Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.12785/pfda/010107

Advanced Generalized Fractional Kinetic Equation in Astrophysics

Manoj Sharma[1,], Mohd. Farman Ali[2,*] and Renu Jain[2,*]*

[1] Department of Mathematics RJIT, BSF Academy, Tekanpur, India
[2] School of Mathematics and Allied Sciences, Jiwaji University, Gwalior, India

Received: 21 Nov. 2014, Revised: 27 Dec. 2014, Accepted: 29 Dec. 2014

Published online: 1 Jan. 2015

Abstract: In recent year's fractional kinetic equation are studied due to their usefulness and importance in mathematical physics, especially in astrophysical problems. The aim of present paper is to find the solution of generalized fractional order kinetic equation, using a new special function. The results obtained here is moderately universal in nature. Special cases, relating to the Mittag-Leffler function is also considered.

Keywords: Fractional kinetic equation, new special function, Riemann-Liouville operator, Laplace transform.

1 Introduction

We give the new special function, called $\mathcal{M}$ function, which is the most generalization of k_4 –function. Here, we give first the notation and the definition of the new special $\mathcal{M}$ function, introduced by the authors as follows:

$$ {}_{p}^{\alpha,\beta,\gamma,\delta,\rho}\mathcal{M}_{q}^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t) = \sum_{n=o}^{\infty} \frac{(a_1)_n \ \ldots \ (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \ldots k_p^n}{l_1^n \ldots l_q^n} \frac{(t-c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}. \quad (1) $$

There are p upper parameters $a_{1,}a_{2,} \ldots a_p$ and q lower parameters $b_{1,}\ b_{2,} \ldots b_q, \alpha,\ \beta,\ \gamma, \delta, \rho \ \epsilon\ C, Re(\alpha) > 0,\ Re(\beta) > 0,$ $Re(\gamma) > 0,\ \ Re(\delta) > 0,\ \ Re(\rho) > 0,\ \ Re(\alpha\gamma - \beta) > 0$ and $(a_j)_k\ (b_j)_k$ are pochammer symbols and $c > 0,\ \ k_1, \ldots k_p,$ $l_{1,} \ldots l_q$ are constants. The function (1) is defined when none of the denominator parameters $b_j s,\ j = 1,2, \ldots q$ is a negative integer or zero. If any parameter a_j is negative then the function (1) terminates into a polynomial in $(t-c)$.

2 Relationship of the ${}_{p}^{\alpha,\beta,\gamma,\delta,\rho}\mathcal{M}_{q}^{k_1,\ldots k_p,l_{1,}\ldots l_q;c}$ function and other special functions

In this section, we defined relationship of $\mathcal{M}$ function and various special functions.

(I). For $k_1 = a, k_2 \ldots\ k_p = 1, l_{1,} \ldots l_q = 1, \delta = 1$ and $\rho = 1$ K_4 –function is given by Sharma [13] ,

* *E-mail: manoj240674@yahoo.co.in, mohdfarmanali@gmail.com, renujain3@rediffmail.com*

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$$ {}^{\alpha,\beta,\gamma,1,1}_{\quad p}\mathcal{M}_q^{a,1;c}(t) = \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \left(a_p\right)_n (\gamma)_n}{(b_1)_n \dots \left(b_q\right)_n} \frac{a^n}{n!} \frac{(t-c)^{(n+\gamma)\alpha-\beta-1}}{\Gamma\left((n+\gamma)\alpha-\beta\right)}. \tag{2} $$

(II). If we take no upper and lower parameter ($p = q = 0$) in equation (2) then the function reduces to the G-Function, which was introduced by Lorenzo and Hartley [15] .

$$ {}^{\alpha,\beta,\gamma,1,1}_{\quad 1}\mathcal{M}_1^{a,1;c}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n (t-c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\left((n+\gamma)\alpha-\beta\right)} = G_{\alpha,\beta,\gamma}(a,c,t). \tag{3} $$

(III). Taking$\gamma = 1$, in equation (3), we get the R $-$function given by introduced by Lorenzo and Hartley [15] .

$$ {}^{\alpha,\beta,1,1,1}_{\quad 1}\mathcal{M}_1^{a,1;c}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)\alpha-\beta-1}}{n!\,\Gamma\left((n+1)\alpha-\beta\right)} = R_{\alpha,\beta}[a,t]\alpha > 0, \beta > 0, (\alpha-\beta) > 0 \tag{4} $$

Now, we take$c = 0$, in various standard functions.

(IV). For $c = 0$, in equation (3),the $\mathcal{M}$ function reduces to New Generalized Mittag-Leffler Function [12]

$$ {}^{\alpha,\beta,\gamma,1,1}_{\quad 1}\mathcal{M}_1^{a,1}(t) = t^{\alpha\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n (t)^{\alpha n}}{n!\,\Gamma\left((n+\gamma)\alpha-\beta\right)} = t^{\alpha\gamma-\beta-1} E_{\alpha,\alpha\gamma-\beta}^{\gamma}[at^{\alpha}]. \tag{5} $$

(V). We take $\gamma = 1$, in (5) obtained Generalized Mittag-Leffler Function [12], we get:

$$ {}^{\alpha,\beta,1,1,1}_{\quad 1}\mathcal{M}_1^{a,1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{(n+1)\alpha-\beta-1}}{\Gamma\left((n+1)\alpha-\beta\right)} = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}[at^{\alpha}]. \tag{6} $$

(VI). Further $\beta = \alpha - 1$in(6), this $\mathcal{M}$ function converts Mittag-Leffler Function [6,7], we have:

$$ {}^{\alpha,\alpha-1,1,1,1}_{\quad 1}\mathcal{M}_1^{a,1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n\alpha}}{\Gamma(n\alpha+1)} = E_{\alpha}[at^{\alpha}]. \tag{7} $$

(VII). When $a = 1, c = 0$ and $\beta = \alpha - \beta$ in (4) then the $\mathcal{M}$ function treats as Agarwal's Function [1]

$$ {}^{\alpha,\alpha-\beta,1,1,1}_{\quad 1}\mathcal{M}_1^{1,1}(t) = \sum_{n=0}^{\infty} \frac{(t)^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}[t^{\alpha}]. \tag{8} $$

(VIII). Rabotnov and Hartley function [15] is obtained from $\mathcal{M}$ function by putting $\beta = 0, a = -a,\ \ c = 0$ in(4), we have:

$$ {}^{\alpha,0,1,1,1}_{\quad 1}\mathcal{M}_1^{-a,1}(t) = \sum_{n=0}^{\infty} \frac{(-a)^n (t)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} = F_{\alpha}[-a,t]. \tag{9} $$

(IX). On substituting $\alpha = 1, \beta = -\beta$ in(4), we get Miller and Ross Function [5].

$$ {}^{1,-\beta,1,1,1}_{\quad 1}\mathcal{M}_1^{a,1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n+\beta}}{\Gamma(n+\beta+1)} = E_t[\beta,a]. \tag{10} $$

(X). Let us consider $c = 0$ in equation(3), this function converts into Wright Function [9]. We have:

$$ {}^{\alpha,\beta,\gamma,1,1}_{\quad 1}\mathcal{M}_1^{a,1}(t) = \frac{t^{\alpha\gamma-\beta-1}}{\Gamma\gamma} {}_1^0\psi_1 \left[\begin{matrix} (\gamma,1) \\ (\alpha\gamma-\beta),\alpha; at^{\alpha} \end{matrix} \right]. \tag{11} $$

Where ${}_1^0\psi_1(t)$ is special case of the wright's generalized Hypergeometric function${}_p^0\psi_q(t)$.

Or

(XI). Thus we get H-Function [9] from last case.

$$ {}_{1}^{\alpha,\beta,\gamma,1,1}\mathcal{M}_{1}^{a,1}(t)=\frac{t^{\alpha\delta-\beta-1}}{\Gamma\gamma}H_{1,2}^{1,1}\left[-at^{\alpha}\left|\begin{matrix}(1-\gamma,1)\\(0,1)(1-\alpha\gamma+\beta),\alpha\end{matrix}\right.\right]. \tag{12}$$

The Laplace transform of(1), from Lorenzo and Hartley [15] with shifting theorem (Wylie, p.281) we have

$$ L\left\{{}_{p}^{\alpha,\beta,\gamma,\delta,\rho}\mathcal{M}_{q}^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t)\right\}=\frac{(a_1)_n\ \ldots\ (a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{1}{l_1^n\ldots l_q^n}\frac{s^{\beta}e^{-cs}}{\{s^{\alpha}+(k_1^n\ldots k_p^n)\}^{\gamma}}. \tag{13}$$

3 Governing fractional kinetic equation

Let us define an arbitrary reaction which is dependent on time $N = N(t)$. It is possible to calculate rate of change dN/dt to a balance between the destruction rate d and the production rate p of N, then

$$\frac{dN}{dt}= -d+p.$$

The production or destruction at time t depends not only on $N(t)$ but also on the previous history$N(t_1)$, $t_1 < t$, of the variable N.

This was represented by Haubold and Mathai [9] as follows:

$$\frac{dN}{dt}= -d\ (Nt\ )+\ p(Nt), \tag{14}$$

where $N(t)$ denotes the function defined by $N_t(t_1) = N(t - t_1), t_1 > 0$.

Haubold and Mathai [2] considered a special case of this equation, when spatial fluctuations in homogeneities in quantity $N(t)$ are neglected. This is given by the equation:

$$\frac{dN_i}{dt}=-c_iN_i(t), \tag{15}$$

where the initial conditions are $N_i\ (t = 0) = N_0$, the number density of species i at time $t = 0$; constant c_i>0, is called standard kinetic equation and $c_i > 0$ is a constant.

The solution of the equation (15) is as follows:

$$N_i(t)\ = N_0e^{-c_it}, \tag{16}$$

or

$$N(t)-\ N_0\ =\ c\,{}_0D_t^{-1}\,N(t). \tag{17}$$

As D_t^{-1} is the integral operator, Haubold and Mathai [2] described the fractional generalization of the standard kinetic equation (15) as:

$$N(t)–\ N_0\ =\ c^{v}\,{}_0D_t^{-v}\,N(t), \tag{18}$$

where D_t^{-v} is the Riemann-Liouville fractional integral operator; Miller and Ross [5]) defined by:

$${}_0D_t^{-v}N(t)=\frac{1}{\Gamma(v)}\int_0^t(t\ -\ u)^{v-1}f(u)du,\quad R(v)>0. \tag{19}$$

The solution of the fractional kinetic equation (18) is given by (see Haubold and Mathai [2])

$$N(t)=N_0\sum_{k=0}^{\infty}\frac{(-1)^{vk}}{\Gamma(vk+1)}(ct)^{vk}. \tag{20}$$

Also, Saxena, Mathai and Haubold [12] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which is the extension of the work of Haubold and Mathai [2].

In the present work, we studied of the generalized fractional kinetic equation. The advanced generalized fractional kinetic equation and its solution, obtained in terms of the $\mathcal{M}$ −function.

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4 Advanced generalized fractional kinetic equations

In this section, we investigate the solution of advanced generalized fractional kinetic equation. The results are obtained in a compact form in terms of $\mathcal{M}-$ function. The result is presented in the form of a theorem as follows:

Theorem 1: If $b \geq 0,\ c > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0,\ \delta > 0,\ \rho > 0$ and $(\gamma\alpha - \beta) > 0$ then for the solution of the Advanced generalized fractional kinetic equation:

$$N(t) - N_0\, {}_p^{\alpha,\beta,\gamma,\delta,\rho}\mathcal{M}_q^{-c^\alpha, b_1,\ldots b_n; b}(t) = -\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha} D_t^{-r\alpha} N(t). \tag{21}$$

Then

$$N(t) = N_0\, {}_p^{\alpha,\beta,(\gamma+n),\delta,\rho}\mathcal{M}_q^{-c^\alpha, b_1,\ldots b_n; b}(t). \tag{22}$$

Proof. We have

$$\begin{aligned} N(t) - N_0 \sum_{n=o}^{\infty} \frac{(a_1)_n \ \ldots\ (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{(-c^\alpha)^n}{b_1^n \ldots b_q^n} \frac{(t-b)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \\ = -\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha} D_t^{-r\alpha} N(t). \end{aligned} \tag{23}$$

Taking the Laplace transforms of both the sides of equation (23), we get:

$$\begin{aligned} L\{N(t)\} - L\left\{N_0 \sum_{n=o}^{\infty} \frac{(a_1)_n \ \ldots\ (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{(-c^\alpha)^n}{b_1^n \ldots b_q^n} \frac{(t-b)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}\right\} \\ = -L\left\{\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha}\, {}_0D_t^{-r\alpha} N(t)\right\}. \end{aligned} \tag{24}$$

From [15] using shifting theorem for Laplace transform, we have,

$$N(s) - N_0 \frac{(a_1)_n \ \ldots\ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{1}{b_1^n \ldots b_q^n} \frac{s^\beta e^{-bs}}{(s^\alpha + c^\alpha)^\gamma} = -\left\{\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha} s^{-r\alpha} N(s)\right\}, \tag{25}$$

or

$$\begin{aligned} N(s) - N_0 \frac{(a_1)_n \ \ldots\ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{1}{b_1^n \ldots b_q^n} \frac{s^\beta e^{-bs}}{(s^\alpha + c^\alpha)^\gamma} \\ = -[{}_0^nc_1 c^\alpha s^{-\alpha} + {}_0^nc_2 c^{2\alpha} s^{-2\alpha} \ldots {}_0^nc_n c^{n\alpha} s^{-n\alpha}]N(s). \end{aligned} \tag{26}$$

$$N(s)(1 + c^\alpha s^{-\alpha})^n = N_0 \frac{(a_1)_n \ \ldots\ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{1}{b_1^n \ldots b_q^n} \frac{s^{\beta-\alpha\gamma} e^{-bs}}{(1 + c^\alpha s^{-\alpha})^\gamma}, \tag{27}$$

$$N(s) = N_0 \frac{(a_1)_n \ \ldots\ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{1}{b_1^n \ldots b_q^n} \frac{s^{\beta-\alpha\gamma} e^{-bs}}{(1 + c^\alpha s^{-\alpha})^{\gamma+n}}, \tag{28}$$

$$N(s) = N_0 \frac{(a_1)_n \ \ldots\ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{1}{b_1^n \ldots b_q^n} \frac{s^{\beta-\alpha(\gamma+n)+n\alpha} e^{-bs}}{(1 + c^\alpha s^{-\alpha})^{\gamma+n}}, \tag{29}$$

$$N(s) = N_0 \frac{(a_1)_n \ \ldots\ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{1}{b_1^n \ldots b_q^n} \sum_{n=o}^{\infty} \left(\frac{-c^\alpha}{s^\alpha}\right)^n \frac{(\gamma+n)_n s^{\beta-\alpha\gamma} e^{-bs}}{n!}. \tag{30}$$

Now, taking inverse Laplace transform, we get:

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$$N(t)=N_0\frac{(a_1)_n\ \cdots\ (a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{1}{b_1^n\ldots b_q^n}\sum_{n=o}^{\infty}(-c^\alpha)^n\frac{(\gamma+n)_n}{n!}L^{-1}\{s^{\beta-\alpha\gamma-\alpha n}e^{-bs}\}, \tag{31}$$

$$N(t)=N_0\sum_{n=o}^{\infty}\frac{(a_1)_n\ \cdots\ (a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{(-c^\alpha)^n}{b_1^n\ldots b_q^n}\frac{(\gamma+n)_n}{n!}\frac{(t-b)^{\alpha\gamma+\alpha n-\beta-1}}{\Gamma((\gamma+n)\alpha-\beta)}, \tag{32}$$

$$N(t)=N_0\,{}^{\alpha,\beta,(\gamma+n),\delta,\rho}{}_p\mathcal{M}_q^{-c^\alpha,b_{1,\ldots}b_n;b}(t). \tag{33}$$

This is the complete proof of the theorem.

5 Special cases

Corollary: 1.

If we take $(a_1)_n\ \ldots\ (a_p)_n=1=(b_1)_n\ldots(b_q)_n,\delta=1,\rho=1$and $b_1^n\ldots b_q^n=1$ then for the solution of the advanced generalized fractional kinetic equation,

$$N(t)-\ N_0\,{}^{\alpha,\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^\alpha,1;b}(t)=-\sum_{r=1}^{n}\binom{n}{r}c^{r\alpha}D_t^{-r\alpha}\,N(t). \tag{34}$$

There holds the result

$$N(t)=\ N_0\,{}^{\alpha,\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^\alpha,1;b}(t). \tag{35}$$

In view of the relation (33), this result coincides with the main result of Chaurasia and Pandey [16].

Corollary: 2.

If we put $b=0$ in corollary (1) then the solution of the Advanced generalized fractional kinetic equation reduces to the special case of theorem (1) in Chaurasia and Pandey [16] , given as follows:

For the solution of

$$N(t)-\ N_0\,{}^{\alpha,\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^\alpha,1;0}(t)=-\sum_{r=1}^{n}\binom{n}{r}c^{r\alpha}D_t^{-r\alpha}\,N(t). \tag{36}$$

There holds the result

$$N(t)=\ N_0\,{}^{\alpha,\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^\alpha,1;0}(t). \tag{37}$$

Corollary: 3.

If we put $\beta=\gamma\alpha-\beta$ in corollary (1) then the solution of the Advanced generalized fractional kinetic equation reduces to the special case of theorem (1) in Chaurasia and Pandey [16], which is given as follows:

For the solution of

$$N(t)-\ N_0\,{}^{\alpha,\gamma\alpha-\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^\alpha,1;b}(t)=-\sum_{r=1}^{n}\binom{n}{r}c^{r\alpha}D_t^{-r\alpha}\,N(t). \tag{38}$$

There holds the formula

$$N(t)=\ N_0\,{}^{\alpha,\gamma\alpha-\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^\alpha,1;b}(t). \tag{39}$$

Corollary: 4.

If we put $b=0$ in corollary (3) then the solution of the Advanced generalized fractional kinetic equation reduces to another special case of theorem (1) in Chaurasia and Pandey [16], which is given as follows:

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For the solution of

$$N(t)- N_0{}^{\alpha,\gamma\alpha-\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^{\alpha},1;0}(t) = -\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha} D_t^{-r\alpha} N(t). \tag{40}$$

There holds the formula

$$N(t) = N_0{}^{\alpha,\gamma\alpha-\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^{\alpha},1;0}(t). \tag{41}$$

This completes the analysis.

6 Conclusions

In this present work, we have introduced a fractional generalization of the standard kinetic equation and a new special function given by authors and also established the solution for the Advanced generalized fractional kinetic equation. The results of the Advanced generalized fractional kinetic equation and its special cease are same as the results of Chaurasia and Panday [16]. And also, the relations function to the various standard functions is discussed in this paper.

7 Acknowledgement

Authors are grateful to referee for his valuable comment and improvement upon the paper.

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