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Variational Constraints of Masses and Radii of $c\bar{c}$-Mesons

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Abstract: Within the non-relativistic quark model framework, the spectra for $c\bar{c}$-mesons are usually derived through the (numerical) solution of the Schrödinger equation with a given potential for quark-antiquark interactions. Cornell potential plus hyperfine corrections are popular models with successful predictions of mass spectra of these heavy mesons. We use the Numerov matrix method to obtain the mass spectra of $c\bar{c}$-mesons with high accuracy. Their (root-mean-squared) radii $r_{\rm rms}$ are then derived through the corresponding radial wavefunctions. Angular momentum quantum numbers and $r_{\rm rms}$ allow to determine the momentum width for these meson states. We then relate the masses and $r_{\rm rms}$ for $c\bar{c}$-mesons through constraints arising from the uncertainty and variational principles. These ideas can be straightforwardly generalized for other meson states.

Keywords: $c\bar{c}$-mesons, meson masses and radii, variational principle, uncertainty principle.

1 Introduction

Understanding the nature of hadron spectra is an exciting problem of non-perturbative hadron physics which demands a modern view of strong interactions among the constituent quarks of these hadrons. A natural framework for the prediction of heavy quarkonium meson masses is the non-relativistic potential model, which albeit being in the market for half a century, still offers a number of possibilities for improvement of the description of strong interactions and development of more accurate numerical and analytical techniques to describe the mass spectra. In this model, Schrödiger equation (SE) is solved given a phenomenological description of quark-antiquark interactions through a static, radial potential. Most known models demand a numerical solution to the SE, and thus the choice of an accurate strategy for this task is highly desirable. Though this problem is already addressed in quantum mechanics textbooks (see, for instance, Ref. [1]), the problem of solving SE in radial potentials is still nowadays a topic of active study.

Approximate methods like WKB, variational principle and perturbation theory have been employed to predict heavy meson spectra in agreement with experimental results [2]. Nevertheless, its applicability is sometimes restricted by the analytic form of the choice of the potential model. A number of robust numerical techniques to solve the SE are also available, among which are the shooting method [3] and several variants of the matrix discretization of this equation [4]. The Qena group has successfully made use of Numerov matrix method [5] to describe bottomonium spectra [6,7,8,9]. Here we adopt this strategy to accurately obtain the $c\bar{c}$-spectra for the Cornell and Cornell plus hyperfine corrections models of quark interactions for S- and P-states. The cornerstone of this method is a matrix representation of the radial kinetic operator which translates the SE to a matrix eigenvalue problem, whose only parameters are the maximum value of the range of integration, r_{max} and the number of equidistant discrete points N for the discretization procedure. Thus, stability can be carried out straightforwardly. Mass spectrum is derived from the eigenvalues of the SE and other meson properties, like the radii of these states are then obtained from the wavefunctions.

Although the Numerov method can be extended directly to higher angular momentum states, in search for a simpler yet accurate calculation of the mass spectra, we combine the uncertainty principle and the variational

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principle to obtain an analytical constraint to the mass and radius of a given state given the momentum width in terms of the angular momentum eigenvalues and root-mean-square $r_{\rm rms}$ separation of constituent quarks in charmonium. To this end, we organized the remaining of this article as follows: In Sect. 2 we describe the non-relativistic framework to derive the mass spectra of charmonia S- and P-states and $r_{\rm rms}$ with Numerov method, for the sake of illustration. We combine the uncertainty and variational principles to derive constraints of these quantities in a given potential in Sect. 3. We discuss our results and conclude in Sect. 4.

2 Non-relativistic framework for mesons

The non-relativistic quark model to the determination of heavy quarkonium meson properties has been successfully applied to determine the mass of bound states (molecular states) of $Q\bar{Q}$-systems. Perturbative quantum chromodynamics (QCD) constraints and lattice simulations have provided valuable information to determine the profile of the $Q\bar{Q}$ interaction potential $\mathscr{V}=V(r)$, considered to be radial, that is then introduced in a stationary Schrödinger equation

$$\mathscr{H}\psi(r,\theta,\phi)=\Delta\psi(r,\theta,\phi)\,, \qquad (1)$$

where $\mathscr{H}=\mathscr{T}+\mathscr{V}$ is the *Hamiltonian* of the system and Δ corresponds to the mass of the bound state. The non-relativistic kinetic term $\mathscr{T}$ is

$$\mathscr{T}=m_Q+m_{\bar{Q}}+\frac{1}{2\mu}\left(p_x^2+p_y^2+p_z^2\right)\,, \qquad (2)$$

where $\mu=m_Q m_{\bar{Q}}/(m_Q+m_{\bar{Q}})$ is the reduced mass of the system and the (cartesian) coordinates describe the center of mass motion of the meson. Thus, for central potentials, in units where $\hbar=1$, Eq. (1) becomes

$$-\frac{1}{2\mu}\frac{d^2\psi(r)}{dr^2}+\left[\frac{l(l+1)}{2\mu r^2}+V(r)+m_Q+m_{\bar{Q}}\right]\psi(r) = \Delta\psi(r)\,, \qquad (3)$$

where the second term represent the centrifugal barrier and l is the orbital angular momentum of the meson. For the potential part, two popular choices are

$$V_1(r)\equiv V_{\rm Cornell}=-\frac{4}{3}\frac{\alpha_s}{r}+br\,, \qquad (4)$$

referred to as the Cornell potential and a second example is of the form

$$V_2(r)=V_1(r)+V_{\rm hyp}(r)\,, \qquad (5)$$

where the hyperfine correction

$$V_{\rm hyp}(r)=\frac{32\pi\alpha_s}{9m_Q m_{\bar{Q}}}\delta_\sigma(r)S_Q\cdot S_{\bar{Q}}\,, \qquad (6)$$

takes into account spin-spin interactions of constituent quarks. Here,

$$\delta_\sigma(r)=\left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2}\,,\quad S_Q\cdot S_{\bar{Q}}=\frac{s(s+1)}{2}-\frac{3}{4}\,, \qquad (7)$$

where s is the spin quantum number of the bound state. For $c\bar{c}$ spectroscopy, we set (see Ref. [4]) $m_Q=m_{\bar{Q}}=m_c$. The parameters that best describe the experimental mass spectra (see below) for $V_1(r)$ are $\alpha_s=0.5317$ and $b=0.1497$ GeV^2, while $m_c=1.4495$ GeV, whereas for $V_2(r)$, $\alpha_s=0.4827$, $b=0.1488$ GeV^2, $\sigma=1.2819$ GeV and $m_c=1.4499$ GeV.

A robust strategy to obtain the heavy meson spectra from Eq. (3) is the Numerov method, that has been widely implemented by the Qena group, particularly for the case of bottonomium [6,7,8,9]. Setting an interval $r\in(0,r_{max})$ and constructing an equidistant set of N points r_i separated a distance d, Eq. (3) can be cast in the matrix form

$$-\frac{1}{2\mu}A_{N,N}B^{-1}_{N,N}\psi_i+\left[V_N(r)+\frac{l(l+1)}{2\mu r^2}\right]\psi_i=\Delta\psi_i\,, \qquad (8)$$

where $\psi_i=\psi(r_i)$ and the matrices

$$A_{N,N}=\frac{I_{-1}-2I_0+I_1}{d^2}\,,\quad B_{N,N}=\frac{I_{-1}+10I_0+I_1}{12}\,, \qquad (9)$$

$V_N(r)=\mathrm{diag}(\ldots,V_{i-1},V_i,V_{i+1},\ldots)$ where $V_i=V(r_i)+l(l+1)/(2\mu r_i^2)+2m_c$ and I_{-1}, I_0 and I_1 are, respectively, the sub-, main-, and up-diagonal unit matrices. By taking $N=200$ and $r_{max}=4$fm for S-states and $N=400$ and $r_{max}=4$fm for P-states, the spectra we find is summarized in Table 1, where a comparison with experimental results [2] is also presented. Corresponding wavefunctions for S states for $V_1(r)$ and $V_2(r)$ are shown in Figs. 1 and 2, respectively.

Table 1: Mass spectra in GeV for $c\bar{c}$ mesons in S- and P-states.

State	Name	Exp. value	V_1	V_2
$1\,^3S_1$	J/ψ	3.09687± 0.0004	3.05027	3.07834
$1\,^1S_0$	$\eta_c(1S)$	2.9792 ± 0.0013	3.05027	2.94518
$2\,^3S_1$	$\psi'(2s)$	3.68609 ± 0.0004	3.65176	3.66535
$2\,^1S_0$	$\eta'{}_c(2S)$	3.637±0.004	3.65176	3.60943
$3\,^3S_1$	$\psi(3S)$	4.039 ± 0.001	4.06608	4.07606
$3\,^1S_0$	$\eta_c(3S)$	–	4.06608	4.03636
$4\,^3S_1$	$\psi(4S)$	4.421±0.004	4.41315	4.42131
$4\,^1S_0$	$\eta_c(4S)$	–	4.41315	4.38935
$1\,^3P_2$	$\chi_2(1P)$	3.55620±0.00009	3.50078	3.52544
$1\,^3P_1$	$\chi_1(1P)$	3.51066±0.00007	3.50078	3.52544
$1\,^3P_0$	$\chi_0(1P)$	3.41475±0.00031	3.50078	3.52544
$1\,^1P_1$	$h_c(1P)$	3.52541±0.00016	3.50078	3.51837

Charmonia root-mean-squared radius $r_{\rm rms}$ is a basic property of these states and can be derived from the radial

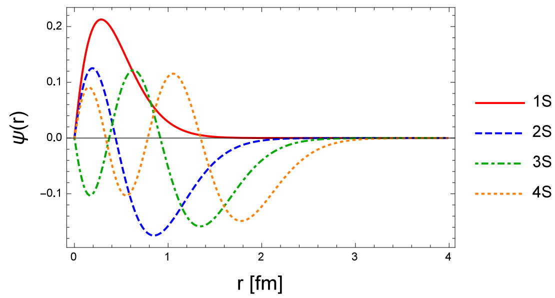

Fig. 1: *S*-state radial wavefunctions for the Cornell potential $V_1(r)$, eq. (4).

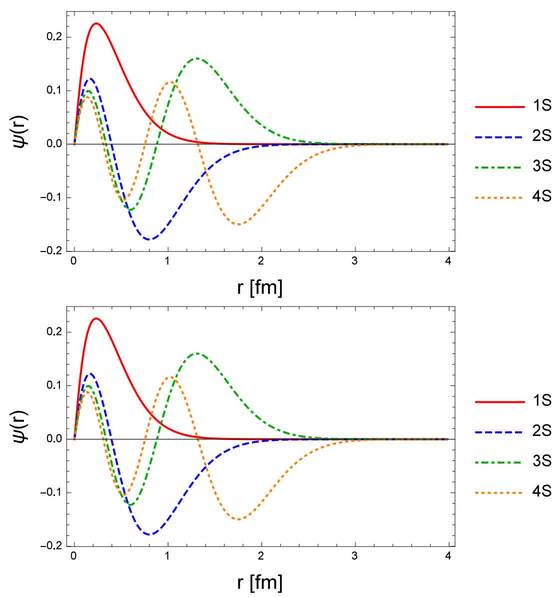

Fig. 2: *S*-state radial wavefunctions for the Cornell potential plus hyperfine corrections, $V_2(r)$ in eqs. (5) and (6). *Upper panel:* $^1S_0$ states. *Lower panel:* $^3S_1$ states.

wavefunction from the definition [10]

$$r_{\rm rms}^2 = \int_0^\infty dr\, r^2 |\psi(r)|^2 \,, \tag{10}$$

It is known and can be straightforwardly observed that the wavefunctions of these mesonic states are characterized by a momentum width of the form

$$\beta = \sqrt{2(n-1)+l+\frac{3}{2}}\,\frac{1}{r_{\rm rms}} \equiv \frac{\delta}{r_{\rm rms}} \tag{11}$$

which along with $r_{\rm rms}$ are summarized in Table 2. Below we show that masses and $r_{\rm rms}$ are tightly constrained quantities from the point of view of the uncertainty and variational principles.

Table 2: $r_{\rm rms}$ and β for $c\bar{c}$ mesons in *S*- and *P*-states.

State	Name	$r_{\rm rms}$ [fm]		β	
		V_1	V_2	V_1	V_2
$1\,^3S_1$	J/ψ	0.417	0.432	0.5874	0.5666
$1\,^1S_0$	$\eta_c(1S)$	0.417	0.369	0.5874	0.6642
$2\,^3S_1$	$\psi'(2s)$	0.87	0.88	0.4248	0.4249
$2\,^1S_0$	$\eta'_c(2S)$	0.87	0.84	0.4299	0.4456
$3\,^3S_1$	$\psi(3S)$	1.238	1.246	0.3787	0.3763
$3\,^1S_0$	$\eta_c(3S)$	1.238	1.216	0.3787	0.3858
$4\,^3S_1$	$\psi(4S)$	1.559	1.565	0.3513	0.3499
$4\,^1S_0$	$\eta_c(4S)$	1.559	1.54	0.3513	0.3555
$1\,^3P_2$	$\chi_2(1P)$	0.6912	0.7025	0.3544	0.3487
$1\,^3P_1$	$\chi_1(1P)$	0.6912	0.7025	0.3544	0.3487
$1\,^3P_0$	$\chi_0(1P)$	0.6912	0.7025	0.3544	0.3487
$1\,^1P_1$	$h_c(1P)$	0.6912	0.6928	0.3544	0.3535

3 Uncertainty and Variational Principles

Heisenberg's uncertainty principle

$$\Delta p_x \Delta x \geq \frac{1}{2} \tag{12}$$

can be combined with the variational principle for a back of the envelope estimate of the ground state energy of a particle of mass m subjected to a spherical central potential. Considering $\Delta p_x = \tilde{\beta}$ as a momentum width of the ground state wavefunction of the said particle and $\Delta x = \bar{x}$ as an effective characteristic *size* in cartesian coordinates of the bound state, we assume that $p_x \equiv \tilde{\beta} = 1/(2\bar{x})$ and consider $\bar{x}$ as a variational parameter (the same along the other cartesian coordinates). Thus, one can write

$$\begin{aligned} H &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(\sqrt{x^2+y^2+z^2}) \\ &= \frac{1}{8m^2\bar{x}^2} + \frac{1}{8m^2\bar{y}^2} + \frac{1}{8m^2\bar{z}^2} + V(\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}) \,. \end{aligned} \tag{13}$$

Minimizing with respect to $\bar{x}$, namely, finding the roots of

$$\frac{\partial H}{\partial \bar{x}} = -\frac{1}{4m\bar{x}^3} + V_{\bar{x}}(\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}) \equiv 0 \,, \tag{14}$$

where $V_{\bar{x}}$ denotes the partial derivative with respect to $\bar{x}$, we obtain the optimum value $x_{\rm min}$. At the symmetric point $\bar{x}=\bar{y}=\bar{z}=x_{\rm min}$, the energy of the ground state is simply

$$E_{\rm g} = \frac{1}{8m^2 r_{\rm min}^2} + V(r_{\rm min}) \,, \tag{15}$$

with $r_{\rm min} = \sqrt{3}x_{\rm min}$. We can use the same reasoning to derive the mass of the η_c meson, which simply becomes

$$\Delta_{\eta_c} = 2m_c + \frac{1}{8m^2 r_{\rm min}^2} + V_{1,2}(r_{\rm min}) \,. \tag{16}$$

The above expression gives $r_{\rm min} = 0.437$ fm for $V_1(r)$, in agreement with other theoretical calculations [11] and the experimental measurement of the η_c interaction radius [12]. Frome here, $\Delta_{\eta_c} = 3.22677$GeV, 6% above of the exact numerical result. For $V_2(r)$, $r_{\min} = 0.44$ fm and $\Delta_{\eta_c} = 3.23702$GeV, with 10% difference. Both results for Δ_{η_c} are in accordance with the variational principle. This approach, however, does not take into account the (actual) momentum width of the wavefunction of charmonia states, nor allows to estimate the masses of higher excited state mesons. We can overcome this situation assuming that for all states (see Eq. (11)), $p_x \to \beta = \delta/(\sqrt{3}\bar{x})$ and similarly to the other cartesian coordinates. Then, minimization with respect to $\bar{x}$ and evaluation of the Hamiltonian at the symmetric point gives the following relation between the mass and $r_{\rm rms}$ $c\bar{c}$-mesons

$$\Delta = 2m_c + \frac{1}{2\mu}\beta^2 + \frac{l(l+1)}{2\mu r_{\rm rms}^2} + V_{1,2}(r_{\rm rms}). \qquad (17)$$

Thus, taking β and $r_{\rm rms}$, we obtain the spectra shown in Table 3 along with the percent error with respect to the numerical results. Again, numbers are in agreement with theoretical calculations presented here and in Refs. [10, 11], for instance and the errors are fair for such a simple minded calculation scheme.

Table 3: Mass spectra for $c\bar{c}$-mesons in S- and P-states from the variational constraint (17).

$c\bar{c}$ state	Name	Δ [GeV]		Error [%]	
		V_1	V_2	V_1	V_2
$1\,^3S_1$	J/ψ	3.10915	3.11616	1.93	1.24
$1\,^1S_0$	$\eta_c(1S)$	3.10915	3.09165	1.93	4.99
$2\,^3S_1$	$\psi'(2s)$	3.51506	3.52171	3.74	3.91
$2\,^1S_0$	$\eta'_c(2S)$	3.51506	3.49553	3.74	3.15
$3\,^3S_1$	$\psi(3S)$	3.81042	3.81574	6.29	6.38
$3\,^1S_0$	$\eta_c(3S)$	3.81042	3.79502	6.29	5.98
$4\,^3S_1$	$\psi(4S)$	4.06008	4.06453	8.0	8.07
$4\,^1S_0$	$\eta_c(4S)$	4.06008	4.04732	8.0	7.79
$1\,^3P_2$	$\chi_2(1P)$	3.41341	3.48758	4.02	1.93
$1\,^3P_1$	$\chi_1(1P)$	3.41341	3.48758	2.77	0.66
$1\,^3P_0$	$\chi_0(1P)$	3.41341	3.48758	0.04	2.13
$1\,^1P_1$	$h_c(1P)$	3.41341	3.48469	3.18	1.16

4 Discussion and Perspectives

The non-relativistic quark model is still a powerful framework to explore the meson mass spectra. For the case of heavy quarkonium, the Numerov method [5] employed to solve the SE in a variety of potential models [6,7,8,9] has proven to be accurate for the case of charmonia. As compared to other methods [3,4], this matrix discretization of the SE is easy to implement and robust to describe details of different proposals for the quark-antiquark potential model of quarkonium states. The main observation is that the different states describe mesons with a momentum width depending upon the angular momentum quantum numbers and the root-mean-square separation between the constituent quark and antiquark. These observations combined with the uncertainty and variational principles set analytic constraints between the meson mass and radius of every state which is very accurate for low-lying states and continues to be accurate for $l \geq 0$ states.

Relations here presented can be straightforward extended to other mesons, including those with light quarks as constituents. Further results along this line are currently under scrutiny and will be presented elsewhere.

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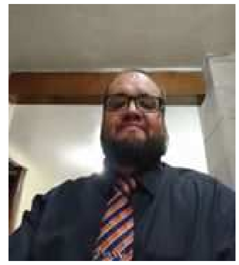

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