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Inverse Problems for a Time-Fractional Diffusion Equation with Unknown Right-Hand Side

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Abstract: This paper is devoted to forward and inverse source problems for a 2D in space variables time-fractional diffusion equation. The forward problem is an initial-boundary/value problem for given equation in a rectangular area. In inverse problem the unknown right side of the equation is assumed to have the form of a product of two functions: one of which depends only on the time variable, while the other one - depends on the spatial variables. Two inverse problems of finding these functions separately under the condition that the other function is known are investigated. For the inverse problem for determining a time-dependence function, Abel's integral equation of the first kind is obtained, which is further reduced to an integral equation of the second kind with the application of fractional differentiation to it. To solve the direct problem and inverse problem of determining a spatial-dependence function, the Fourier spectral method is used. Theorems of unique solvability of the formulated problems are proved. The existence and uniqueness results are based on the Fourier method, fractional calculus and properties of Mittag-Lefller function.

Keywords: Inverse problem, Gerasimov-Caputo fractional derivative, Fourier spectral method, Fourier coefficient, Bessel inequality.

1 Introduction, Motivation and Preliminaries

Fractional heat equations are more sufficient than whole-order models for describing anomalous diffusion phenomena because fractional order derivatives allow the description of memory and hereditary properties of heterogeneous substances [1,2]. For the past several decades, fractional diffusion equations have involved great attention not only from mathematicians and engineers but also from many scientists from fields like biology, physics, chemistry and biochemistry, medicine and finance [2,3,4,5,6].

The fractional diffusion equations appear when replacing the standard time derivative with time fractional derivatives and can be used to describe superdiffusion and subdiffusion phenomena [1], [7,8,9]. Forward problems, i.e. correct initial value problems (Cauchy problem), initial boundary/value problems for time-fractional diffusion equations, have attracted much more attention in recent years, for example, on some uniqueness and existence results we refer readers to works [10,11,12,13,14,15,16,17,18] and on exact solutions of these problems to [19,20]. In papers [19,20] the equivalence of homogeneous second-order parabolic and hyperbolic integro-differential equations with a convolution type integral, whose kernel is Mittag-Leffler type functions, to fractional diffusion-wave equations were also proved. Inverse problems of determining kernels from integro-differential equations of parabolic equations are studied in [21,22,23], and similar problems for hyperbolic integro-differential equations are investigated in [24,25,26]. Conditions for the unique solvability of problems are obtained. Efficient numerical methods for solving direct and inverse problems for equations of fractional and integer orders are applied in the works [27,28,29,30,31].

But, in some practical cases, a part of boundary data, or initial data, or heat coefficient, or source term may not be given and we want to find them by additional measurement data which will yield some fractional heat inverse problems. As we know, research on inverse problems for time fractional diffusion equations yet deficiencies wide observation [32,

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33]. On the uniqueness of inverse problems, in [34] and in [35] gave the uniqueness results for determining the order of the fractional derivative and space-dependent diffusion coefficient in a fractional diffusion equation by using different types of initial data. In [11,12,13,14,15,16,17,18] established a few uniqueness results for several inverse problems. In [36] provided a conditional stability for determining a zeroth-order coefficient in a half-order fractional heat equation. On numerical computations of inverse problems, in [37] solved a Cauchy problem by a mollification regularization and space marching algorithm. Besides, in [38] proposed a quasi-reversibility method for solving the backward problem of a fractional diffusion equation. Other type of problems, such as, in [39,40] solved the Cauchy problems for the time fractional diffusion equations on a strip domain by a Fourier truncation method and a convolution regularization method. Wei and Wang in [41,42] solved a backward problem and an inverse space-dependent source problem by a modified quasi-boundary value method.

In this paper, for a two-dimensional inhomogeneous time-fractional diffusion equation in a bounded rectangle, we investigate two inverse inverse problems, in first of which it is determined the time-dependent source function and in second one the two-dimensional source function of spatial variables is to be determined.

Consider the following 2D time-fractional heat equation:

$$Lu \equiv \partial_t^\alpha u - a^2(u_{xx} + u_{yy}) = F(x,y,t), \tag{1}$$

in the domain

$$Q := D \times (0,T), \quad D := \{(x,y) : 0 < x < l, 0 < y < q\},$$

as state the following problems.

The first initial-boundary value problem. Find in the domain Q a function $u(x,y,t)$ such that

$$u(x,y,t) \in C(\overline{Q}) \cap C_\gamma^{\alpha,2}(Q); \tag{2}$$

$$Lu = F(x,y,t), \quad (x,y,t) \in Q; \tag{3}$$

$$u(0,y,t) = u(l,y,t) = 0, \quad 0 \le y \le q,\ 0 \le t \le T; \tag{4}$$

$$u(x,0,t) = u(x,q,t) = 0, \quad 0 \le x \le l,\ 0 \le t \le T; \tag{5}$$

$$u(x,y,0) = \varphi(x,y), \quad 0 \le x \le l,\ 0 \le y \le q; \tag{6}$$

where φ, F are those given and ∂_t^α stands for Gerasimov-Caputo fractional derivative of order $n-1 < \alpha \le n$ in the time variable (see [43])

$$\partial_t^\alpha g(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} g^{(n)}(\tau)d\tau, & n-1 < \alpha < n, \\ g^{(n)}(t), & \alpha = n, \end{cases}$$

and

$$C_\gamma^{\alpha,n}[0,T] = \Big\{ u(x,y,t) : u(\cdot,\cdot,t) \in C^{(n)}(D),\ t \in [0,T], \text{ and}$$
$$\partial_t^\alpha u(x,y,\cdot) \in C_\gamma[0,T],\ (x,y) \in D \Big\}, \quad C_\gamma^{\alpha,0}[0,T] = C_\gamma^\alpha[0,T],$$

where $\alpha > 0$, $n \in \mathbb{N}$, $0 \le \gamma < 1$ be such that $\gamma \le \alpha$ (see [43], p. 199) and here

$$C_\gamma[0,T] = \{f(t) : t^\gamma f(t) \in C[0,T]\}.$$

But in this work, we only mention at case $0 < \alpha < 1$.

On the basis of this direct problem for Eq. (1) we consider the following inverse problems.

Inverse problem 1. Suppose $F(x,y,t) = f(x,y)g(t)$. It is required to find functions $u(x,y,t)$ and $g(t)$ which satisfy conditions (2)-(6) and, in addition, the following ones:

$$g(t) \in C[0,T], \tag{7}$$

$$u(x_0,y_0,t) = h(t), \quad 0 \le t \le T, \tag{8}$$

where (x_0, y_0) is a given fixed point in domain D, $\varphi(x,y)$, $h(t)$, $f(x,y)$ are given sufficiently smooth functions, wherein $\varphi(x_0, y_0) = h(0)$.

Inverse problem 2. Suppose $F(x,y,t) = f(x,y)g(t)$. Find functions $u(x,y,t)$ and $f(x,y)$ which satisfy equalities (2)-(6) and, in addition, the following ones:

$$f(x,y) \in C(\overline{D}), \tag{9}$$

$$u(x,y,t_0) = \psi(x,y), \quad (x,y) \in \overline{D}, \tag{10}$$

where t_0 is a given fixed point in the half-interval $(0,T]$, wherein $\varphi(x,y)$, $\psi(x,y)$, $g(t)$ are given functions.

When $\varphi(x,y) \equiv 0$, $(x,y) \in \overline{D}$, then these inverse problems were investigated in [44].

2 Preliminaries

In this section, we give some notations which will be repeatedly used in the sequent sections.

Two parameter Mittag-Leffler (M-L) function. The two parameter M-L function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$−denote the real part of the complex number α. The Mittag-Leffler function has been studied by many authors who have proposed and investigated various generalizations and applications. A very interesting work that meets many results about this function is due to Haubold et al. (see [45]).

Proposition 1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{C}{1+|z|}, \quad \kappa \leq |arg(z)| \leq \pi.$$

For the proof, we refer to [46] for example.

Proposition 2. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

Proposition 3. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0.$$

Proposition 4. *Let $\alpha > 0$, $\beta > 0$ and $\lambda > 0$, then we have*

$$\frac{d}{dt} t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) = t^{\beta-2} E_{\alpha,\beta-1}(-\lambda t^\alpha), \quad t > 0.$$

The proof of these assertions come from the definition of Caputo fractional derivative and differentiation of the two-parameter M-L function.

Proposition 5. (see [47]) *For $0 < \alpha < 1$ and $\lambda > 0$, if $g(t) \in AC[0,T]$, we have*

$$\partial_{0+}^\alpha \int_0^t g(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha) d\tau$$

$$= g(t) - \lambda \int_0^t g(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha) d\tau, \quad 0 < t \leq T.$$

In particular, if $\lambda = 0$, we have

$$\partial_{0+}^\alpha \int_0^t g(\tau)(t-\tau)^{\alpha-1} d\tau = \Gamma(\alpha) g(t), \quad 0 < t \leq T.$$

Proposition 6. (see [48]) *For* $0 < \alpha < 1$, $t > 0$, *we have* $0 < E_\alpha(-t) < 1$. *Moreover,* $E_\alpha(-t)$ *is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dt^n} E_\alpha(-t) \geq 0, \quad \forall n \in \mathbb{N}.$$

Proposition 7. *For* $0 < \alpha < 1$, $\eta > 0$, *we have* $0 \leq E_{\alpha,\alpha}(-\eta) \leq \frac{1}{\Gamma(\alpha)}$. *Moreover,* $E_{\alpha,\alpha}(-\eta)$ *is a monotonic decreasing function with* $\eta > 0$.

Lemma 1. (see [49]) *Consider a sequence of functions* $\{f_k\}_{k\geq 0}$ *defined on* $(0,T]$. *Suppose the following conditions are fulfilled:*

(i) for a given $\alpha > 0$, $\partial_t^\alpha f_k(t)$, $k \geq 0$, $t \in (0,T]$ *exists;*

(ii) $\sum\limits_{k\geq 0} f_k(t)$ *and* $\sum\limits_{k\geq 0} \partial_t^\alpha f_k(t)$ *are uniformly convergent on* $[\varepsilon_0, T]$ *for any* $\varepsilon_> 0$.

Then the function defined by the series $\sum\limits_{k\geq 0} f_k(t)$ *is* $\alpha-$*differentiable and satisfies*

$$\partial_t^\alpha \sum_{k\geq 0} f_k(t) = \sum_{k\geq 0} \partial_t^\alpha f_k(t).$$

Let $L^2(D)$ be a usual L^2-space. Since $-\Delta := \partial_x^2 + \partial_y^2$ is a symmetric uniformly elliptic operator, the spectrum of $-\Delta$ is entirely composed of eigenvalues and counting to the multiplicities, we can set: $0 < \lambda_{11} \leq \lambda_{12} \leq \ldots$.

Now, we will solve the following boundary-value problem:

$$\begin{cases} v_{xx} + v_{yy} + \lambda^2 v = 0; \\ v(0,y) = 0, \quad v(l,y) = 0; \\ v(x,0) = 0, \quad v(x,q) = 0. \end{cases} \tag{11}$$

Thus the eigenvalue problem reduces to the solution of the homogeneous equation with homogeneous boundary conditions. We shall solve this problem by the method of separation of variables, assuming

$$v(x,y) = X(x)Y(y).$$

Problem (11) is an interesting equation since each side can be set to a fixed constant λ as that is the only solution that works for all values of t and x. Therefore, the equation can be separated into two ordinary differential equations:

$$\begin{cases} X'' + \mu^2 X = 0; \\ X(0) = 0, \quad X(l) = 0; \end{cases} \tag{12}$$

$$\begin{cases} Y'' + \nu^2 Y = 0; \\ Y(0) = 0, \quad Y(q) = 0; \end{cases} \tag{13}$$

where μ and ν are constants of the separation of variables connected by the relationship $\mu^2 + \nu^2 = \lambda^2$. The boundary conditions for $X(x)$ and $Y(y)$ follows from the corresponding conditions for function v. For example from

$$v(0,y) = X(0)Y(y) = 0$$

it follows $X(0) = 0$, since $Y(y) \neq 0$ (we have only non-trivial solutions).

The solutions of equations (12) and (13) have the form

$$X_m(x) = \sin(\mu_m x), \quad Y_m(y) = \sin(\nu_n y);$$

$$\mu_m = \left(\frac{m\pi}{l}\right)^2, \quad \nu_n = \left(\frac{n\pi}{q}\right)^2, \quad m,n \in \mathbb{N}.$$

The eigenvalues

$$\lambda_{mn}^2 = \mu_m^2 + \nu_n^2$$

have corresponding eigenfunctions

$$v_{mn}(x,y) = A_{mn} \sin(\mu_m x) \sin(\nu_n y),$$

where A_{mn} is some constant. We choose this so that the norm in $L^2(D)$ of function v_{mn} with weight 1 equals unity

$$\iint\limits_D v_{mn}^2 dxdy = A_{mn}^2 \int_0^l \sin^2(\mu_m x)dx \int_0^q \sin^2(\nu_n y)dy = 1.$$

Hence $A_{mn} = \frac{2}{\sqrt{lq}}$. So,

$$v_{mn}(x,y) = \frac{2}{\sqrt{lq}} \sin(\mu_m x)\sin(\nu_n y) \tag{14}$$

is the orthonormal eigenfunction corresponding to $-\lambda_{mn}$. Then the sequence $\{v_{mn}\}_{m,n\in\mathbb{N}}$ is orthonormal basis in $L^2(D)$.

Lemma 2. *Let us given the ordinary fractional differential equation for $\alpha \in (0,1)$,*

$$\begin{cases} \partial_t^\alpha v(t) + \lambda v(t) = f(t), & 0 < t \le T, \\ v(0) = a, \end{cases}$$

where $\partial_t^\alpha -$ Caputo fractional derivative operator, λ, a are constants. Then there is a explict solution which is given in the integral form

$$v(t) = aE_\alpha(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha) f(\tau)d\tau.$$

and this solution is unique $v(t) \in C_\gamma^\alpha[0,T]$, where $0 \le \gamma < \alpha$, $E_{\alpha,\beta}(\cdot)$ is the M-L function.

Proof. See Ex.4.9 in [43].

3 Well-Posedness of the Direct Problem (2)-(6)

In this section, we construct of the solution to the forward problem and investigate the uniqueness and existence of that solution.

3.1 Uniqueness of the Solution

The following theorem is an analogue of the result in [50].

Theorem 1. *If problem (2)-(6) has a solution such that*

$$\lim_{x\to+0} u_x \sin\frac{m\pi}{l}x = \lim_{x\to l-0} u_x \sin\frac{m\pi}{l}x = 0, \quad 0 \le y \le q, \tag{15}$$

$$\lim_{y\to+0} u_y \sin\frac{n\pi}{q}y = \lim_{x\to q-0} u_y \sin\frac{n\pi}{q}y = 0, \quad 0 \le x \le l, \tag{16}$$

then this solution is unique.

Proof. Applying the method of separation of variables, we seek a solution of (2)-(6) with the form

$$u(x,y,t) = \rho(t)v(x,y). \tag{17}$$

Substituting (17) into (2) with $F(x,y,t) \equiv 0$, we require that $v(x,y)$ satisfies the spectral problem (13) and its non-trivial solution gives by (14).

Let a function $u(x,y,t)$ stand for a solution to problem (2)-(5) which satisfies conditions (15), (16).

Following [51], we consider the integral

$$\rho_{mn}(t) = \iint\limits_D u(x,y,t)v_{mn}(x,y)dxdy. \tag{18}$$

Introduce an auxiliary integral, namely,

$$\rho_{mn}^{\varepsilon,\delta}(t) = \iint\limits_{D_{\varepsilon\delta}} u(x,y,t)v_{mn}(x,y)dxdy,$$

here ε, δ are given sufficiently small positive values,

$$D_{\varepsilon\delta} = \{(x,y) | \varepsilon \leq x \leq l-\varepsilon, \delta \leq y \leq q-\delta\}.$$

Applying the fractional derivative to equality (18) and making use of Eq. (1), we get the correlation

$$\left(\partial_t^\alpha \rho_{mn}^{\varepsilon,\delta}\right)(t) = \iint\limits_{D_{\varepsilon\delta}} \partial_t^\alpha u(x,y,t) v_{mn}(x,y) dxdy = \iint\limits_{D_{\varepsilon\delta}} \left(a^2 \triangle u + F(x,y,t)\right) v_{mn}(x,y) dxdy$$

$$= a^2 \left(\iint\limits_{D_{\varepsilon\delta}} u_{xx} v_{mn}(x,y) dxdy + \iint\limits_{D_{\varepsilon\delta}} u_{yy} v_{mn}(x,y) dxdy \right)$$

$$+ \iint\limits_{D_{\varepsilon\delta}} F(x,y,t) v_{mn}(x,y) dxdy = a^3 (I_1 + I_2) + F_{mn}^{\varepsilon,\delta}(t), \tag{19}$$

where

$$F_{mn}^{\varepsilon,\delta}(t) = \iint\limits_{D_{\varepsilon\delta}} F(x,y,t) v_{mn}(x,y) dxdy.$$

Calculating integrals I_1 and I_2 by parts, we conclude that as [51], the limit in integrals I_1 and I_2 as $\varepsilon \to 0$ and $\delta \to 0$, taking into account conditions (4), (5), (15) and (16), from formula (19) arrives

$$\left(\partial_t^\alpha \rho_{mn}\right)(t) = -a^2 \lambda_{mn}^2 \iint\limits_D u(x,y,t) v_{mn}(x,y) dxdy + F_{mn}(t) = -a^2 \lambda_{mn}^2 \rho_{mn}(t) + F_{mn}(t)$$

or

$$\left(\partial_t^\alpha \rho_{mn}\right)(t) + a^2 \lambda_{mn}^2 \rho_{mn}(t) = F_{mn}(t), \tag{20}$$

where

$$F_{mn}(t) = \iint\limits_D F(x,y,t) v_{mn}(x,y) dxdy. \tag{21}$$

The initial condition (6) gives:

$$\rho_{mn}(0) = \iint\limits_D u(x,y,0) v_{mn}(x,y) dxdy = \iint\limits_D \varphi(x,y) v_{mn}(x,y) dxdy = \varphi_{mn}. \tag{22}$$

By lemma 3, the problem (20), (22) has a unique solution in $\rho_{mn}(t) \in C_\gamma^\alpha[0,T]$ and it is defined by the formula

$$\rho_{mn}(t) = \varphi_{mn} E_\alpha(-a^2 \lambda_{mn}^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-a^2 \lambda_{mn}^2 (t-\tau)^\alpha\right) F_{mn}(\tau) d\tau. \tag{23}$$

It means that the solution to problem (2)-(6) is unique, because with $\varphi(x,y) \equiv 0$ and $F(x,y,t) \equiv 0$ we get identities $\varphi_{mn} \equiv 0$, $F_{mn}(t) \equiv 0$, and from the equality (23) comes that $u_{mn}(t) \equiv 0$. According to the formula (18) the latter equality is equivalent to that

$$\iint\limits_D u(x,y,t) v_{mn}(x,y) dxdy = 0.$$

Since the system $v_{mn}(x,y)$ is complete in the space $L^2(D)$, the function $u(x,y,t) = 0$ almost everywhere in D and with any $t \in [0,T]$. Since in view of condition (2) the function $u(x,y,t)$ is continuous on $\overline{Q}$, we make sure that $u(x,y,t) \equiv 0$ on $\overline{Q}$. Thus, we have proved the uniqueness of the solution to problem (2)-(6).

Remark. Note that conditions (15) and (16) mean that first derivatives u_x, u_y near the corresponding faces of the parallelepiped Q can have singularities of order less than one.

3.2 Existence

Under certain requirements to functions $F(x,y,t)$ and $\varphi(x,y)$ we can prove that the function

$$u(x,y,t)=\sum_{m,n=1}^{\infty}\rho_{mn}(t)v_{mn}(x,y), \tag{24}$$

where $v_{mn}(x,y)$ obeys formula (14), (while $\rho_{mn}(t)$ does (23)), is a solution to problem (2)-(6).

Lemma 3. *The next estimates are valid with large m and n:*

$$|\rho_{mn}(t)|\le C_1\left(|\varphi_{mn}|+\frac{1}{\lambda_{mn}^2}|F_{mn}(t_M)|\right),\quad t\in[0,T], \tag{25}$$

$$|\rho_{mn}(t)|\le C_2\left(\frac{|\varphi_{mn}|}{1+a^2\lambda_{mn}^2\varepsilon_0^{\alpha}}+\frac{1}{\lambda_{mn}^2}|F_{mn}(t_M)|\right),\quad t\in[\varepsilon_0,T], \tag{26}$$

$$|\partial_t^{\alpha}\rho_{mn}(t)|\le C_3\left(\frac{|\varphi_{mn}|}{1+a^2\lambda_{mn}^2\varepsilon_0^{\alpha}}\lambda_{mn}^2+|F_{mn}(t_M)|\right),\quad t\in[\varepsilon_0,T], \tag{27}$$

hereinafter C_i are positive constant values independent of $\varphi(x,y)$ and $F(x,y,t)$, $F_{mn}(t_M)=\max_{0\le t\le T}|F_{mn}(t)|$, $t_M\in[0,T]$, and ε_0 is a positive sufficiently small value.

Proof. The estimate (25) is obtained directly by calculating the equality (23) and using the proposition 1. The estimate (26) is also obtained as above. In estimating (26) we use Eq. (20) and the estimate (26).

Lemma 4. *If $\varphi(x,y)\in C^2(\overline{D})$, $F(x,y,t)\in C_{\gamma}^{\alpha,2}(\overline{Q})$, and*

$$\varphi(0,y)=\varphi(l,y)=0,\quad 0\le y\le q,\quad \varphi(x,0)=\varphi(x,q)=0,\quad 0\le x\le l, \tag{28}$$

$$F(0,y,t)=F(l,y,t)=0,\quad 0\le y\le q,\quad 0\le t\le T,$$

$$F(x,0,t)=F(x,q,t)=0,\quad 0\le x\le l,\quad 0\le t\le T, \tag{29}$$

then the following representations are valid:

$$\varphi_{mn}=\frac{\varphi_{mn}^{(2)}}{\mu_m\nu_n},\quad F_{mn}(t)=\frac{F_{mn}^{(2)}(t)}{\mu_m\nu_n}; \tag{30}$$

here $\varphi_{mn}^{(2)}$, $F_{mn}^{(2)}(t)$ are coefficients of the expansion of functions $\varphi_{xy}(x,y)$, $F_{xy}(x,y,t)$ in series with respect to the function system $\left\{\frac{1}{\sqrt{lq}},\frac{2}{\sqrt{lq}}\cos\mu_m x\cos\nu_n y\right\}_{m,n\ge 0}$ such that

$$\sum_{m,n\ge 0}|\varphi_{mn}^{(2)}|^2\le\iint_D(\varphi_{xy}(x,y))^2\,dxdy,$$

$$\sum_{m,n\ge 0}|F_{mn}^{(2)}|^2\le\iint_D(F_{xy}(x,y,t))^2\,dxdy,\quad 0\le t\le T. \tag{31}$$

In view of lemmas 3 and 4 series (24) with any $(x,y,t)\in\overline{Q}$ is majorized by the convergent series

$$C_4\sum_{m,n=1}^{\infty}\left(\frac{1}{mn}|\varphi_{mn}^{(2)}|+\frac{1}{mn}|F_{mn}(t_M)|\right). \tag{32}$$

Consequently, the function $u(x,y,t)$ is continuous on $\overline{Q}$.

Termwise differentiating the series in formula (24), we have

$$u_{xx}(x,y,t)=\sum_{m,n=1}^{\infty}(-\mu_m^2)\rho_{mn}(t)v_{mn}(x,y),\quad t>0, \tag{33}$$

$$u_{yy}(x,y,t)=\sum_{m,n=1}^{\infty}(-\nu_n^2)\rho_{mn}(t)v_{mn}(x,y),\quad t>0, \tag{34}$$

$$\partial_t^{\alpha}u(x,y,t)=\sum_{m,n=1}^{\infty}\left(\partial_t^{\alpha}\rho_{mn}\right)(t)v_{mn}(x,y),\quad t>0; \tag{35}$$

To get the last equality, we have used lemma 1 and 3.

In view of lemmas 3 and 4 with $t\geq\varepsilon_0>0$ they are majorized by the convergent series

$$C_5\sum_{m,n=1}^{\infty}\left(\frac{\lambda_{mn}^2}{1+a^2\lambda_{mn}^2\varepsilon_0^{\alpha}}\frac{|\varphi_{mn}^{(2)}|}{mn}+\frac{|F_{mn}^{(2)}(t_M)|}{mn}\right).$$

Then series (33)-(35) converge absolutely and uniformly on $\overline{Q}_{\varepsilon_0}=\overline{Q}\cap\{t\geq\varepsilon_0\}$, i.e., function (24) satisfies conditions (2) and (3).

Therefore, we have proved the following assertion.

Theorem 2. *If functions $\varphi(x,y)$ and $F(x,y,t)$ satisfy conditions of Lemma 4, then the initial-boundary/value problem (2)-(6) has a unique solution, which represents the sum of series (24), whose coefficients obey formulas (23).*

If we replace the class of functions (2) with the space of functions

$$u(x,y,t)\in C_{\gamma}^{\alpha,1}(\overline{Q})\cap C_{xy}^{2}(Q), \tag{36}$$

then we should impose additional smoothness conditions on the function $\varphi(x,y)$. The estimate (27) takes the form

$$|\partial_t^{\alpha}\rho_{mn}(t)|\leq C_6\left(\lambda_{mn}^2|\varphi_{mn}|+|F_{mn}(t_M)|\right),\quad t\in[0,T], \tag{37}$$

In this case, series (24), (33)-(35) with any $(x,y,t)\in\overline{Q}$ are majorized by the series

$$C_7\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2|\varphi_{mn}|+|F_{mn}(t_M)|\right). \tag{38}$$

The following lemma is also obtained by [50].

Lemma 5. *Let $\varphi(x,y)\in C^4(\overline{D})$,*

$$\varphi(0,y)=\varphi_{xx}(0,y)=\varphi(l,y)=\varphi_{xx}(l,y),\quad 0\leq y\leq q,$$

$$\varphi(x,0)=\varphi_{yy}(x,0)=\varphi(x,q)=\varphi_{yy}(x,q),\quad 0\leq x\leq l.$$

Then we get the representation

$$|\varphi_{mn}|=\frac{1}{[\mu_m+\nu_n]^4}\left(\left|\varphi_{mn}^{(4,0)}\right|+4\left|\varphi_{mn}^{(3,1)}\right|+6\left|\varphi_{mn}^{(2,2)}\right|+4\left|\varphi_{mn}^{(1,3)}\right|+\left|\varphi_{mn}^{(0,4)}\right|\right)$$
$$=\frac{1}{[\mu_m+\nu_n]^4}\sum_{i+j=4}\binom{4}{j}|\varphi_{mn}^{(i,j)}|, \tag{39}$$

where

$$\varphi_{mn}^{(4,0)}=\frac{2}{\sqrt{lq}}\iint_D\varphi_x^{(4)}(x,y)\sin(\mu_mx)\sin(\nu_ny)dxdy,$$

$$\varphi_{mn}^{(3,1)}=-\frac{2}{\sqrt{lq}}\iint_D\varphi_{xy}^{(3,1)}(x,y)\cos(\mu_mx)\cos(\nu_ny)dxdy,$$

$$\varphi_{mn}^{(2,2)}=\frac{2}{\sqrt{lq}}\iint_D\varphi_{x,y}^{(2,2)}(x,y)\sin(\mu_mx)\sin(\nu_ny)dxdy,$$

$$\varphi_{mn}^{(1,3)}=-\frac{2}{\sqrt{lq}}\iint_D\varphi_{x,y}^{(1,3)}(x,y)\cos(\mu_mx)\cos(\nu_ny)dxdy,$$

$$\varphi_{mn}^{(0,4)} = \frac{2}{\sqrt{lq}} \iint\limits_D \varphi_y^{(4)}(x,y)\sin(\mu_m x)\sin(\nu_n y)dxdy,$$

while

$$\sum_{m,n=1} |\varphi_{m,n}^{(i,j)}|^2 \leq \left\| \frac{\partial^4 \varphi(x,y)}{\partial x^i \partial y^j} \right\|_{L_2(D)}^2, \quad i+j=4. \tag{40}$$

The series (38) is estimated by the convergent series

$$C_8 \sum_{m,n=1}^{\infty} \left(\frac{|\varphi_{mn}^{(i,j)}|}{mn} + \frac{|F_{mn}^{(2)}(t_M)|}{mn} \right).$$

The given statement implies the next theorem.

Theorem 3. *Let the function $\varphi(x,y)$ satisfy conditions of Lemma 5, and $F(x,y,t)$ does conditions of Lemma 4, then problem (2)-(6) has a unique solution, which is defined by series (24) and belongs to class (36).*

4 Inverse Problem 1

In this section, we present several results regarding inverse problems of determining the right hand in the fractional diffusion equation from two observation conditions.

If the function $g(t)$ exists, then the solution to problem (2)-(8) is found by series (24) with coefficients (23), where

$$F_{mn}(t) = g(t) f_{mn},$$

$$f_{mn} = \iint\limits_D f(x,y) v_{mn}(x,y)dxdy. \tag{41}$$

If we put $x = x_0$, $y = y_0$ to (24) and change the order of integration and summation, then for the desired function $g(t)$ we get the Volterra integral equation of the first kind with weak singular kernel:

$$\int_0^t K(t,\tau)g(\tau)d\tau = \widetilde{h}(t), \quad 0 \leq t \leq T, \tag{42}$$

with the kernel

$$K(t,\tau) = \sum_{m,n=1}^{\infty} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-a^2\lambda_{mn}^2 (t-\tau)^{\alpha}\right) f_{mn} v_{mn}(x_0,y_0), \ 0 \leq \tau \leq t \leq T, \tag{43}$$

and the right-hand side

$$\widetilde{h}(t) = h(t) - \varphi_0(t),$$

$$\varphi_0(t) = \sum_{m,n=1}^{\infty} \varphi_{mn} E_{\alpha}\left(-a^2\lambda_{mn}^2 t^{\alpha}\right) v_{mn}(x_0,y_0). \tag{44}$$

Lemma 6. *If $\varphi(x,y)$ satisfies conditions of Lemma 5, while $f(x,y) \in C^4(\overline{D})$,*

$$f(0,y) = f_{xx}(0,y) = f(l,y) = f_{xx}(l,y), \quad 0 \leq y \leq q,$$

$$f(x,0) = f_{yy}(x,0) = f(x,q) = f_{yy}(x,q), \quad 0 \leq x \leq l$$

then series (43) and (44) and their α^{th} fractional derivatives in t uniformly converge in C_γ, correspondingly, on $0 \leq \tau \leq t \leq T$ and $0 \leq t \leq T$, where $1-\alpha \leq \gamma < 1$.

This lemma is proved analogously to lemma 5 (see [50]). Then we have the equality

$$|f_{mn}| = \frac{1}{[\mu_m + \nu_n]^4} \sum_{i+j=4} \binom{4}{j} |f_{mn}^{(i,j)}|, \tag{45}$$

where $f_{mn}^{i,j}$, $i+j=4$, are coefficients of the extension of functions $\frac{\partial^4 f}{\partial x^i \partial y^j}$ in the series with respect to the system of functions (2.4) and their derivatives $\frac{\partial^4 f}{\partial x^i \partial y^j}$. Besides, in view of the Bessel inequality we get the estimate

$$\sum_{m,n=1}^{\infty} \left(f_{mn}^{(i,j)}\right)^2 \leq \left\| \frac{\partial^4 f(x,y)}{\partial x^i \partial y^j} \right\|_{L_2(D)}^2, \quad i+j=4. \tag{46}$$

According to with correlations (39), (40), (45) and (46), series (43) and (44), as well as their α^{th} fractional derivatives in t, are majorized, correspondingly, by convergent series

$$C_9 \sum_{m,n=1}^{\infty} \frac{1}{mn} \sum_{i+j=4} |f_{mn}^{(i,j)}|, \quad C_{10} \sum_{m,n=1}^{\infty} \frac{1}{mn} \sum_{i+j=4} |\varphi_{mn}^{(i,j)}|$$

Therefore, series (43) and (44) and their $\alpha - th$ derivatives with respect to t uniformly converge in C_γ, respectively, for $0 \leq \tau \leq t \leq T$ and $0 \leq t \leq T$.

Theorem 4. *Let conditions of Lemma 6 be fulfilled. Besides*

$$f(x_0,y_0) \neq 0, \quad h(t) \in AC[0,T], h(0) = \varphi(x_0,y_0),$$

then Eq. (42) has s unique solution $g(t)$ in the functional class $AC[0,T]$.

Proof. Taking into account assertion of Lemma 6, series (43) converges uniformly and it can be termwise differentiated in the sense fractional derivative of Gerasimov-Caputo in t with $0 \leq \tau \leq t \leq T$. Therefore the function $K(t,\tau)$ is in C_γ on the mention set. In view of Proposition 5 and giving the Gerasimov-Caputo derivative from both sides Eq. (42), we get the correlation

$$f(x_0,y_0)g(t) - a^2 \int_0^t K(t,\tau)g(\tau)d\tau = \partial_t^\alpha \tilde{h}(t), \quad 0 \leq t \leq T. \tag{47}$$

where $f(x_0,y_0)$ represents the expansion of the function $f(x,y)$ in the series with respect to system (14) at the point (x_0,y_0). By condition $f(x_0,y_0) \neq 0$, Eq. (47) represents the Volterra integral equation of the second kind with a absolute continuous kernel and a absolute continuous right hand side. Consequently, Eq. (42) has a unique solution $g(t) \in AC[0,T]$.

Let us now prove that the condition $f(x_0,y_0) \neq 0$ is essential. Assume that for some $m = m_0$ or $n = n_0$ and $(x_0,y_0) \in D$, it holds that $\sin \mu_{m_0} x_0 = 0$ or $\sin \nu_{n_0} y_0 = 0$. Let for definiteness, $\sin \nu_{n_0} y_0 = 0$. Then for the function $f(x,y) = \sin \mu_m x \sin \nu_{n_0} y$ and any $h(t) \in AC[0,T]$ there exists a nonzero solution to the inverse problem 1 (while $\varphi(x,y) \equiv 0$), namely,

$$u(x,y,t) = \sin \mu_m x \sin \nu_{n_0} y \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-a^2 \lambda_{mn_0}^2 (t-\tau)^\alpha) g(\tau) d\tau.$$

There arise the question on the existence of roots of the equation

$$\sin \frac{\pi n_0}{q} y_0 = 0 \Leftrightarrow \frac{y_0}{q} = \frac{k}{n_0}, \quad k \in \mathbb{N},$$

i.e., when the ration takes on rational values, the uniqueness of the solution to inverse problem (with the mentioned choice of the function $f(x,y)$) is violated.

Therefore, if conditions of Lemma 6 are fulfilled, then the function $g(t)$ from the class $AC[0,T]$ is defined uniquely. Then fulfilled are all conditions of Theorem 3, in view of which the function $u(x,y,t)$ is defined by series (24) as a solution to the direct problem (3)-(6) in the functional class (36). The mentioned function always satisfies conditions (15) and (16).

5 Inverse Problem 2

In section five, we analyze the inverse problem of determining the time dependent source function.

Let $u(x,y,t)$ and $f(x,y)$ be a solution to problem (2-6), (9), (10). We look for these functions as sums of series

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \rho_{mn}(t) v_{mn}(x,y), \tag{48}$$

$$f(x,y)=\sum_{m,n=1}^{\infty} f_{mn}(t)v_{mn}(x,y), \tag{49}$$

As in section 3, we have an equation with respect to function $\rho_{mn}(t)$ defined by (18):

$$\partial_t^\alpha \rho_{mn}(t)+a^2\lambda_{mn}^2\rho_{mn}(t)=f_{mn}g(t). \tag{50}$$

In view of the lemma 3, its general solution with $m,n\in\mathbb{N}$ is given by the formula

$$\rho_{mn}(t)=d_{mn}E_\alpha\left(-(a\lambda_{mn})^2t^\alpha\right)+f_{mn}g_{mn}(t), \tag{51}$$

where d_{mn} are arbitrary constants and

$$g_{mn}(t)=\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-(a\lambda_{mn})^2(t-\tau)^\alpha\right)g(\tau)d\tau. \tag{52}$$

For finding coefficients d_{mn} and f_{mn}, we use boundary conditions (6) and (10) and formula (18):

$$\rho_{mn}(0)=\iint\limits_D \varphi(x,y)v_{mn}(x,y)dxdy=\varphi_{mn}, \tag{53}$$

$$\rho_{mn}(t_0)=\iint\limits_D \psi(x,y)v_{mn}(x,y)dxdy=\psi_{mn}. \tag{54}$$

As a result, by the boundary conditions (53) and (54), we find unknown constants

$$d_{mn}=\varphi_{mn},\quad f_{mn}=\frac{1}{g_{mn}(t_0)}\left[\psi_{mn}-\varphi_{mn}E_\alpha\left(-(a\lambda_{mn})^2t_0^\alpha\right)\right], \tag{55}$$

provided that

$$g_{mn}(t_0)\neq 0 \tag{56}$$

with all $m,n\in\mathbb{N}$. Substituting formula (55) in (51), we analytically construct functions

$$\rho_{mn}(t)=\left[E_\alpha\left(-(a\lambda_{mn})^2t^\alpha\right)-\frac{g_{mn}(t)}{g_{mn}(t_0)}E_\alpha\left(-(a\lambda_{mn})^2t_0^\alpha\right)\right]\varphi_{mn}+\frac{g_{mn}(t)}{g_{mn}(t_0)}\psi_{mn}. \tag{57}$$

Suppose now $\varphi(x,y)\equiv 0$ and $\psi(x,y)\equiv 0$ on $\overline{D}$. Then $\varphi_{mn}=\psi_{mn}=0$ and, according to relations (55) and (57), $\rho_{mn}(t)\equiv 0$ on the segment $[0,T]$ and $f_{mn}=0$ for all $m,n\in\mathbb{N}$. Therefore, taking into account formulas (57) and (41) it follows that

$$\iint\limits_D u(x,y,t)v_{mn}(x,y)dxdy=0,\quad \iint\limits_D f(x,y)v_{mn}(x,y)dxdy=0,\quad m,n\in\mathbb{N}.$$

Since the system $v_{mn}(x,y)$ is complete in the space $L^2[0,l]$, the letter equalities imply that $u(x,y,t)=0$, $f(x,y)=0$ almost everywhere on $\overline{D}$ with any $t\in[0,T]$. According to the condition (2) functions $u(x,y,t)$ and $f(x,y)$ are continuous, respectively, on $\overline{Q}$ and $\overline{D}$, the identity $u(x,y,t)\equiv 0$ takes place in $\overline{Q}$ and $f(x,y)\equiv 0$ on $\overline{D}$.

If with certain t_0 and $m=m_0$ or $n=n_0$ the expansion $g_{m_0n}(t_0)=0$ or $g_{mn_0}(t_0)=0$, then the homogeneous problem (3) (where $\varphi(x,y)=\psi(x,y)=0$), for example, with $g_{m_0n}(t_0)=0$, has a nonzero solution:

$$u(x,y,t)=f_{m_0n}g_{m_0n}(t)v_{mn}(x,y),\quad f(x,y)=f_{m_0n}v_{m_0n}(x,y),$$

where $f_{m_0n}\neq 0$ is an arbitrary constant.

There is also a question on the existence of zeros of the expression $g_{m_0n}(t_0)$. If the function $g(t)$ has a constant sing on $[0,t_0]$, then $g_{m_0n}(t_0)\neq 0$ with $m_0,n\in\mathbb{N}$; but if the function $g(t)$ changes its sing on $[0,t_0]$, i.e., it has zeros, for example, $g(t)=\sin(at+b)$, $a,b\in\mathbb{R}$, $a\neq 0$, then so does $g_{m_0n}(t_0)$.

Hence, we have established the uniqueness criterion for the solution to inverse problem 2.

Theorem 5. *If inverse problem 2 has a solution, then it is unique if and only if conditions (56) are true for all $m,n\in\mathbb{N}$.*

The following lemma help us to estimating the Fourier coefficients from above.

Lemma 7. *If $g(t) \in C[0,T]$ and $|g(t)| \geq g_0 = const > 0$, then there exists a constant C_0 such that for all $m,n \in \mathbb{N}$ the estimate*

$$|g_{mn}(t_0)| \geq \frac{C_0}{m^2+n^2}$$

is fulfilled.

Proof. In view of the mean value theorem, from formula (52) we deduce that

$$g_{mn}(t_0) = g(\xi)\int_0^{t_0}(t_0-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-(a\lambda_{mn})^2(t_0-\tau)^{\alpha}\right)d\tau$$

$$= g(\xi)t_0^{\alpha}E_{\alpha,\alpha+1}\left(-(a\lambda_{mn})^2 t_0^{\alpha}\right), \quad \xi \in [0,t_0].$$

According to the proposition 6, we have

$$1-E_{\alpha}\left(-(a\lambda_{mn})^2 t^{\alpha}\right) \geq C_0' > 0.$$

Taking into identity

$$t_0^{\alpha}E_{\alpha,\alpha+1}\left(-(a\lambda_{mn})^2 t_0^{\alpha}\right) = \frac{1}{a^2\lambda_{mn}^2}\left[1-E_{\alpha}\left(-(a\lambda_{mn})^2 t_0^{\alpha}\right)\right], \tag{58}$$

we obtain the lower estimate

$$|g_{mn}(t_0)| \geq \left(\frac{l_0}{\pi a}\right)^2 g_0\frac{\frac{C_0'}{2}}{m^2+n^2} \geq \frac{C_0}{m^2+n^2}, \quad l_0 = \min\{l,q\}.$$

Using the above result, we obtain the following assertion.

Lemma 8. *For any $m,n \in \mathbb{N}$ and $t \in [0,T]$, the following estimates take place:*

$$|f_{mn}| \leq C_{15}\lambda_{mn}^2\left(|\varphi_{mn}|+|\psi_{mn}|\right),$$

$$|\rho_{mn}(t)| \leq C_{16}\left(|\varphi_{mn}|+|\psi_{mn}|\right),$$

$$|\partial_t^{\alpha}\rho_{mn}(t)| \leq C_{17}\lambda_{mn}^2\left(|\varphi_{mn}|+|\psi_{mn}|\right).$$

Proof. We show that the first estimation of the lemma is true. Indeed, according to the (58) and proposition 6, and lemma 7, then the validity of the first estimate follows from formula (55).

The validity of the second estimate is shown by the above considerations, proposition 1 and from formula (57). The validity of the third estimate follows from (51) and preliminary two estimates of this lemma.

Lemma 9. *Let $\varphi(x,y)$ satisfy Lemma 5, while $\psi(x,y) \in C^4(\overline{D})$,*

$$\psi(0,y) = \psi_{xx}(0,y) = \psi(l,y) = \psi_{xx}(l,y), \quad 0 \leq y \leq q,$$

$$\psi(x,0) = \psi_{yy}(x,0) = \psi(x,q) = \psi_{yy}(x,q), \quad 0 \leq x \leq l.$$

Then we get the representation

$$|\psi_{mn}| = \frac{1}{[\mu_m+\nu_n]^4}\sum_{i+j=4}\binom{4}{j}|\psi_{mn}^{(i,j)}|,$$

where $\psi_{mn}^{(i,j)}$, $i+j=4$, are coefficients of the extension of functions $\frac{\partial^4\psi(x,y)}{\partial x^i\partial y^j}$ in the series with respect to the system of function (14) and their derivatives $\frac{\partial^4\psi(x,y)}{\partial x^i\partial y^j}$. In addition, the following estimate takes place:

$$\sum_{m,n=1}|\psi_{m,n}^{(i,j)}|^2 \leq \left\|\frac{\partial^4\psi(x,y)}{\partial x^i\partial y^j}\right\|_{L_2(D)}^2, \quad i+j=4.$$

Termwise differentiating the series in formula (48), we have

$$u_{xx}(x,y,t) = -\sum_{m,n=1}^{\infty}\mu_m^2\rho_{mn}(t)v_{mn}(x,y), \tag{59}$$

$$u_{yy}(x,y,t) = -\sum_{m,n=1}^{\infty} \nu_n^2 \rho_{mn}(t) v_{mn}(x,y), \tag{60}$$

$$\partial_t^{\alpha} u(x,y,t) = \sum_{m,n=1}^{\infty} \partial_t^{\alpha} \rho_{mn}(t) v_{mn}(x,y). \tag{61}$$

In view of lemmas 8 and 9, with any $(x,y,t) \in \overline{Q}$ series (48) and (59)-(61) are majorized by the convergent numerical series

$$C_{14} \sum_{m,n=1}^{\infty} \frac{1}{mn} \left(\sum_{i+j=4} |\varphi_{mn}^{(i,j)}| + \sum_{i+j=4} |\psi_{mn}^{(i,j)}| \right). \tag{62}$$

Hence it follows that series (48) and (59)-(61) converge absolutely and uniformly on $\overline{Q}$. Then functions $u(x,y,t)$ and $f(x,y)$ that are defined by formulas (48) and (49) belong to classes (36) and (9), respectively, and satisfy conditions (3), (15) and (16).

Furthermore, analogously, termwise differentiating series (48), we obtain, respectively series (59)-(61), where $\rho_{mn}(t)$ is determined by formula (57). These series, as well as that (49), are also majorized by the convergent numerical series (62).

In this way, we have the following final result.

Theorem 6. *Let functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy conditions of Lemma 9, at the same time, the function $g(t)$ is subject to the conditions of Lemma 7, then the inverse problem two has a unique solution, which is defined by series (48) and (49), whose coefficients are determined by formulas (55) and (57).*

6 Conclusion

In this paper, we investigate inverse one dimensional time-dependent and 2D spatial-dependent source problems for a two-dimensional fractional heat equation. The existence and uniqueness theorems for the solutions of the stated problems are proved. It should be noted that when $\alpha = 1$, the fractional derivative $\partial_t^{\alpha} u$ coincides with the usual derivative $\partial/\partial t$ and the results obtained will be true in this case as well. As still open problems, we indicate the case of equation (1) with $1 < \alpha \leq 2$ and the Laplacian of higher dimensions in space variables.

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