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Exponential Chain Ratio Cum Dual to Ratio Estimator of Finite Population Mean under Double Sampling Scheme

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Abstract: In this paper an exponential chain ratio cum dual to ratio estimator has been considered for estimating population mean of the study variate using two auxiliary variables under double sampling procedure, when the information on another additional auxiliary variate is available along with the main auxiliary variate. The asymptotically optimum estimators (AOE) are identified in two different cases with their biases and variances. The optimum values of the first and second phase sample sizes have been obtained for the fixed cost of survey. Theoretical and empirical studies have also been done to demonstrate the efficiency of the proposed estimator with respect to strategies which utilized the information on two auxiliary variables..

Keywords: Exponential, Chain, Dual to Ratio, Asymptotically optimum estimators, Bias, Variances.

1 Introduction

The use of auxiliary variable in the estimation of population mean of the study variate has been a common phenomenon in sampling theory of surveys. Auxiliary information may be fruitfully utilized either at planning stage or at design stage or at the information stage to arrive at improved estimator compared to those, not utilizing auxiliary information. The ratio method introduced by [6] has been widely used when the correlation between the character under study y and the auxiliary character x is positive. If this correlation is negative, a product estimator envisaged by [15] and [11] may be used instead of a ratio estimator. The use of ratio and product strategies in survey sampling solely depends upon the knowledge of population mean $\bar{X}$ of the auxiliary character x. In many situations of practical importance, the population mean $\bar{X}$ is not known before the start of a survey. In such a situation, the usual thing to do is to estimate it by the sample mean $\bar{x_1}$ based on a preliminary sample of size n_1 of which n is a subsample $(n < n_1)$. If the population mean $\bar{Z}$ of another auxiliary variate z, closely related to study variate y is known, it is advisable to estimate $\bar{X}$ by $\bar{X} = \bar{x_1}\bar{Z}/\bar{z_1}$, which would provide better estimate of $\bar{X}$ than $\bar{x_1}$ to the terms of order $O(n^{-1})$ if $\rho_{xz}C_x/C_z > 1/2$.

[5] and [27] proposed a technique of chaining the available information on auxiliary characteristics with the main characteristic. [9], [10], [23] also proposed some chain type ratio and regression estimators based on two auxiliary variables. Using proper information on parameters of auxiliary variate, [24], [25], [8], [20], [13], [14], [2] defined two classes of estimators of S_y^2 by using prior information on parameters of one of the two auxiliary variables under double sampling scheme. [1] gave some chain ratio-type as well as chain product type estimators of S_y^2 under two-phase sampling scheme. [16] worked on ratio cum dual to ratio estimator. [23] proposed a chain ratio and regression type estimators for median estimation. Using known coefficient of kurtosis of second auxiliary variable in double sampling, [21] defined a chain-type estimator of population variance.

Consider a finite population $U = (U_1, U_2, ..., U_N)$ of N units, y be the study variate, x and z are two auxiliary variates. Let $\bar{X}$ is not known, but $\bar{Z}$, the population mean of another cheaper auxiliary variate z is closely related to x but compared to x remotely relatede to y $(i.e.\rho_{yx} > \rho_{yz})$ is available. In this case, [5] defined the chain ratio estimator $\bar{y}_R^{dc} = \bar{y}\frac{\bar{x_1}}{\bar{x}}\frac{\bar{Z}}{\bar{z_1}}$, where $\bar{x}$ and $\bar{y}$ are the sample means of x and y respectively based on the sample size n out of the population N units and

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$\bar{x}_1 = (1/n_1)\sum_{i=1}^{n_1} x_i$ and $\bar{z}_1 = (1/n_1)\sum_{i=1}^{n_1} z_i$ denote the sample means based on $n_1 > n$ units of the auxiliary variates x and z.

Using the transformation $\bar{x}_i^{\sigma} = (N\bar{X} - nx_i)/(N-n)$, $i = 1,2,3,...,N$, [26] obtained dual to ratio estimator as

$$\bar{y}_R = \bar{y}\frac{\bar{x^{\sigma}}}{\bar{X}}$$

where $\bar{x^{\sigma}} = (N\bar{X} - n\bar{x})/(N-n)$.

[3] suggested the exponential ratio type estimator

$$\hat{\bar{Y}}_{Re} = \bar{y}exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$$

and the exponential product type estimator

$$\hat{\bar{Y}}_{Pe} = \bar{y}exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$$

for the population mean $\bar{Y}$

[22] suggested the modified exponential ratio and product estimators for $\bar{Y}$ in double sampling respectively as

$$\hat{\bar{Y}}_{ReMd} = \bar{y}exp\left(\frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}}\right)$$

$$\hat{\bar{Y}}_{PeMd} = \bar{y}exp\left(\frac{\bar{x} - \bar{x}'}{\bar{x} + \bar{x}'}\right)$$

[19] suggested exponential ratio cum dual to ratio estimator in double sampling as

$$t = \bar{y}\left\{\alpha exp\left(\frac{\bar{x}_1 - \bar{x}}{\bar{x}_1 + \bar{x}}\right) + \beta exp\left(\frac{\bar{x}^{*d} - \bar{x}_1}{\bar{x}^{*d} + \bar{x}_1}\right)\right\}$$

where $\bar{x}^{*d} = (1+g')\bar{x}_1 - g'\bar{x}$ and $g' = \frac{n}{n_1 - n}$

[17] suggested exponential chain ratio and product estimators under double sampling scheme as

$$\hat{\bar{Y}}_{Re}^{dc} = \bar{y}exp\left(\frac{\bar{x}_1\frac{\bar{Z}}{\bar{z}_1} - \bar{x}}{\bar{x}_1\frac{\bar{Z}}{\bar{z}_1} + \bar{x}}\right)$$

$$\hat{\bar{Y}}_{Pe}^{dc} = \bar{y}exp\left(\frac{\bar{x} - \bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}{\bar{x} + \bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}\right)$$

[18] again utililized the above estimators to a class of exponential chain ratio-product type estimator in double sampling scheme as

$$\hat{\bar{Y}}_{RPe}^{dc} = \bar{y}\left[\alpha exp\left(\frac{\bar{x}_1\frac{\bar{Z}}{\bar{z}_1} - \bar{x}}{\bar{x}_1\frac{\bar{Z}}{\bar{z}_1} + \bar{x}}\right) + \beta exp\left(\frac{\bar{x} - \bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}{\bar{x} + \bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}\right)\right]$$

where α and β are unknown constants such that $\alpha + \beta = 1$

Motivated by [19] and [18], we have proposed an exponential chain ratio cum dual to ratio estimator in double sampling for estimating finite population mean $\bar{Y}$ using two auxiliary characters. The properties of the proposed estimator are studied in two cases. Numerical illustrations are also shown in support of the present study.

2 The Proposed Estimator

In the use of two auxiliary variables x and z, we consider the population mean $\overline{X}$ as unknown and the population mean $\overline{Z}$ of second auxiliary variable z which has a positive correlation with x(i.e. $\rho_{xz} > 0$) as known. Further we assume that $\rho_{yx} > \rho_{yz} > 0$. Let $\overline{x_1}$ and $\overline{z_1}$ be the sample means of x and z respectively based on a priliminary sample size n_1 drawn with simple random sampling without replacement(SRSWOR) strategy in order to get an estimate of $\overline{X}$. Then the proposed estimator for estimating $\overline{Y}$ is

$$\hat{\overline{Y}}_{EC}^{RdR} = \overline{y}\left[\alpha I_1 + (1-\alpha) I_2\right] \tag{1}$$

where α is unknown constant to be determined,

$$I_1 = \hat{\overline{Y}}_{Re}^{dc} = exp\left(\frac{\frac{\overline{x_1}}{\overline{z_1}}\overline{Z} - \overline{x}}{\frac{\overline{x_1}}{\overline{z_1}}\overline{Z} + \overline{x}}\right)$$

and

$$I_2 = \hat{\overline{Y}}_{EdR}^{dc} = exp\left(\frac{\frac{N\frac{\overline{x_1}}{\overline{z_1}}\overline{Z} - n\overline{x}}{N-n} - \frac{\overline{x_1}}{\overline{z_1}}\overline{Z}}{\frac{N\frac{\overline{x_1}}{\overline{z_1}}\overline{Z} - n\overline{x}}{N-n} + \frac{\overline{x_1}}{\overline{z_1}}\overline{Z}}\right)$$

To obtain the bias and MSE of $\hat{\overline{Y}}_{EC}^{RdR}$, we write

$$e_0 = \left(\overline{y} - \overline{Y}\right)/\overline{Y}, \quad e_1 = \left(\overline{x} - \overline{X}\right)/\overline{X}, \quad e_2 = \left(\overline{x_1} - \overline{X}\right)/\overline{X} \text{ and } e_3 = \left(\overline{z_1} - \overline{Z}\right)/\overline{Z}$$

Expressing $\hat{\overline{Y}}_{EC}^{RdR}$ in terms of e's, we have

$$\begin{aligned}
\hat{\overline{Y}}_{EC}^{RdR} =& \overline{y}\left[\alpha\left(1+w_1\right) + \left(1-\alpha\right)\left(1+w_2\right)\right] \\
=& \overline{Y}\left(1+e_0\right)\left[1+w_2+\alpha\left(w_1-w_2\right)\right] \\
=& \overline{Y}\left[1+e_0+W_1+\alpha W_2\right] \\
\hat{\overline{Y}}_{EC}^{RdR} - \overline{Y} =& \overline{Y}\left[e_0+W_1+\alpha W_2\right]
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
w_1 =& \frac{1}{8}\left(4e_2 - 4e_1 - 4e_3 + 3e_1^2 - e_2^2 + 3e_3^2 - 2e_1e_2 + e_1e_3 - 2e_2e_3\right) \\
w_2 =& \frac{1}{8}\left(4ge_2 - 4ge_3 - 4ge_1 - g^2e_1^2 - g\left(4+g\right)e_2^2 - g^2e_3^2 - 2g\left(2+g\right)e_1e_3 + \right. \\
& \left. 2g\left(2+g\right)e_1e_2 + 2g\left(2+g\right)e_2e_3\right) \\
W_1 =& \frac{1}{8}\left(4ge_2 - 4ge_3 - 4ge_1 - g^2e_1^2 - g\left(4+g\right)e_2^2 - g^2e_3^2 - 2g\left(2+g\right)e_1e_3 + \right. \\
& \left. 2g\left(2+g\right)e_1e_2 + 2g\left(2+g\right)e_2e_3 + 4ge_0e_2 - 4ge_0e_3 - 4ge_0e_1\right) \\
W_2 =& \frac{1}{8}\left(4\left(1-g\right)e_2 - 4\left(1-g\right)e_3 - 4\left(1-g\right)e_1 + \left(3+g^2\right)e_1^2 - \left(1-4g-g^2\right)e_2^2 + \right. \\
& \left(3+g^2\right)e_3^2 - \left(2+4g+2g^2\right)e_1e_2 + \left(2+4g+2g^2\right)e_1e_3 - \left(2+4g+2g^2\right)e_2e_3 + \\
& \left. 4\left(1-g\right)e_0e_2 - 4\left(1-g\right)e_0e_3 - 4\left(1-g\right)e_0e_1\right) \\
g =& \frac{n}{N-n}
\end{aligned}$$

To find the bias and MSE of $\hat{\overline{Y}}_{EC}^{RdR}$, the following notations are used

$$C_y^2 = S_y^2/\overline{Y}^2, \quad C_x^2 = S_x^2/\overline{X}^2, \quad C_z^2 = S_z^2/\overline{Z}^2, \quad \rho_{yx} = S_{yx}/S_yS_x, \quad \rho_{yz} = S_{yz}/S_yS_z,$$
$$\rho_{zx} = S_{xz}/S_zS_x$$

where

$$S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{X})^2,\quad S_y^2=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2,\quad S_z^2=\frac{1}{N-1}\sum_{i=1}^{N}(z_i-\bar{Z})^2$$

$$S_{yx}=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})(x_i-\bar{X}),\quad S_{yz}=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})(z_i-\bar{Z})\text{ and}$$

$$S_{xz}=\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{X})(z_i-\bar{Z})$$

The following two cases will be considered separately:
Case I: When the second phase sample of size n is a subsample of the first phase of size n_1
Case II: When the second phase sample of size n is drawn independently of the first phase sample of size n_1
The case where the second sample is drawn independently of the first was considered by [4]

3 Case I

3.1 Bias, MSE and Optimum α of $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_I$

In case I, we have

$$\begin{aligned}
&E(e_0)=E(e_1)=E(e_2)=E(e_3)=0,\quad E\left(e_0^2\right)=\frac{1-f}{n}C_y^2,\quad E\left(e_1^2\right)=\frac{1-f}{n}C_x^2\\
&E\left(e_2^2\right)=\frac{1-f_1}{n_1}C_x^2,\quad E\left(e_3^2\right)=\frac{1-f_1}{n_1}C_z^2,\quad E(e_0e_1)=\frac{1-f}{n}K_{yx}C_x^2\\
&E(e_0e_2)=\frac{1-f_1}{n_1}K_{yx}C_x^2,\quad E(e_0e_3)=\frac{1-f_1}{n_1}K_{yz}C_z^2,\quad E(e_1e_2)=\frac{1-f_1}{n_1}C_x^2\\
&E(e_1e_3)=\frac{1-f_1}{n_1}K_{xz}C_z^2,\quad E(e_2e_3)=\frac{1-f_1}{n_1}K_{xz}C_z^2
\end{aligned}\tag{3}$$

where $f=\frac{n}{N}$, $f_1=\frac{n_1}{N}$, $K_{yx}=\frac{\rho_{yx}C_y}{C_x}$, $K_{yz}=\frac{\rho_{yz}C_y}{C_z}$ and $K_{xz}=\frac{\rho_{xz}C_x}{C_Z}$

Taking expectations in (2) and using results from (3), we get the bias of the estimator $\hat{\bar{Y}}_{EC}^{RdR}$ to the first order of approximation as

$$B\left(\hat{\bar{Y}}_{EC}^{RdR}\right)_I=\bar{Y}\frac{1}{8}\left[M_1-M_2+\alpha(M_3+M_4)\right]\tag{4}$$

where

$$\begin{aligned}
M_1&=\frac{1-f^*}{n}g^2C_x^2\left(1+\frac{4}{g}K_{yx}\right)\\
M_2&=\frac{1-f_1}{n_1}g^2C_x^2\left(1+\frac{4}{g}K_{yz}\right)\\
M_3&=\frac{1-f^*}{n}C_x^2\left\{3+g^2-4(1-g)K_{yx}\right\}\\
M_4&=\frac{1-f_1}{n_1}C_z^2\left\{\left(3+g^2\right)-4(1-g)K_{yz}\right\}
\end{aligned}$$

where $f^*=\frac{n}{n_1}$.

Again from (2), we have

$$\hat{\bar{Y}}_{EC}^{RdR}-\bar{Y}=\bar{Y}\left[e_0+\frac{g}{2}-\frac{g}{2}e_3-\frac{g}{2}e_1+\alpha\left(\frac{1-g}{2}e_2-\frac{1-g}{2}e_3-\frac{1-g}{2}e_1\right)\right]\tag{5}$$

Squaring both the sides in (5), taking expectations and using the results from (3), we obtain the MSE of the estimator $\hat{\bar{Y}}_{EC}^{RdR}$ to the first order of approximations, as

$$MSE\left(\hat{\bar{Y}}_{EC}^{RdR}\right)_I = \bar{Y}^2\left[\frac{(1-f)}{n}C_y^2 + N_1 + N_2 + (1-g)\,\alpha\,(N_3+N_4) + \frac{(1-g)^2\,\alpha^2}{4}N_5\right] \tag{6}$$

where

$$\begin{aligned}
N_1 &= \frac{(1-f^*)}{n}\frac{g^2}{4}C_x^2\left(1-\frac{g}{4}K_{yx}\right)\\
N_2 &= \frac{(1-f_1)}{n_1}\frac{g^2}{4}C_z^2\left(1-\frac{g}{4}K_{yz}\right)\\
N_3 &= \frac{(1-f^*)}{n}\frac{g}{2}C_x^2\left(1-\frac{2}{g}K_{yx}\right)\\
N_4 &= \frac{(1-f_1)}{n_1}\frac{g}{2}C_z^2\left(1-\frac{2}{g}K_{yz}\right)\\
N_5 &= \frac{(1-f^*)}{n}C_x^2 + \frac{(1-f_1)}{n_1}C_z^2
\end{aligned}$$

The MSE of $\hat{\bar{Y}}_{EC}^{RdR}$ is minimum when

$$\alpha = -\frac{2}{(1-g)}\frac{N_3+N_4}{N_5} = \alpha_{opt} \quad (say) \tag{7}$$

Substituting the the value of (7) in (1) yields the 'asymptotically optimum estimator' (AOE) as

$$\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)} = \bar{Y}\left[\alpha_{I(opt)}I_1 + \left(1-\alpha_{I(opt)}\right)I_2\right]$$

Thus, the resulting MSE of $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)}$ is given by

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)} = \bar{Y}^2\left[\frac{(1-f)}{n}C_y^2 + N_1 + N_2 - \frac{(N_3+N_4)^2}{N_5}\right] \tag{8}$$

Remarks:

1. For $\alpha = 1$ the estimator $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}$ in (1) boils down to the exponential chain ratio estimator $\hat{\bar{Y}}_{Re}^{dc}$ suggested by [17] in double sampling. The bias and MSE of $\hat{\bar{Y}}_{Re}^{dc}$ can be obtained by putting $\alpha = 1$ in (4) and (6) respectively as

$$B\left\{\hat{\bar{Y}}_{Re}^{dc}\right\}_I = \bar{Y}\left[\left(\frac{1-f^*}{n}\right)C_x^2\left(\frac{3}{8}+\frac{1}{2}K_{yx}\right) + \left(\frac{1-f_1}{n_1}\right)C_z^2\left(\frac{3}{8}-K_{yx}\right)\right]$$

and

$$\begin{aligned}
MSE\left\{\hat{\bar{Y}}_{Re}^{dc}\right\}_I = \bar{Y}^2\Bigg[\left(\frac{1-f}{n}\right)C_y^2 &+ \left(\frac{1-f^*}{n}\right)\frac{C_x^2}{4}(1-4K_{yx})\\
&+ \left(\frac{1-f_1}{n_1}\right)\frac{C_z^2}{4}(1-4K_{yx})\Bigg]
\end{aligned} \tag{9}$$

2. For $\alpha = 0$, the estimator $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}$ in (1) reduces to the exponential chain dual to ratio estimator $\hat{\bar{Y}}_{EdR}^{dc}$ in double sampling. The bias and MSE of $\hat{\bar{Y}}_{EdR}^{dc}$ can be obtained by puting $\alpha = 0$ in (4) and (6) respectively as

$$B\left(\hat{\bar{Y}}_{EdR}^{dc}\right)_I = -\frac{\bar{Y}}{8}[B_1+B_2]$$

where

$$B_1 = \left(\frac{1-f^*}{n}\right) g^2 C_x^2 \left(1+\frac{4}{g}K_{yx}\right),$$
$$B_2 = \left(\frac{1-f_1}{n_1}\right) g^2 C_z^2 \left(1+\frac{4}{g}K_{yz}\right)$$

and

$$MSE\left(\hat{\bar{Y}}_{EdR}^{dc}\right)_I = \bar{Y}^2\left[\left(\frac{1-f}{n}\right)C_y^2 + N_1 + N_2\right] \tag{10}$$

4 Efficiency Comparisons in Case I

4.1 Comparison with sample mean per unit estimator $\bar{y}$

The variance of usual unbiased estimator $\bar{y}$ is given by

$$V(\bar{y}) = \left(\frac{1-f}{n}\right)C_y^2 \tag{11}$$

From (11) and (8), we have

$$V(\bar{y}) - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)} = -\bar{Y}^2\left[N_1+N_2-\frac{(N_3+N_4)^2}{N_5}\right] > 0 \tag{12}$$

if $N_1+N_2 < \frac{(N_3+N_4)^2}{N_5}$

4.2 Comparison with chain ratio estimator $\hat{\bar{Y}}_R^{dc}$

The MSE of chain ratio estimator $\hat{\bar{Y}}_R^{dc}$ suggested by Chand (1975) in double sampling is given by

$$MSE\left(\hat{\bar{Y}}_R^{dc}\right)_I = \bar{Y}^2\left[\left(\frac{1-f}{n}\right)C_y^2 + \left(\frac{1-f^*}{n}\right)C_x^2(1-2K_{yx}) + \left(\frac{1-f_1}{n_1}\right)C_z^2(1-2K_{yz})\right] \tag{13}$$

From (13) and (8), we have

$$MSE\left(\hat{\bar{Y}}_R^{dc}\right)_I - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)} = \bar{Y}^2\left[K_1+K_2-\frac{(N_3+N_4)^2}{N_5}\right] > 0 \tag{14}$$

if $K_1+K_2 > \frac{(N_3+N_4)^2}{N_5}$
where

$$K_1 = \left(\frac{1-f^*}{n}\right)\left\{C_x^2\left(1-\frac{g^2}{4}\right) - (2-g)K_{yx}C_x^2\right\}$$
$$K_2 = \left(\frac{1-f_1}{n_1}\right)\left\{C_z^2\left(1-\frac{g^2}{4}\right) - (2-g)K_{yz}C_z^2\right\}$$

4.3 Comparison with chain linear regression estimator $\hat{\bar{Y}}^{dc}_{reg}$

The MSE of chain regression estimator $\hat{\bar{Y}}^{dc}_{reg} = \bar{y} + b_{yx}\left[\bar{x}_1 + b_{xz}\left(\bar{Z} - \bar{z}_1\right) - \bar{x}\right]$,
where b_{yx} and b_{xz} are the regression coefficients of y on x and x on z respectively suggested by [10] is given by

$$MSE\left(\hat{\bar{Y}}^{dc}_{reg}\right)_I = \bar{Y}^2\left[\left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right) + \left(\frac{1-f_1}{n_1}\right)\left(K_{yx}^2 C_x^2 + K_{yx}K_{xz}C_z^2\left(K_{yx}K_{xz} - 2K_{yz}\right)\right)\right] \tag{15}$$

From (15) and (8), we have

$$MSE\left(\hat{\bar{Y}}^{dc}_{reg}\right)_I - MSE\left\{\hat{\bar{Y}}^{RdR}_{EC}\right\}_{I(opt)} = \bar{Y}^2\left[K_3 - K_4 + N_2 + \frac{\left(N_3 + N_4\right)^2}{N_5}\right] > 0 \tag{16}$$

if $K_3 + N_2 + \frac{\left(N_3+N_4\right)^2}{N_5} > K_4$
where

$$K_3 = \left(\frac{1-f_1}{n_1}\right)\left\{K_{yx}K_{xz}C_z^2\left(K_{yx}K_{xz} - 2K_{yz}\right)\right\}$$
$$K_4 = \left(\frac{1-f^*}{n_1}\right)\left(\frac{g^2}{C_x^2} - g\rho_{yx}C_yC_x + K_{yx}C_x^2\right)$$

4.4 Comparison with exponential chain ratio estimator$\hat{\bar{Y}}^{dc}_{Re}$

From (9) and (8), we have

$$MSE\left\{\hat{\bar{Y}}^{dc}_{Re}\right\}_I - MSE\left\{\hat{\bar{Y}}^{RdR}_{EC}\right\}_{I(opt)} = \bar{Y}^2\left[K_5 + K_6 + \frac{\left(N_3 + N_4\right)}{N_5}\right] > 0 \tag{17}$$

if $K_5, K_6, N_5 > 0$ where

$$K_5 = \left(\frac{1-f^*}{n}\right)\left\{\frac{\left(g^2-1\right)}{4}C_x^2 - (g-1)\rho_{yx}C_yC_x\right\}$$
$$K_6 = \left(\frac{1-f_1}{n_1}\right)\left\{\frac{\left(g^2-1\right)}{4}C_z^2 - (g-1)\rho_{yz}C_yC_z\right\}$$

4.5 Comparison with chain dual to ratio estimator $\hat{\bar{Y}}^{dc}_{dR}$

The MSE of chain dual to ratio estimator $\hat{\bar{Y}}^{dc}_{dR}$ is given by

$$MSE\left(\hat{\bar{Y}}^{dc}_{dR}\right)_I = \bar{Y}^2\left[\left(\frac{1-f}{n}\right)C_y^2 + L_1 + L_2 + \frac{\left(N_3 + N_4\right)^2}{N_5}\right] \tag{18}$$

where $L_1 = \left(\frac{1-f^*}{n}\right)g^2C_x^2\left(1 - \frac{2}{g}K_{yx}\right)$, $\quad L_2 = \left(\frac{1-f_1}{n_1}\right)g^2C_z^2\left(1 - \frac{2}{g}K_{yz}\right)$

From (18) and (8), we have

$$MSE\left(\hat{\bar{Y}}^{dc}_{dR}\right)_I - MSE\left\{\hat{\bar{Y}}^{RdR}_{EC}\right\}_{I(opt)} = \bar{Y}^2\left[L_3 + L_4 + \frac{\left(N_3 + N_4\right)^2}{N_5}\right] > 0 \tag{19}$$

if $L_3, L_4, N_5 > 0$,
where

$$L_3 = \left(\frac{1-f^*}{n}\right)\left(\frac{3}{4}C_x^2 - g\rho_{yx}C_yC_x\right)$$
$$L_4 = \left(\frac{1-f_1}{n_1}\right)\left(\frac{3}{4}g^2C_x^2 - g\rho_{yz}C_yC_z\right)$$

4.6 Comparison with exponential chain dual to ratio estimator $\hat{\bar{Y}}_{EdR}^{dc}$

From (10) and (8), we have

$$MSE\left(\hat{\bar{Y}}_{EdR}^{dc}\right)_I - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)} = \bar{Y}^2 \frac{(N_3+N_4)^2}{N_5} > 0 \tag{20}$$

if $N_5 > 0$

Now, we state the following theorem

5 Theorem1

To the first degree of approximation, the proposed strategy under the optimality condition (7), is always more efficient than $V(\bar{y})$, $MSE\left(\hat{\bar{Y}}_R^{dc}\right)_I$, $MSE\left(\hat{\bar{Y}}_{reg}^{dc}\right)_I$, $MSE\left\{\hat{\bar{Y}}_{Re}^{dc}\right\}_I$, $MSE\left(\hat{\bar{Y}}_{dR}^{dc}\right)_I$ and $MSE\left(\hat{\bar{Y}}_{EdR}^{dc}\right)_I$ under the conditions $N_1+N_2 < \frac{(N_3+N_4)^2}{N_5}$; $K_1+K_2 > \frac{(N_3+N_4)^2}{N_5}$;, $K_3+N_2+\frac{(N_3+N_4)^2}{N_5} > K_4$; $K_5, K_6, N_5 > 0$; $L_3, L_4, N_5 > 0$; and $N_5 > 0$.

6 Case II

6.1 Bias, MSE and Optimum α of $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II}$

In this case II, we have

$$\begin{aligned}
&E(e_0) = E(e_1) = E(e_2) = E(e_3) = 0, \quad E\left(e_0^2\right) = \frac{1-f}{n}C_y^2, \\
&E\left(e_1^2\right) = \frac{1-f}{n}C_x^2, \quad E\left(e_2^2\right) = \frac{1-f_1}{n_1}C_x^2, \quad E\left(e_3^3\right) = \frac{1-f_1}{n_1}C_z^2, \\
&E(e_0e_1) = \frac{1-f}{n}K_{yx}C_x^2, \quad E(e_2e_3) = \frac{1-f_1}{n_1}K_{xz}C_z^2, \\
&E(e_0e_2) = E(e_0e_3) = E(e_1e_2) = E(e_1e_3) = 0
\end{aligned} \tag{21}$$

Taking expectations in (2) and using the results of (21), we get the bias of $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II}$ upto the first order of approximations as

$$B\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II} = \frac{\bar{Y}}{8}\left[-M_1^{'} + M_2^{'} + \alpha\left(M_3^{'} + M_4^{'}\right)\right] \tag{22}$$

where

$$\begin{aligned}
M_1^{'} &= \left(\frac{1-f}{n}\right) g^2 C_x^2 \left(1 + \frac{4}{g}K_{yx}\right) \\
M_2^{'} &= \left(\frac{1-f_1}{n_1}\right)\left\{g(4+g)C_x^2 + g^2C_z^2 - 2g(2+g)K_{xz}C_z^2\right\} \\
M_3^{'} &= \left(\frac{1-f}{n}\right)\left\{\left(3+g^2\right)C_x^2 - 4(1-g)K_{yx}C_x^2\right\} \\
M_4^{'} &= \left(\frac{1-f_1}{n_1}\right)\left\{\left(3+g^2\right)C_z^2 - \left(1-4g-g^2\right)C_x^2 - \left(2+4g+2g^2\right)K_{xz}C_z^2\right\}
\end{aligned}$$

Squaring both the sides in (2), taking the expectations and using the results of (21), we obtain the MSE of the estimator $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II}$ to the first order of approximations as

$$\begin{aligned}
MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II} = \bar{Y}^2 \Bigg[&\left(\frac{1-f}{n}\right)C_y^2 + N_1^{'} + N_2^{'} + (1-g)\alpha\left(N_3^{'} + N_4^{'}\right) + \\
&\frac{(1-g)^2}{4}\alpha^2\left\{\left(\frac{1-f}{n}\right)C_x^2 + N_5^{'}\right\}\Bigg]
\end{aligned} \tag{23}$$

where

$$N_1' = \left(\frac{1-f_1}{n_1}\right)\left(\frac{g^2}{4}C_x^2 + \frac{g^2}{4}C_z^2 - \frac{g^2}{2}K_{xz}C_z^2\right)$$
$$N_2' = \left(\frac{1-f}{n}\right)\left(\frac{g^2}{4}C_x^2 - gK_{yx}C_x^2\right)$$
$$N_3' = \left(\frac{1-f}{n}\right)\left(\frac{g}{2}C_x^2 - K_{yx}C_x^2\right)$$
$$N_4' = \left(\frac{1-f_1}{n_1}\right)\left(\frac{g}{2}C_x^2 + \frac{g}{2}C_z^2 - gK_{xz}C_z^2\right)$$
$$N_5' = \left(\frac{1-f_1}{n_1}\right)\left(C_x^2 + C_z^2\left(1-2K_{xz}\right)\right)$$

Differentiation of (23) with respect to α yields its optimum value as

$$\alpha = -\left(\frac{2}{1-g}\right)\left(\frac{N_3'+N_4'}{\left(\frac{1-f}{n}\right)C_x^2 + N_5'}\right) = \alpha_{II(opt)} \quad (say) \tag{24}$$

Thus, the resulting optimum MSE of $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II}$ is given by

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \bar{Y}^2\left[\left(\frac{1-f}{n}\right)C_y^2 + N_1' + N_2' - \frac{\left(N_3'+N_4'\right)^2}{\left(\frac{1-f}{n}\right)C_x^2 + N_5'}\right] \tag{25}$$

Remarks: 1. For $\alpha = 1$, the estimator $\hat{\bar{Y}}_{EC}^{RdR}$ in (1) boils down to the exponential chain ratio estimator $\hat{\bar{Y}}_{Re}^{dc}$ [17] in double sampling. Thus, putting $\alpha = 1$ in (23), we get the MSE of $\hat{\bar{Y}}_{Re}^{dc}$ to the first degree of approximation as

$$MSE\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II} = \bar{Y}^2\left[\frac{1-f}{n}\left(C_y^2 + \frac{C_x^2}{4} - K_{yx}C_x^2\right) + \frac{1-f_1}{n_1}\left(\frac{C_x^2}{4} - \frac{K_{xz}C_z^2}{2}\right)\right] \tag{26}$$

2. For $\alpha = 0$, the estimator $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}$ in (1) reduces to the exponential chain dual to ratio estimator $\hat{\bar{Y}}_{EdR}^{dc}$ in double sampling. The MSE of $\hat{\bar{Y}}_{EdR}^{dc}$ can be obtained by puting $\alpha = 0$ in (23) respectively as

$$MSE\left(\hat{\bar{Y}}_{EdR}^{dc}\right)_{II} = \bar{Y}^2\left[\frac{1-f}{n}\left(C_y^2 + \frac{g^2}{4}C_x^2 - gK_{yx}C_x^2\right) + \frac{1-f_1}{n_1}\left(\frac{g^2}{4}C_x^2 + \frac{g^2}{4}C_z^2 - \frac{g^2}{2}K_{xz}C_z^2\right)\right] \tag{27}$$

7 Efficiency Comparisons in Case II

7.1 Comparison with sample mean per unit estimator $\bar{y}$

The variance of usual unbiased estimator $\bar{y}$ is given by

$$V\left(\bar{y}\right) = \left(\frac{1-f}{n}\right)C_y^2 \tag{28}$$

From equation (28) and (25), we have

$$V\left(\bar{y}\right) - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = -\left(N_1'+N_2'\right) + \frac{\left(N_3'+N_4'\right)^2}{\frac{1}{n}C_x^2 + N_5'} > 0 \tag{29}$$

if $\frac{\left(N_3'+N_4'\right)^2}{\frac{1}{n}C_x^2+N_5'} > \left(N_1'+N_2'\right)$

7.2 Comparison with chain ratio estimator $\left\{\hat{\bar{Y}}_R^{dc}\right\}_{II}$

The MSE of chain ratio estimator $\left\{\hat{\bar{Y}}_R^{dc}\right\}_{II}$ in double sampling is given by

$$MSE\left\{\hat{\bar{Y}}_R^{dc}\right\}_{II} = \bar{Y}^2\left[\frac{1-f}{n}\left(C_y^2+C_x^2\left(1-2K_{yx}\right)\right)+\frac{1-f_1}{n_1}\left(C_x^2+C_z^2\left(1-2K_{xz}\right)\right)\right] \tag{30}$$

From equation (30) and (25), we have

$$MSE\left\{\hat{\bar{Y}}_R^{dc}\right\}_{II} - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \bar{Y}^2\left[K_1'+K_2'+\frac{\left(N_3''+N_4''\right)^2}{K_3'}\right] > 0 \tag{31}$$

if $K_1', K_2', K_3' > 0$, where

$$K_1' = \left(\frac{1-f}{n}\right)\left\{C_x^2\left(1-\frac{g^2}{4}\right)-(2-g)K_{yx}C_x^2\right\}$$
$$K_2' = \left(\frac{1-f1}{n_1}\right)\left\{\left(1-\frac{g^2}{4}\right)\left(C_x^2+C_z^2\right)-\left(2-\frac{g^2}{2}\right)K_{xz}C_z^2\right\}$$
$$K_3' = \left\{\left(\frac{1}{n}+\frac{1}{n_1}\right)C_x^2+\frac{1}{n_1}C_z^2\left(1-2K_{xz}\right)\right\}$$

7.3 Comparison with chain linear regression estimator $\left\{\hat{\bar{Y}}_{reg}^{dc}\right\}_{II}$

The MSE of the chain linear regression estimator $\left\{\hat{\bar{Y}}_{reg}^{dc}\right\}_{II}$ in double sampling suggested by [**?**] is given by

$$MSE\left\{\hat{\bar{Y}}_{reg}^{dc}\right\}_{II} = \bar{Y}^2\left[\frac{1-f}{n}\left(C_y^2-K_{yx}^2C_x^2\right)+\frac{1-f_1}{n_1}K_{yx}^2\left(C_x^2-K_{xz}^2C_z^2\right)\right] \tag{32}$$

From equation (32) and (25), we have

$$MSE\left\{\hat{\bar{Y}}_{reg}^{dc}\right\}_{II} - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \bar{Y}^2\left[\frac{1-f}{n}C_y^2-K_4'+K_5'+\frac{\left(N_3'+N_4'\right)^2}{K_3'}\right] > 0 \tag{33}$$

if $K_3', K_5' > 0$ and $K_4' < 0$

7.4 Comparison with exponential chain ratio estimator $\left\{\hat{\bar{Y}}_{Re}^{dc}\right\}$

From equation (26) and (25), we have

$$MSE\left\{\hat{\bar{Y}}_{Re}^{dc}\right\}_{II} - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \bar{Y}^2\left[-\left(K_6'+K_7'\right)+\frac{\left(N_3'+N_4'\right)^2}{K_3'}\right] > 0 \tag{34}$$

if $\frac{\left(N_3'+N_4'\right)^2}{K_3'} > K_6'+K_7'$, where

$$K_6' = \left(\frac{1-f}{n}\right)C_x^2\left\{\frac{1-g^2}{4}-(1-g)K_{yx}\right\}$$
$$K_7' = \left(\frac{1-f_1}{n_1}\right)\left\{\frac{1-g^2}{4}C_{x^2}-\frac{g^2}{4}C_z^2-\frac{(1-g^2)}{2}\rho_{xz}C_xC_z\right\}$$

7.5 Comparison with chain dual to ratio estimator $\left\{\hat{\bar{Y}}_{dR}^{dc}\right\}_{II}$

The MSE of chain dual to ratio estimator $\left\{\hat{\bar{Y}}_{dR}^{dc}\right\}_{II}$ is given by

$$MSE\left\{\hat{\bar{Y}}_{dR}^{dc}\right\}_{II} = \bar{Y}^2\left[\frac{1-f}{n}C_y^2 + \frac{1-f}{n}g^2C_x^2\left(1-\frac{2}{g}K_{yx}\right) + \frac{1-f_1}{n_1}\left\{g^2C_x^2 + g^2C_z^2\left(1-\frac{2}{g}K_{xz}\right)\right\}\right] \quad (35)$$

From equation (35) and (25), we have

$$MSE\left\{\hat{\bar{Y}}_{dR}^{dc}\right\}_{II} - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \bar{Y}^2\left[K_8^{'} + K_9^{'} + \frac{\left(N_3^{'}+N_4^{'}\right)^2}{K_3^{'}}\right] > 0 \quad (36)$$

if $K_8^{'}, K_9^{'}, K_3^{'} > 0$, where

$$K_8^{'} = \left(\frac{1-f}{n}\right)\frac{3}{4}g^2C_x^2\left(1-\frac{4}{3g}K_{yx}\right)$$
$$K_9^{'} = \left(\frac{1-f_1}{n_1}\right)\frac{3}{4}g^2\left\{C_x^2 + C_z^2\left(1-2K_{xz}\right)\right\}$$

7.6 Comparison with exponential chain dual to ratio estimator $\left\{\hat{\bar{Y}}_{EdR}^{dc}\right\}_{II}$

From equation (27) and (25), we have

$$MSE\left\{\hat{\bar{Y}}_{EdR}^{dc}\right\}_{II} - MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \bar{Y}^2\left[\frac{\left(N_3^{'}+N_4^{'}\right)^2}{K_3^{'}}\right] > 0 \quad (37)$$

if $K_3^{'} > 0$

8 Theorem II

To the first order of approximations, the proposed strategy under optimality condition (24) is always more efficient than $V\left(\bar{y}\right)$, $MSE\left(\hat{\bar{Y}}_{R}^{dc}\right)_{II}$, $MSE\left(\hat{\bar{Y}}_{reg}^{dc}\right)_{II}$, $MSE\left\{\hat{\bar{Y}}_{Re}^{dc}\right\}_{II}$, $MSE\left(\hat{\bar{Y}}_{dR}^{dc}\right)_{II}$ and $MSE\left(\hat{\bar{Y}}_{EdR}^{dc}\right)_{II}$ under the conditions $\frac{\left(N_3^{'}+N_4^{'}\right)^2}{\frac{1}{n}C_x^2+N_5^{'}} > \left(N_1^{'}+N_2^{'}\right)$; $K_1^{'}, K_2^{'}, K_3^{'} > 0$; $K_3^{'}, K_5^{'} > 0$ and $K_4^{'} < 0$; $\frac{\left(N_3^{'}+N_4^{'}\right)^2}{K_3^{'}} > K_6^{'} + K_7^{'}$; $K_3^{'}, K_8^{'}, K_9^{'} > 0$; and $K_3^{'} > 0$.

9 Cost aspect

The different estimators reported in this paper have so far been compared with respect to their variances. In practical applications, the cost aspect should also be taken into account. In the literature, therefore, convention is to fix the total cost of the survey and then to find optimum sizes of priliminary and final samples so that the variance of the estimator is minimized. In most of the practical situations, total cost is a linear function of samples selected at first and second phases.

In this section, we shall consider the cost of the survey and find the optimum sizes of the preliminary and second-phase samples in Case I and Case II separately.

Case I: When we use one auxiliary variate x then the cost function is given by $C = nC_1 + n_1C_2$, where C, C_1 and C_2 are the total cost, cost per unit of collecting information on the study variate y and the cost per unit of collecting information on the auxiliary variate x respectively of the survey.

When we use additional auxiliary variate z to estimate $\hat{\bar{Y}}_{EC}^{RdR}$, then the cost function is given by

$$C = nC_1 + n_1\left(C_2 + C_3\right) \tag{38}$$

where C_3 is the cost per unit collecting information on auxiliary variate z.

Ignoring FPC, the MSE of $\hat{\bar{Y}}_{EC}^{RdR}$ in (6) can be expressed as

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_I = \frac{1}{n}V_1 + \frac{1}{n_1}V_2$$

where $V_1 = C_y^2 + \frac{g^2}{4}C_x^2\left(1 - \frac{4}{g}K_{yx}\right) + \frac{(1-g)g}{2}\alpha C_x^2\left(1 - \frac{2}{g}K_{yx}\right) + \frac{(1-g)^2}{4}\alpha^2 C_x^2$

$V_2 = -\frac{g^2}{4}C_x^2\left(1 - \frac{4}{g}K_{yx}\right) + \frac{g^2}{4}C_z^2\left(1 - \frac{4}{g}K_{yz}\right) - \frac{(1-g)g}{2}\alpha C_x^2\left(1 - \frac{2}{g}K_{yx}\right) + \frac{(1-g)^2}{4}\alpha^2\left(C_z^2 - C_x^2\right) + \frac{g}{2}C_z^2\left(1 - K_{yz}\right)$

It is assumed that $C_1 > C_2 > C_3$. The optimum values of n and n_1 for fixed cost $C = C_0$ which minimizes the MSE of $\hat{\bar{Y}}_{EC}^{RdR}$ in (6) under cost function are given by

$$n_{opt} = \frac{C_0\sqrt{V_1/C_1}}{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2 + C_3\right)}}$$

$$n_{1opt} = \frac{C_0\sqrt{V_2/\left(C_2 + C_3\right)}}{\sqrt{V_1C_1} + \sqrt{V_2}\sqrt{C_2 + C_3}}$$

Thus the resulting MSE of $\hat{\bar{Y}}_{EC}^{RdR}$ is given by

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{I(opt)} = \frac{1}{C_0}\left\{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2 + C_3\right)}\right\}^2 \tag{39}$$

If all the resources were diverted towards the study variate y only, then we would have optimum sample size as below

$$n^{**} = C/C_1$$

Thus, the variance of sample mean $\bar{y}$ for a given fixed cost $C = C_0$ in case of large population is given by

$$V_{opt}\left(\bar{y}\right) = \frac{C_1}{C_0}S_y^2 \tag{40}$$

From (39) and (40), the proposed sampling strategy would be profitable as long as

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_I < V_{opt}\left(\bar{y}\right)$$

or equivalently,

$$\frac{C_2 + C_3}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_2}}\right]^2$$

Case II: We assume that y measured on n units, and x and z are measured on n_1 units. We consider a simple cost function

$$C = nC_1 + n_1\left(C_2^{'} + C_3^{'}\right) \tag{41}$$

where $C_2^{'}$ and $C_3^{'}$ denote cost per unit of observing x and z values respectively.

The MSE of $\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II}$ in (23) can be written as

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II} = \frac{1}{n}V_1 + \frac{1}{n_1}V_3 \tag{42}$$

where $V_3 = \left\{\frac{g^2}{4} + \frac{(1-g)g}{2} + \frac{(1-g)^2}{4}\alpha^2\right\}\left\{C_x^2 + C_z^2\left(1-2K_{xz}\right)\right\}$
To obtain the optimum allocation of sample between phases for a fixed cost $C = C_0$, we minimized (23) with condition (41). It is easily found that this minimum is attained for

$$n_{opt} = \frac{C_0\sqrt{V_1/C_1}}{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2^{'} + C_3^{'}\right)}}$$

$$n_{1opt} = \frac{C_0\sqrt{V_2/\left(C_2^{'} + C_3^{'}\right)}}{\sqrt{V_1C_1} + \sqrt{V_2}\sqrt{C_2^{'} + C_3^{'}}}$$

Thus, the optimum MSE corresponding to these optimum values of n and n_1 are given by

$$MSE\left\{\hat{\bar{Y}}_{EC}^{RdR}\right\}_{II(opt)} = \frac{1}{C_0}\left\{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2^{'} + C_3^{'}\right)}\right\}^2 \tag{43}$$

From (40) and (43), it is obtained that the proposed estimator $\hat{\bar{Y}}_{EC}^{RdR}$ yields less variance than that of sample mean $\bar{y}$ for the same fixed cost if

$$\frac{C_2^{'} + C_3^{'}}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_2}}\right]^2$$

10 Empirical Study

To examine the merits of the proposed estimator, we have considered three natural population data sets. The description of the population are given as follows:

Population I[7]
Y: Number of placebo children
X: Number of paralytic polio cases in the placebo group.
Z: Number of paralytic polio cases in the notinoculated groups.
$N = 34$, $n = 15$, n_{15}, $\bar{Y} = 4.92$, $\bar{X} = 2.59$, $\bar{Z} = 2.91$, $\rho_{yx} = 0.7326$, $\rho_{yz} = 0.6430$, $\rho_{xz} = 0.6837$, $C_y^2 = 1.0248$, $C_x^2 = 1.5175$, $C_z^2 = 1.1492$

Population II:[27]
Y:Apples trees of bearing age in 1964.
X:Bushels of apples harvested in 1964.
Z:Bushels of apples harvested in 1959.
$N = 200$, $n = 20$, $n_1 = 30$, $\bar{Y} = 0.103182X\,10^4$, $\bar{X} = 0.293458X\,10^4$, $\bar{Z} = 0.365149X\,10^4$ $\rho_{yx} = 0.93$, $\rho_{yz} = 0.77$, $\rho_{xz} = 0.84$, $C_y^2 = 2.55280$, $C_x^2 = 4.02504$, $C_z^2 = 2.09379$

Population III [12]
Y:Area under wheat in 1964.
X:Area under wheat in 1963.
Z: Cultivated area in 1961.
$N = 34$, $n = 7$, $n_1 = 10$, $\bar{Y} = 199.44$ acre, $\bar{X} = 208.89$acre, $\bar{Z} = 747.59$ acre, $\rho_{yx} = 0.9801$, $\rho_{yz} = 0.9043$, $\rho_{xz} = 0.9097$, $C_y^2 = 0.5673$, $C_x^2 = 0.5191$, $C_z^2 = 0.3527$

Table 1: Percent relative efficiencies of different estimators w.r.t $\bar{y}$

Estimators	Population I	Population II	Population III
Case I			
$\bar{y}$	100	100	100
$\{\hat{\bar{Y}}_{R}^{dc}\}_{I}$	136.91	279.93	730.78
$\{\hat{\bar{Y}}_{reg}^{dc}\}_{I}$	185.53	326.33	778.93
$\{\hat{\bar{Y}}_{Re}^{dc}\}_{I}$	184.35	150.97	259.55
$\{\bar{Y}_{dR}^{dc}\}_{I}$	174.26	*	*
$\{\bar{Y}_{EdR}^{dc}\}_{I}$	136.82	*	102.30
$\{\hat{\bar{Y}}_{EC}^{RdR}\}_{I(opt)}$	230.17	322.9 4	763.29
Case II			
$\bar{y}$	100	100	100
$\{\hat{\bar{Y}}_{R}^{dc}\}_{II}$	*	182.67	730.78
$\{\hat{\bar{Y}}_{reg}^{dc}\}_{II}$	162.83	338.50	701.87
$\{\hat{\bar{Y}}_{Re}^{dc}\}_{II}$	207.03	574.91	454.36
$\{\hat{\bar{Y}}_{dR}^{dc}\}_{II}$	*	130.93	171.75
$\{\hat{\bar{Y}}_{EdR}^{dc}\}_{II}$	*	113.69	130.42
$\{\hat{\bar{Y}}_{EC}^{RdR}\}_{II(opt)}$	196.16	362.61	735.15

* Data is not applicable

11 Conclusion

Table1 clearly indicates that there is a substantial gain in efficiency by using the proposed estimator $\hat{\bar{Y}}_{EC}^{RdR}$ over the conventional estimator $\bar{y}$, $\{\hat{\bar{Y}}_{R}^{dc}\}$, $\{\hat{\bar{Y}}_{reg}^{dc}\}$, $\{\hat{\bar{Y}}_{Re}^{dc}\}$, $\{\hat{\bar{Y}}_{dR}^{dc}\}$ and $\{\hat{\bar{Y}}_{EdR}^{dc}\}$ in both the cases with respect to all sets of data except $\hat{\bar{Y}}_{reg}^{dc}$ for population II and III in Case I and $\hat{\bar{Y}}_{Re}^{dc}$ for population I and II in case II.

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