Test examples and numerical results

To demonstrate the efficiency of the method, two numerical examples are considered. Since the exact solution of the considered problems are given so the maximum absolute errors are estimated by using $E_{N,\epsilon} = \max_{0\leq i\leq N} | y_i^N - S_i^N |$, where y_i^N is the exact solution and S_i^N is the computed solution. If the exact solution is not known then the maximum absolute errors can be obtained by using the double mesh principle.

Example 1. Consider the boundary value problem

$$-\epsilon y''(x) - \mu y'(x) + y(x) = x, \ y(0) = 1, \ y(1) = 0.$$

The exact solution of the problem is given by

$$y(x) = \frac{(1+\mu) + (1-\mu)e^{m_2}}{e^{m_2} - e^{m_1}}e^{m_1 x} \quad (52)$$
$$+ \frac{(1+\mu) + (1-\mu)e^{m_1}}{e^{m_1} - e^{m_2}}e^{m_2 x} + x + \mu, \quad (53)$$

Table 1: Maximum absolute error at nodal points for Example 1 for $\epsilon = 10^{-6}$ and $\mu = 10^{-2}$

Nodes	k			
	1	10	20	25
0.0000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.0000E-05	5.38960E-04	3.87545E-07	3.04298E-06	4.75461E-06
2.0000E-05	1.02676E-03	6.64424E-07	5.50136E-06	8.59576E-06
3.0000E-05	1.46829E-03	8.48286E-07	7.45937E-06	1.16551E-05
4.0000E-05	1.86802E-03	9.54410E-07	8.99046E-06	1.40473E-05
5.0000E-05	2.22993E-03	9.95999E-07	1.01586E-05	1.58724E-05
6.0000E-05	2.55767E-03	9.84435E-07	1.10193E-05	1.72172E-05
7.0000E-05	2.85450E-03	9.29511E-07	1.16210E-05	1.81572E-05
8.0000E-05	3.12340E-03	8.39627E-07	1.20054E-05	1.87577E-05
9.0000E-05	3.36704E-03	7.21971E-07	1.22087E-05	1.90753E-05
1.0000E-04	3.58785E-03	5.82670E-07	1.22622E-05	1.91588E-05
2.0000E-04	—	1.17851E-06	8.93357E-06	1.39570E-05
3.0000E-04	—	2.55479E-06	4.88174E-06	7.62567E-06
4.0000E-04	—	3.33986E-06	2.37164E-06	3.70349E-06
5.0000E-04	—	3.74546E-06	1.08063E-06	1.68623E-06
6.0000E-04	—	3.95320E-06	4.73167E-07	7.37044E-07
7.0000E-04	—	4.06733E-06	2.01927E-07	3.13214E-07
8.0000E-04	—	4.13967E-06	8.49324E-08	1.30390E-07
9.0000E-04	—	4.19415E-06	3.56951E-08	5.34361E-08
1.0000E-03	—	4.24135E-06	1.53545E-08	2.16320E-08
1.5000E-03	—	—	1.72512E-09	2.17642E-10
2.0000E-03	—	—	1.67393E-09	1.19676E-11
2.5000E-03	—	—	—	1.07341E-11
O_{max}	3.62474E-03	4.2801E-06	3.7031E-10	3.85780E-12
7.5000E-01	—	—	—	4.91700E-04
8.0000E-01	—	—	2.53277E-09	2.58682E-14
8.5000E-01	—	—	1.68565E-12	2.64777E-12
9.0000E-01	—	4.67764E-06	1.59341E-10	2.49347E-10
9.1000E-01	—	9.63063E-11	3.86025E-10	6.04063E-10
9.2000E-01	—	2.30447E-10	9.23640E-10	1.44533E-09
9.3000E-01	—	5.42783E-10	2.17545E-09	3.40416E-09
9.4000E-01	—	1.25234E-09	5.01926E-09	7.85416E-09
9.5000E-01	—	2.80917E-09	1.12589E-08	1.76179E-08
9.6000E-01	—	6.04931E-09	2.42449E-08	3.79386E-08
9.7000E-01	—	1.22125E-08	4.89462E-08	7.65912E-08
9.8000E-01	—	2.19154E-08	8.78343E-08	1.37443E-07
9.9000E-01	3.30908E-03	2.94955E-08	1.18214E-07	1.84982E-07
9.9100E-01	1.49558E-07	2.93090E-08	1.17467E-07	1.83813E-07
9.9200E-01	2.93511E-10	2.87642E-08	1.15284E-07	1.80396E-07
9.9300E-01	2.77633E-10	2.77885E-08	1.11373E-07	1.74277E-07
9.9400E-01	2.62739E-10	2.62979E-08	1.05399E-07	1.64929E-07
9.9500E-01	2.41754E-10	2.41960E-08	9.69747E-08	1.51746E-07
9.9600E-01	2.13540E-10	2.13716E-08	8.56549E-08	1.34033E-07
9.9700E-01	1.76838E-10	1.76971E-08	7.09279E-08	1.10988E-07
9.9900E-01	7.18654E-11	7.19099E-09	2.88206E-08	4.50986E-08
1.0000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00

Table 2: Maximum absolute error at nodal points for Example 2 for $\epsilon = 10^{-6}$ and $\mu = 10^{-2}$

Nodes	k			
	1	10	20	25
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-05	1.37308E-04	1.38923E-06	5.55763E-06	8.68492E-06
2.00000E-05	2.50032E-04	2.27032E-06	9.08228E-06	1.41928E-05
3.00000E-05	3.42622E-04	2.78266E-06	1.11317E-05	1.73952E-05
4.00000E-05	4.18724E-04	3.03167E-06	1.21276E-05	1.89514E-05
5.00000E-05	4.81323E-04	3.09652E-06	1.23868E-05	1.93563E-05
6.00000E-05	5.32863E-04	3.03626E-06	1.21455E-05	1.89791E-05
7.00000E-05	5.75346E-04	2.89448E-06	1.15781E-05	1.80923E-05
8.00000E-05	6.10414E-04	2.70301E-06	1.08120E-05	1.68950E-05
9.00000E-05	6.39408E-04	2.48477E-06	9.93878E-06	1.55304E-05
1.00000E-04	6.63430E-04	2.25595E-06	9.02331E-06	1.40998E-05
2.00000E-04	—	5.98898E-07	2.39413E-06	3.74076E-06
3.00000E-04	—	1.19486E-07	4.76420E-07	7.44336E-07
4.00000E-04	—	2.14468E-08	8.42716E-08	1.31651E-07
5.00000E-04	—	3.87671E-09	1.39749E-08	2.18296E-08
6.00000E-04	—	9.46114E-10	2.22496E-09	3.47457E-09
7.00000E-04	—	4.83770E-10	3.44543E-10	5.37378E-10
8.00000E-04	—	4.18796E-10	5.23901E-11	8.11162E-11
9.00000E-04	—	4.15949E-10	7.94195E-12	1.17553E-11
1.00000E-03	—	4.22711E-10	1.26660E-12	1.38656E-12
1.50000E-03	—	—	4.82947E-14	3.89855E-13
2.00000E-03	—	—	0.00000E+00	2.11442E-13
2.50000E-03	—	—	—	0.00000E+00
O_{max}	6.63451E-04	9.25778E-11	0.00000E+00	0.00000E+00
7.5000E-01	—	—	—	0.00000E+00
8.0000E-01	—	—	0.00000E+00	7.77156E-16
8.5000E-01	—	—	3.33067E-16	8.88178E-16
9.0000E-01	—	1.88076E-09	4.74065E-14	7.39964E-14
9.1000E-01	—	7.89369E-14	3.11196E-13	4.88609E-13
9.2000E-01	—	4.99378E-13	2.00612E-12	3.15242E-12
9.3000E-01	—	3.15709E-12	1.27205E-11	1.99918E-11
9.4000E-01	—	1.95986E-11	7.90040E-11	1.24165E-10
9.5000E-01	—	1.18328E-10	4.77026E-10	7.49711E-10
9.6000E-01	—	6.85878E-10	2.76507E-09	4.34568E-09
9.7000E-01	—	3.72718E-09	1.50259E-08	2.36153E-08
9.8000E-01	—	1.80037E-08	7.25809E-08	1.14071E-07
9.9000E-01	2.01006E-03	6.52235E-08	2.62945E-07	4.13255E-07
9.9100E-01	7.16987E-10	7.15573E-08	2.88480E-07	4.53386E-07
9.9200E-01	7.73468E-10	7.75369E-08	3.12586E-07	4.91273E-07
9.9300E-01	8.24999E-10	8.27035E-08	3.33415E-07	5.24009E-07
9.9400E-01	8.62005E-10	8.64140E-08	3.48374E-07	5.47519E-07
9.9500E-01	8.75659E-10	8.77830E-08	3.53893E-07	5.56192E-07
9.9600E-01	8.53941E-10	8.56066E-08	3.45119E-07	5.42403E-07
9.9700E-01	7.80726E-10	7.82665E-08	3.15528E-07	4.95896E-07
9.9800E-01	6.34468E-10	6.36051E-08	2.56421E-07	4.03002E-07
9.9900E-01	3.86719E-10	3.87676E-08	1.56290E-07	2.45632E-07
1.0000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00

where $m_1 = \frac{-\mu-\sqrt{\mu^2+4\epsilon}}{2\epsilon}$, $m_2 = \frac{-\mu+\sqrt{\mu^2+4\epsilon}}{2\epsilon}$.

For this problem the zeroth order asymptotic expansion W_0 is given by

$$W_0(x) = x - \exp((x-1)/\mu) \quad (54)$$
$$+ (1-\exp(-1/\mu))\exp(-x/(\epsilon/\mu)). \quad (55)$$

Example 2. Consider the boundary value problem

$$-\epsilon y''(x) - 2\mu y'(x) + 4y(x) = 1,\ y(0) = 0,\ y(1) = 1.$$

The exact solution of the problem is given by

$$y(x) = \frac{3+e^{m_2}}{4(e^{m_1}-e^{m_2})}e^{m_1 x} - \frac{3+e^{m_1}}{4(e^{m_1}-e^{m_2})}e^{m_2 x} + \frac{1}{4}$$

where $m_1 = \frac{-\mu-\sqrt{\mu^2+4\epsilon}}{\epsilon}$, $m_2 = \frac{-\mu+\sqrt{\mu^2+4\epsilon}}{\epsilon}$.

For this example the zeroth order asymptotic expansion W_0 is given by

$$W_0(x) = (1/4)[1-\exp(-2x/(\epsilon/\mu))] \quad (56)$$
$$+ (3/4)[\exp(2x/\mu) - \exp(-2x/(\epsilon/\mu))]\exp(-2/\mu). \quad (57)$$

4. Conclusion

In this paper, we present an approximate method based on asymptotic expansion for two parameters singularly perturbed boundary value problems. The B-spline method is used in the inner region and for rest of the region we use the zeroth order asymptotic expansion approximation. To demonstrate the applicability of the method two numerical examples have been considered. For inner regions the absolute errors at nodal points are presented in the tables while for the outer region the maximum absolute error denoted by O_{max} is presented at the middle of the table. It can be seen from the tables that the numerical solutions are very closed to exact solutions. The solution profiles for the considered examples for a fix μ and different values of ϵ are given in Figures 1 and 2.

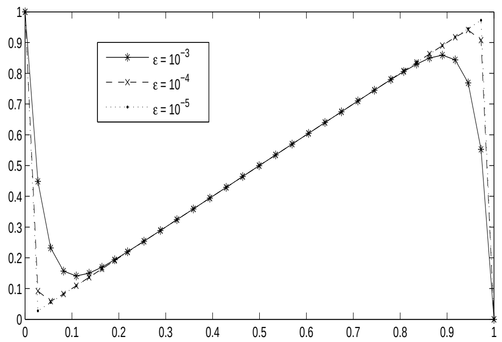

Figure 1: Solution profile for Example 1 for $\mu = 10^{-4}$ using $N = 32$.

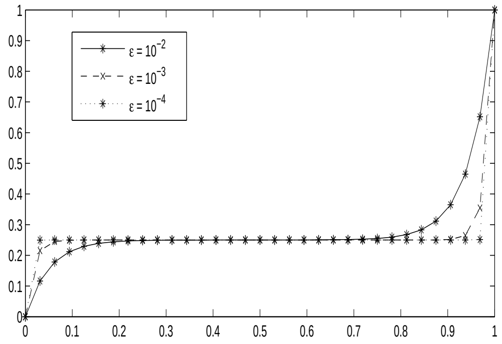

Figure 2: Solution profile for Example 2 for $\mu = 10^{-4}$ using $N = 32$.

References

[1] S.M. Roberts, J. Math. Anal. Appl. **113**, 411 (1988).
[2] R.E. O'Malley Jr., J. Math. Anal. Appl. **19**, 291 (1967).
[3] R.E. O'Malley Jr., J. Math. Mech. **16**, 1143 (1967).
[4] small H.G. Roos and Z. Uzelac, Comput. Methods Appl. Math. **3**, 443 (2003).
[5] J.L. Gracia, E. O'Riordan and M.L. Pickett, Appl. Numer. Math. **56**, 962 (2006).
[6] T. Linßand H.G. Roos, J. Math. Anal. Appl. **289**, 355 (2004).
[7] R.E. O'Malley Jr., J. Math. Mech. **18**, 835 (1969).
[8] R.E. O'Malley Jr., Introduction to Singular Perturbations (Academic Press, New York, 1974).
[9] R.E. O'Malley Jr., Singular Perturbation Methods for Ordinary Differential Equations (Springer, New York, 1990).
[10] E. O'Riordan, M.L. Pickett and G. I. Shishkin, Comput. Methods Appl. Math. **3**, 424 (2003).
[11] E. O'Riordan, M.L. Pickett and G.I. Shishkin, Math. Comp. **75**, 1135 (2006).
[12] G.I. Shishkin and V.A. Titov, Chisl. Metody Mekh. Sploshn. Sredy **2**, 145 (1976).
[13] R. Vulanović, Computing **67**, 287 (2001).
[14] C. De Boor, A Practical Guide to Splines (Springer-Verlag, New York, 1978).

D. Kumar is presently employed as an assistant professor in the department of Mathematics at BITS Pilani, India. He obtained his PhD from IIT Kanpur (INDIA). He is an active researcher coupled with the teaching experience in various Institutes in India. He has published more than 10 research articles in reputed international journals of mathematical and engineering sciences.

A. S. Yadaw is presently a postdoc fellow in the department of Pharmacology & Systems Biology Mount Sinai School of Medicine, New York. He obtained his PhD from IIT Kanpur (INDIA). He is an active researcher and has published more than 6 research articles in reputed international journals of mathematical and engineering sciences.

M. K. Kadalbajoo is presently employed as a professor in the department of Mathematics at IIT Kanpur, India. He obtained his PhD from IIT Bombay (INDIA). He is an active researcher coupled with the vast (40 years) teaching experience at IIT Bombay and IIT Kanpur in India. He has been an invited speaker of number of conferences and has published more than 200 (Two hundred) research articles in reputed international journals of mathematical and engineering sciences.