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The Simulation Studies of Stress-Strength Model for ELED

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Abstract: In this paper, the estimation of R = P [Y < X], namely Stress- Strength model is studied when both X and Y are two independent random variables with the extended linear exponential distribution (ELED), under different assumptions about their parameters, Maximum likelihood estimator in the case of fixed two parameters $(a = 1, b = 2)$, common unknown parameters $(a_1 = a_2 = a,\ b_1 = b_2 = b)$, and all unknown parameters $(a_1, a_2, b_1, b_2, \alpha, \beta)$ can also be obtained in explicit form. Estimating R with Bayes estimator with non-informative prior in the same previous cases with the same parameters , we obtain the asymptotic distribution of the maximum likelihood estimator and it can be used to construct confidence interval of R. Different methods are compared using simulations and one data analysis has been performed for illustrative purposes.

Keywords: Extended linear exponential distribution (ELED), Stress –Strength, Bayes and Maximum Likelihood.

1 Introduction

The estimation of R = p (Y< X), when X and Y are random variables following the specified distributions have been extensively discussed in the literature including quality control, engineering statistics, reliability, medicine, psychology, and biostatistics. This quantity can be obviously seen as a function of the parameters of the distribution of the random vector (X, Y) and could be calculated in the closed form for a limited number of cases as [1,2,3]. For instance, the estimation of R when X and Y are independent and normally-distributed has been considered as in the papers [4] and [5].

In this paper, the main objective is to focus on the inference of R= p[Y< X], where X and Y follow the (ELED) are independently distributed. In Section 3, the estimation of R is studied when the parameters (a, b) are common and fixed. In this section, we derive the MLE of the Stress-Strength model and Bayes estimator of R. In Section 4, we carry out a similar inference, made in the previous section, about R when the parameters (a, b) are common and unknown. We consider inference about R for the general case when the parameters of both distributions are not known and non-common, we derive MLE of R and Bayes estimator in Section 5. Simulation results will be studied in sections 6.

2 An Extension of the Generalized Linear Exponential Distribution

The exponential, generalized exponential, and Rayleigh distribution are among the most commonly used distributions for analyzing lifetime data. The researchers can be effectively used in modeling strength and general lifetime data. [6] Used different methods to estimate the parameters of the generalized Rayleigh on their observed data. In analyzing lifetime data, the exponential, Rayleigh, linear failure rate or generalized exponential distributions are normally used.

The (ELED) with three parameters (a, b, α) which developed in [7] with probability density function as follows:

$$f(x; a, b, \alpha) = \left[1 + \alpha e^{-\left(ax+\frac{b}{2}x^2\right)}\right](a + bx)e^{-\left(ax+\frac{b}{2}x^2\right)}e^{-\alpha\left[1-e^{-\left(ax+\frac{b}{2}x^2\right)}\right]}, x \geq 0, \alpha \geq 0, \quad (1)$$

And the cumulative distribution function can be written as:

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$$F(x;a,b,\alpha) = 1 - e^{-(ax+\frac{b}{2}x^2)} e^{-\alpha\left[1-e^{-\left(ax+\frac{b}{2}x^2\right)}\right]}, x \geq 0, \alpha \geq 0. \quad (2)$$

3 Estimation of R with Fixed Parameters $(a = 1, b = 2)$

In this section, the main aim is the estimation of R = p [Y <X], where the independent random variables X and Y follow the (ELED) with fixed parameters(a, b), that is, X ∼ELED (1, 2, α) and Y ∼ELED (1, 2, β).
The stress-strength parameter, R is defined as the following relation:

$$R = p[Y < X] = \int_0^\infty p(Y < X|X = x) f_X(x) dx, \quad (3)$$

$$R = \int_0^\infty \left[1 + \alpha e^{-(x+x^2)}\right](1 + 2x)\, e^{-(x+x^2)} e^{-\alpha\left(1-e^{-(x+x^2)}\right)} \left[1 - e^{-(x+x^2)} e^{-\beta\left(1-e^{-(x+x^2)}\right)}\right] dx. \quad (4)$$

3.1. Maximum Likelihood Estimation of R with Fixed Parameters $(a = 1, b = 2)$

Suppose that $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ random samples of n and munits which are observed from ELED. The Maximum Likelihood estimator of R denoted by $\hat{R}_{ML1}$. To compute the MLE of R, the corresponding log-likelihood of the observed sample is given by:

$$ln\, L(\alpha, \beta) = \sum_{i=1}^{n} \ln\left(1 + \alpha e^{-(x_i+x_i^2)}\right) + \sum_{i=1}^{n} \ln(1 + 2x_i) - \sum_{i=1}^{n}(x_i + x_i^2) - \alpha(n - t) +$$
$$\sum_{j=1}^{m} \ln\left(1 + \beta e^{-\left(y_j+y_j^2\right)}\right) + \sum_{j=1}^{m} \ln\left(1 + 2y_j\right) - \sum_{j=1}^{m}(y_j + y_j^2) - \beta(m - l), \quad (5)$$

where,

$t=\sum_{i=1}^{n} e^{-(x_i+x_i^2)}$, and $l = \sum_{j=1}^{m} e^{-\left(y_j+y_j^2\right)}$,

The MLE of (α, β) denoted by $(\hat{\alpha}, \hat{\beta})$ can be derived by solving the following equations:

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n} \frac{e^{-(x_i+x_i^2)}}{(1+\alpha e^{-(x_i+x_i^2)})} - (n - t)\,, \quad (6)$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{j=1}^{m} \frac{e^{-(y_j+y_j^2)}}{(1+\beta e^{-(y_j+y_j^2)})} - (m - l), \quad (7)$$

Then, the maximum likelihood estimates can be obtained by solving the non-linear equations numerically for α, β. This can be done using MATHCAD15. The relatively large number of parameters can be cause the problems, especially when the sample size is not large. Once we get the values of the parameters, we can evaluate the reliability function as the form:

$$\hat{R}_{ML1} = \int_0^\infty \left[1 + \hat{\alpha} e^{-(x+x^2)}\right](1 + 2x) e^{-(x+x^2)} e^{-\hat{\alpha}\left[1-e^{-(x+x^2)}\right]} \left[1 - e^{-(x+x^2)} e^{-\hat{\beta}\left[1-e^{-(x+x^2)}\right]}\right] dx. \quad (8)$$

3.2. Bayes Estimation of R with Fixed Parameters $(a = 1, b = 2)$

In this section, the Baysian estimator of R denoted by $\hat{R}_{BE1}$ is obtained with non-informative prior distribution. The equation to find fisher Information can be written as follow:

$$I(\theta) = -E\left(\frac{\partial^2 logL(\theta)}{\partial \theta^2}\right) \quad (9)$$

The Jeffrey's prior distribution [8] considered as a prior distribution for the likelihood function L(θ) is justified on the grounds of its invariance under parameterization according to Sinha [9]. The prior distribution of (α) and (β) are, respectively:

$$g(\alpha) \propto \frac{1}{\alpha};\ \alpha > 0, \quad (10)$$

and

$$g(\beta) \propto \frac{1}{\beta}\ ;\ \beta > 0, \quad (11)$$

Combining the prior densities of (α), (β) and the likelihood function to obtain the joint posterior density of (α, β) as:

$$\pi\,(\alpha, \beta/x, y) = \frac{g(\alpha)g(\beta)L(x,y/\alpha\,,\beta\,)}{\int_0^\infty \int_0^\infty g(\alpha)g(\beta)L(x,y/\alpha\,,\beta)d\alpha d\beta}\,, \tag{12}$$

$$\pi(\alpha, \beta/x, y) = \frac{k^{-1}}{\alpha.\beta}\prod_{i=1}^{n}\left[1 + \alpha e^{-(x_i+x_i^2)}\right](1 + 2x_i)e^{-(x_i+x_i^2)}e^{-\alpha\left(1-e^{-(x_i+x_i^2)}\right)}\prod_{j=1}^{m}\left[1 + \beta e^{-\left(y_j+y_j^2\right)}\right](1 + 2y_j)e^{-\left(y_j+y_j^2\right)}e^{-\beta\left(1-e^{-\left(y_j+y_j^2\right)}\right)}, \tag{13}$$

where,

$k = \int_0^\infty \int_0^\infty g(\alpha, \beta)L(x, y/\alpha\,, \beta)d\alpha d\beta,$

Therefore, the (BE) of R under squared error loss function is given by:

$$\hat{R}_{BS1} = E(R/x, y) = \int_0^1 R\pi(R/x, y)dR. \tag{14}$$

The estimated value of R under squared error loss cannot be computed analytically. Alternatively, numerical solution based on MATHCAD15 program is employed to evaluate $\hat{R}_{BS1}$ for different values of the parameters.

4 Maximum Likelihood Estimation of R with Common Unknown Parameters ($a_1 = a_2 = a,\ b_1 = b_2 = b$).

In this section, we aim at making inference about R, when the parameters (a, b) are unknown, and then investigate its properties. Let $X_1, X_2, \dots, X_n$ be a random sample from ELED (a, b, α) and $Y_1, Y_2, \dots, Y_m$ be a random sample from ELED(a, b, β), we can find the Stress-Strength parameter, as equation (3), then R can be written as:

$$R = \int_0^\infty \left[1 + \alpha e^{-\left(ax+\frac{b}{2}x^2\right)}\right](a + bx)\,e^{-\left(ax+\frac{b}{2}x^2\right)}e^{-\alpha\left(1-e^{-\left(ax+\frac{b}{2}x^2\right)}\right)}\left[1 - e^{-\left(ax+\frac{b}{2}x^2\right)}e^{-\beta\left(1-e^{-\left(ax+\frac{b}{2}x^2\right)}\right)}\right]dx. \tag{15}$$

The corresponding log-likelihood of the observed sample is given by:

$$ln\,L(a, b, \alpha, \beta) = \sum_{i=1}^{n} ln\left[1 + \alpha e^{-(ax_i+\frac{b}{2}x_i^2)}\right] + \sum_{j=1}^{m} ln\left[1 + \beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right] + \sum_{i=1}^{n} ln(a + bx_i) + \sum_{j=1}^{m} ln(a\ + by_j) - (aT_1 + bT_2) - \alpha(n - s_1) - (av_1 + bv_2) - \beta(m - s_2), \tag{16}$$

where,

$s_1 = \sum_{i=1}^{n} e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}, \qquad s_2 = \sum_{j=1}^{m} e^{-\left(ay_j+\frac{b}{2}y_j^2\right)},$

$T_j = \frac{1}{j}\sum_{i=1}^{n} x_i^j\,, j = 1{,}2\ ,\ v_i = \frac{1}{i}\sum_{i=1}^{n} y_j^i\,, i = 1{,}2$

The MLE of (a, b, α, β) denoted by $(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})$ can be derived by solving the following equations:

$$\frac{\partial \ln L}{\partial a} = -\alpha \sum_{i=1}^{n} \frac{x_i e^{-(ax_i+\frac{b}{2}x_i^2)}}{\left[1+\alpha e^{-(ax_i+\frac{b}{2}x_i^2)}\right]} - \beta \sum_{j=1}^{m} \frac{y_j e^{-(ay_j+\frac{b}{2}y_j^2)}}{\left[1+\beta e^{-(ay_j+\frac{b}{2}y_j^2)}\right]} - (T_1 + v_1) - \alpha \sum_{i=1}^{n} x_i e^{-\left(ax_i+\frac{b}{2}x_i^2\right)} - \beta \sum_{j=1}^{m} y_j e^{-\left(ay_j+\frac{b}{2}y_j^2\right)} + \sum_{i=1}^{n} \frac{1}{(a+bx_i)} + \sum_{j=1}^{m} \frac{1}{(a+by_j)}, \tag{17}$$

$$\frac{\partial \ln L}{\partial b} = -\frac{\alpha}{2}\sum_{i=1}^{n}\frac{x_i^2 e^{-(ax_i+\frac{b}{2}x_i^2)}}{\left[1+\alpha e^{-(ax_i+\frac{b}{2}x_i^2)}\right]} - \frac{\beta}{2}\sum_{j=1}^{m}\frac{{y_j}^2 e^{-(ay_j+\frac{b}{2}y_j^2)}}{\left[1+\beta e^{-(ay_j+\frac{b}{2}y_j^2)}\right]} + \sum_{i=1}^{n}\frac{x_i}{(a+bx_i)} + \sum_{j=1}^{m}\frac{y_j}{(a+by_j)} - (T_2+v_2) - \frac{\alpha}{2}\sum_{i=1}^{n}{x_i}^2 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)} - \frac{\beta}{2}\sum_{j=1}^{m}{y_j}^2 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}, \tag{18}$$

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n}\frac{e^{-(ax_i+\frac{b}{2}x_i^2)}}{\left[1+\alpha e^{-(ax_i+\frac{b}{2}x_i^2)}\right]} - n + s_1, \tag{19}$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{j=1}^{m}\frac{e^{-(ay_j+\frac{b}{2}y_j^2)}}{\left[1+\beta e^{-(ay_j+\frac{b}{2}y_j^2)}\right]} - m + s_2, \tag{20}$$

From the non-linear equations (19) and (20), we can obtain the estimated values of α and β as function of a and b by replacing $\hat{\alpha}, \hat{\beta}$ in equations (17) and (18). The estimated values of a and b can be then achieved.

Finally, due to the invariance property of the MLE, the MLE of R, which denoted by $\hat{R}_{ML2}$, can be given by:

$$\hat{R}_{ML2} = \int_0^\infty \left[1+\hat{\alpha} e^{-\left(\hat{a}x+\frac{\hat{b}}{2}x^2\right)}\right]\left(\hat{a}+\hat{b}x\right) e^{-\left(\hat{a}x+\frac{\hat{b}}{2}x^2\right)} e^{-\hat{\alpha}\left(1-e^{-\left(\hat{a}x+\frac{\hat{b}}{2}x^2\right)}\right)}\left[1-e^{-\left(\hat{a}x+\frac{\hat{b}}{2}x^2\right)} e^{-\hat{\beta}\left(1-e^{-\left(\hat{a}x+\frac{\hat{b}}{2}x^2\right)}\right)}\right] dx. \tag{21}$$

4.1 Asymptotic Distributions

Since the exact distribution of $\hat{R}$ does not exist, it is essential to investigate the asymptotic behavior of the derived MLE of R, which is considered in this section. We first derive the asymptotic distribution of $\hat{\theta} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})$ and then the asymptotic distribution of $\hat{R}$ will be obtained accordingly. We then, based on the asymptotic distribution of $\hat{R}$, calculate the asymptotic confidence interval of R. We denote the observed information matrix of $\theta = (a, b, \alpha, \beta)$ by:

$$I = [I_{ij}]_{i,j=1,2,3,4},$$

where,

$$I(\theta) = \begin{pmatrix} \frac{\partial^2 l}{\partial a^2} & \frac{\partial^2 l}{\partial a \partial b} & \frac{\partial^2 l}{\partial a \partial \alpha} & \frac{\partial^2 l}{\partial a \partial \beta} \\ \frac{\partial^2 l}{\partial b \partial a} & \frac{\partial^2 l}{\partial b^2} & \frac{\partial^2 l}{\partial b \partial \alpha} & \frac{\partial^2 l}{\partial b \partial \beta} \\ \frac{\partial^2 l}{\partial \alpha \partial a} & \frac{\partial^2 l}{\partial \alpha \partial b} & \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} \\ \frac{\partial^2 l}{\partial \beta \partial a} & \frac{\partial^2 l}{\partial \beta \partial b} & \frac{\partial^2 l}{\partial \beta \partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} \end{pmatrix} \tag{22}$$

$$I_{11} = -\frac{\partial^2 l}{\partial a^2} = \sum_{i=1}^{n}\frac{\left[x_i \alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2 - \left[\alpha x_i^2 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]}{\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2} + \sum_{j=1}^{m}\frac{\left[y_j \beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2 - \left[\beta y_j^2 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]}{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2} + \sum_{i=1}^{n}\frac{1}{[a+bx_i]^2} + \sum_{i=1}^{n}\frac{1}{[a+by_j]^2} - \alpha\sum_{i=1}^{n}x_i^2 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)} - \beta\sum_{j=1}^{m}y_j^2 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}, \tag{23}$$

$$I_{12}=I_{21}=-\frac{\partial^2 l}{\partial a\partial b}=-\frac{\partial^2 l}{\partial b\partial a}=\sum_{i=1}^{n}\frac{\frac{\alpha^2 x_i^3}{2}e^{-2\left(ax_i+\frac{b}{2}x_i^2\right)}-\frac{\alpha x_i^3}{2}e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]}{\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2}+$$
$$\sum_{j=1}^{m}\frac{\frac{\beta^2 y_j^3}{2}e^{-2\left(ay_j+\frac{b}{2}y_j^2\right)}-\frac{\beta y_j^3}{2}e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]}{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2}+\sum_{i=1}^{n}\frac{x_i}{(a+bx_i)^2}+\sum_{i=1}^{n}\frac{y_j}{(a+by_j)^2}-\frac{\alpha}{2}\sum_{i=1}^{n}x_i^3 e^{-(ax_i+\frac{b}{2}x_i^2)}-$$
$$\frac{\beta}{2}\sum_{j=1}^{m}y_j^3 e^{-(ay_j+\frac{b}{2}y_j^2)}, \qquad (24)$$

$$I_{13}=I_{31}=-\frac{\partial^2 l}{\partial a\partial \alpha}=-\frac{\partial^2 l}{\partial \alpha\partial a}=\sum_{i=1}^{n}\frac{x_i e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]-\alpha x_i e^{-2(ax_i+\frac{b}{2}x_i^2)}}{\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2}+\sum_{i=1}^{n}x_i e^{-(ax_i+\frac{b}{2}x_i^2)}, \qquad (25)$$

$$I_{14}=I_{41}=-\frac{\partial^2 l}{\partial a\partial \beta}=-\frac{\partial^2 l}{\partial \beta\partial a}=+\sum_{j=1}^{m}\frac{y_j e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]-\beta y_j e^{-2\left(ay_j+\frac{b}{2}y_j^2\right)}}{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2}+\sum_{j=1}^{m}y_j e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}, \qquad (26)$$

$$I_{22}=-\frac{\partial^2 l}{\partial b^2}=\sum_{i=1}^{n}\frac{\left[\frac{\alpha}{2}x_i^2 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2-\frac{\alpha}{4}x_i^4 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]}{\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2}+$$
$$\sum_{j=1}^{m}\frac{\left[\frac{\beta}{2}y_j^2 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2-\frac{\beta}{4}y_j^4 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]}{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2}+\sum_{i=1}^{n}\frac{x_i^2}{[a+bx_i]^2}+\sum_{j=1}^{m}\frac{y_j^2}{[a+by_j]^2}-\frac{\alpha}{4}\sum_{i=1}^{n}x_i^4 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}-$$
$$\frac{\beta}{4}\sum_{j=1}^{m}y_j^4 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}, \qquad (27)$$

$$I_{23}=I_{32}=-\frac{\partial^2 l}{\partial b\partial \alpha}=-\frac{\partial^2 l}{\partial \alpha\partial b}=\sum_{i=1}^{n}\frac{\frac{1}{2}x_i^2 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]-\frac{\alpha}{2}x_i^2 e^{-2\left(ax_i+\frac{b}{2}x_i^2\right)}}{\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2}+\frac{1}{2}\sum_{i=1}^{n}{x_i}^2 e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}, \qquad (28)$$

$$I_{24}=I_{42}=-\frac{\partial^2 l}{\partial b\partial \beta}=-\frac{\partial^2 l}{\partial \beta\partial b}=\sum_{j=1}^{m}\frac{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]\left[\frac{y_j^2}{2}e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]-\frac{\beta}{2}y_j^2 e^{-2\left(ay_j+\frac{b}{2}y_j^2\right)}}{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2}+\frac{1}{2}\sum_{j=1}^{m}{y_j}^2 e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}, \qquad (29)$$

$$I_{33}=-\frac{\partial^2 l}{\partial \alpha^2}=\sum_{i=1}^{n}\frac{e^{-2\left(ax_i+\frac{b}{2}x_i^2\right)}}{\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right]^2}, \qquad (30)$$

$$I_{34}=I_{43}=-\frac{\partial^2 l}{\partial \alpha\partial \beta}=-\frac{\partial^2 l}{\partial \beta\partial \alpha}=0, \qquad (31)$$

$$\mathrm{I}_{44}=-\frac{\partial^2 l}{\partial \beta^2}=\sum_{j=1}^{m}\frac{e^{-2\left(ay_j+\frac{b}{2}y_j^2\right)}}{\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right]^2}, \qquad (32)$$

Theorem 1:

If $\mathrm{n},\mathrm{m}\to\infty$ *and* ${}^{n}/_{m}\to p$, then we have:

$$\left[\sqrt{n}(\hat{a}-a), \sqrt{m}(\hat{b}-b), \sqrt{n}(\hat{\alpha}-\alpha), \sqrt{m}(\hat{\beta}-\beta)\right] \to \mathrm{N}_4\left(0, \mathrm{u}^{-1}(\theta)\right),$$

where,

$$\mathrm{U}(\theta) = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{pmatrix}, \tag{33}$$

and

$$u_{11} = \frac{1}{n} I_{11}, \quad u_{12}=u_{21} = \frac{1}{n} I_{12}, \quad u_{13} = u_{31} = \frac{1}{n} I_{13}, \; u_{14} = u_{41} = \frac{\sqrt{p}}{n} I_{14}$$
$$u_{22} = \frac{1}{n} I_{22}, \quad u_{23}=u_{32} \frac{1}{n} I_{23}, \quad u_{24} = u_{42} = \frac{\sqrt{p}}{n} I_{24}, \; u_{33} = \frac{1}{n} I_{33}, \; u_{44} = \frac{1}{m} I_{44},$$

Proof: the proof follows from the asymptotic normality of MLE as in [10] and the references listed below.

Theorem 2:

If $(n, m) \to \infty$ and $^n/_m \to p$, then

$$\sqrt{n}\left(\hat{R} - R\right) \to N(0, \sigma^2), \tag{34}$$

where,

$$\sigma^2 = \frac{1}{k(\alpha+\beta)^4}\left[\beta^2 a_{33} - 2\sqrt{p}\alpha\beta a_{34} + \alpha^2 p a_{44}\right], \tag{35}$$

$$\begin{aligned} K = {} & u_{11}u_{22}u_{33}u_{44} + u_{12}u_{23}u_{31}u_{44} + u_{12}u_{24}u_{33}u_{41} + u_{13}u_{21}u_{32}u_{44} + u_{13}u_{24}u_{31}u_{42} + u_{14}u_{21}u_{33}u_{42} + u_{14}u_{23}u_{32}u_{41} \\ & - u_{11}u_{23}u_{32}u_{44} - u_{11}u_{24}u_{33}u_{42} - u_{12}u_{21}u_{33}u_{44} - u_{13}u_{22}u_{31}u_{44} - u_{13}u_{24}u_{32}u_{41} - u_{14}u_{22}u_{33}u_{41} \\ & - u_{14}u_{23}u_{31}u_{42}, \end{aligned}$$

$$a_{33} = u_{11}u_{22}u_{44} + u_{12}u_{24}u_{41} + u_{14}u_{21}u_{42} - u_{11}u_{24}u_{42.} - u_{12}u_{21}u_{44} - u_{14}u_{22}u_{41},$$

$$a_{34} = u_{11}u_{24}u_{32} + u_{14}u_{22}u_{31} - u_{12}u_{24}u_{31.} - u_{14}u_{21}u_{32},$$

$$a_{44} = u_{11}u_{22}u_{33} + u_{12}u_{23}u_{31} + u_{13}u_{21}u_{32} - u_{11}u_{23}u_{32.} - u_{12}u_{21}u_{33} - u_{13}u_{22}u_{31}.$$

The motivation behind the asymptotic distribution presented above for $\hat{R}$ is to construct an asymptotic confidence interval for R. In order to construct this confidence interval, we first need to estimate σ^{2*}**.** Due to the invariance property of the MLE, we can estimate σ^{2*} by estimating its elements via replacing (a, b, α, β) by their estimated values$(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})$.

4.2 Bayes Estimation of R with Common Unknown Parameters $(a_1 = a_2 = a, \; b_1 = b_2 = b)$

The Jeffrey's prior in [8] is justified on the grounds of its invariance under parameterization according to Sinha as in [9]. The prior distribution ofa, b, α, β, are, respectively:

$$g(a) \propto \frac{1}{a}; \; a > 0, \tag{36}$$
$$g(b) \propto \frac{1}{b}; \; b > 0, \tag{37}$$
$$g(\alpha) \propto \frac{1}{\alpha}; \; \alpha > 0, \tag{38}$$
and
$$g(\beta) \propto \frac{1}{\beta}; \; \beta > 0, \tag{39}$$

where, all of these are independent.
Then,

$$g(a, b, \alpha, \beta) \propto \frac{1}{a.b.\alpha.\beta}; \; a, b, \alpha, \beta > 0, \tag{40}$$

Combining the joint prior density of (a, b, α, β) and the likelihood function to obtain the joint posterior density of (a, b, α, β) as follows:

$$\pi(a,b,\alpha,\beta/x,y)=\frac{g(a,b,\alpha,\beta)L(x,y/a,b,\alpha,\beta)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(a,b,\alpha,\beta)L(x,y/a,b,\alpha,\beta)da\,db\,d\alpha d\beta}, \quad (41)$$

$$\pi(a,b,\alpha,\beta/x,y)=\frac{k^{-1}}{a.b.\alpha.\beta}\prod_{i=1}^{n}\left[1+\alpha e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right](a+bx_i)e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}e^{-\alpha\left(1-e^{-\left(ax_i+\frac{b}{2}x_i^2\right)}\right)}\prod_{j=1}^{m}\left[1+\beta e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right](a+by_j)e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}e^{-\beta\left(1-e^{-\left(ay_j+\frac{b}{2}y_j^2\right)}\right)},$$
(42)

where,

$$k=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(a,b,\alpha,\beta)L(x,y/a,b,\alpha,\beta)da\,db\,d\alpha d\beta,$$

Therefore, the Bayes estimator of R denoted by $\hat{R}_{BS2}$ under squared error loss function is given by:

$$\hat{R}_{BS2}=E(R/x,y)=\int_0^1 R\pi(R/x,y)dR. \quad (43)$$

The estimated value of R under squared error loss function cannot be computed analytically. Alternatively, numerical solution based on MATHCAD15 program is employed to evaluate $\hat{R}_{BS2}$ for different values of the parameters.

5 Estimation of R in the General Case $(a_1,a_2,b_1,b_2,\alpha,\beta)$

In this section, the Stress-Strength model $R=P(Y<X)$ will be estimated, when X ~ELED $(a_1,\ b_1,\ \alpha)$ and Y ~ELED$(a_2,\ b_2,\ \beta)$. We present the MLE of R and its associated confidence intervals in the next subsection.

5.1. Maximum Likelihood Estimation of R in the General Case $(a_1,a_2,b_1,b_2,\alpha,\beta)$

Suppose further $(X_1,X_2,\ldots,X_n)$ is a random sample from ELED (a_1,b_1,α) and $(Y_1,Y_2,\ldots\ldots,Y_m)$ is an-other random sample from ELED(a_2,b_2,β). We can find the Stress-Strength parameter, as equation (3), then R can be written as:

$$R=\int_0^\infty\left(1+\alpha e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)(a_1+b_1x)\,e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}e^{-\alpha\left[1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right]}\left[1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)})e^{-\beta\left[1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right]}\right]. \quad (44)$$

The log-likelihood function of the observed samples is presented as:

$$ln\left(L(a_1,b_1,\alpha,a_2,b_2,\beta)\right)=\sum_{i=1}^{n}ln\left[1+\alpha e^{-(a_1x_i+\frac{b_1}{2}x_i^2)}\right]+\sum_{j=1}^{m}ln\left[1+\beta e^{-(a_2y_j+\frac{b_2}{2}y_j^2)}\right]+\sum_{i=1}^{n}ln(a_1+b_1x_i)+\sum_{j=1}^{m}ln(a_2+b_2y_j)-(a_1T_1+b_1T_2)-(a_2v_1+b_2v_2)-\alpha[n-s_1]-\beta[m-s_2], \quad (45)$$

where

$$s_1=\sum_{i=1}^{n}e^{-\left(ax_i+\frac{b}{2}x_i^2\right)},\quad s_2=\sum_{j=1}^{m}e^{-\left(ay_j+\frac{b}{2}y_j^2\right)},$$
$$T_j=\frac{1}{j}\sum_{i=1}^{n}x_i^j,\ j=1,2\ ,\ v_i=\frac{1}{i}\sum_{i=1}^{n}y_j^i,\ i=1,2$$

The estimated values of $(a_1,b_1,\alpha_1,\beta_1,a_2,b_2,\alpha_2,\beta_2)$ denoted $(\widehat{a_1},\widehat{b_1},\widehat{\alpha_1},\widehat{\beta_1},\widehat{a_2},\widehat{b_2},\widehat{\alpha_2},\widehat{\beta_2})$ can be derived as follows:

$$\frac{\partial\ln L}{\partial a_1}=\sum_{i=1}^{n}\frac{1}{(a_1+b_1x_i)}-\alpha\sum_{i=1}^{n}\frac{x_ie^{-\left(a_1x_i+\frac{b_1}{2}x_i^2\right)}}{\left[1+\alpha e^{-\left(a_1x_i+\frac{b_1}{2}x_i^2\right)}\right]}-T_1-\alpha\sum_{i=1}^{n}x_ie^{-\left(a_1x_i+\frac{b_1}{2}x_i^2\right)}, \quad (46)$$

$$\frac{\partial \ln L}{\partial a_2} = \sum_{j=1}^{m} \frac{1}{(a_2+b_2 y_j)} - \beta \sum_{j=1}^{m} \frac{y_j e^{-(a_2 y_i + \frac{b_2}{2} y_j^2)}}{\left[1+\beta e^{-(a_2 x_i + \frac{b_2}{2} y_j^2)}\right]} - v_1 - \beta \sum_{j=1}^{m} y_j e^{-\left(a_2 y_j + \frac{b_2}{2} y_j^2\right)}, \quad (47)$$

$$\frac{\partial \ln L}{\partial b_1} = \sum_{i=1}^{n} \frac{x_i}{(a_1+b_1 x_i)} - \frac{\alpha}{2} \sum_{i=1}^{n} \frac{x_i^{\,2} e^{-\left(a_1 x_i + \frac{b_1}{2} x_i^2\right)}}{\left[1+\alpha e^{-\left(a_1 x_i + \frac{b_1}{2} x_i^2\right)}\right]} - T_2 - \frac{\alpha}{2} \sum_{i=1}^{n} x_i^{\,2} e^{-\left(a_1 x_i + \frac{b_1}{2} x_i^2\right)}, \quad (48)$$

$$\frac{\partial \ln L}{\partial b_2} = \sum_{j=1}^{m} \frac{y_j}{(a_2+b_2 y_j)} - \frac{\beta}{2} \sum_{j=1}^{m} \frac{y_j^{\,2} e^{-\left(a_2 x_i + \frac{b_2}{2} y_j^2\right)}}{\left[1+\beta e^{-\left(a_2 y_i + \frac{b_2}{2} y_j^2\right)}\right]} - v_2 - \frac{\beta}{2} \sum_{j=1}^{m} y_j^{\,2} e^{-\left(a_2 y_j + \frac{b_2}{2} y_j^2\right)}, \quad (49)$$

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n} \frac{e^{-(a_1 x_i + \frac{b_1}{2} x_i^2)}}{\left[1+\alpha e^{-(a_1 x_i + \frac{b_1}{2} x_i^2)}\right]} - n + s_1, \quad (50)$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{j=1}^{m} \frac{e^{-(a_2 y_i + \frac{b_2}{2} y_j^2)}}{\left[1+\beta e^{-(a_2 y_i + \frac{b_2}{2} y_j^2)}\right]} - m + s_2. \quad (51)$$

Then, the MLE of R is given by:

$$\hat{R}_{ML3} = \int_0^\infty \left\{1 + \hat{\alpha} e^{-\left(\widehat{a_1} x + \frac{\widehat{b_1}}{2} x^2\right)}\right\} \left(\widehat{a_1} + \widehat{b_1} x\right) e^{-\left(\widehat{a_1} x + \frac{\widehat{b_1}}{2} x^2\right)} e^{-\hat{\alpha}\left[1 - e^{-\left(\widehat{a_1} x + \frac{\widehat{b_1}}{2} x^2\right)}\right]} \left[1 - e^{-\left(\widehat{a_2} x + \frac{\widehat{b_2}}{2} x^2\right)} e^{-\hat{\beta}\left[1 - e^{-\left(\widehat{a_2} x + \frac{\widehat{b_2}}{2} x^2\right)}\right]}\right] dx. \quad (52)$$

5.2 Bayes Estimation of R in the General Case

The non-informative prior distribution of $(a_1, a_2, b_1, b_2, \alpha, \beta)$ is:

$$g(a_1, a_2, b_1, b_2, \alpha, \beta) \propto \frac{1}{a_1} \frac{1}{a_2} \frac{1}{b_1} \frac{1}{b_2} \frac{1}{\alpha} \frac{1}{\beta}; \; a_1 > 0 \;\; a_2 > 0 \;\; b_1 > 0 \;\; b_2 > 0 \;\; \alpha > 0 \;\; \beta > 0. \quad (53)$$

Combining the joint prior density of $(a_1, a_2, b_1, b_2, \alpha, \beta)$ and the likelihood function to obtain the joint posterior density of (a, b, α, β) as the form:

$$L\,(a_1, a_2, b_1, b_2, \alpha, \beta | X, Y) = \frac{g(a_1, a_2, b_1, b_2, \alpha, \beta)\, L\,(X, Y | a_1, a_2, b_1, b_2, \alpha, \beta)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty g(a_1, a_2, b_1, b_2, \alpha, \beta) L\,(X, Y | a_1, a_2, b_1, b_2, \alpha, \beta)\, da_1\, db_1 da_2 db_2\, d\alpha\, d\beta}, \quad (54)$$

$$\pi(a_1, a_2, b_1, b_2, \alpha, \beta / x, y) = \frac{k^{-1}}{a_1.a_2.b_1.b_2.\alpha.\beta} \prod_{i=1}^{n} \left[1 + \alpha e^{-\left(a_1 x_i + \frac{b_1}{2} x_i^2\right)}\right] (a_1 + b_1 x_i) e^{-\left(a_1 x_i + \frac{b_1}{2} x_i^2\right)} e^{-\alpha\left(1 - e^{-\left(a_1 x_i + \frac{b_1}{2} x_i^2\right)}\right)} \prod_{j=1}^{m} \left[1 + \beta e^{-\left(a_2 y_j + \frac{b_2}{2} y_j^2\right)}\right] \left(a_2 + b_2 y_j\right) e^{-\left(a_2 y_j + \frac{b_2}{2} y_j^2\right)} e^{-\beta\left(1 - e^{-\left(a_2 y_j + \frac{b_2}{2} y_j^2\right)}\right)},$$

(55)

where,

$$k = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty g(a_1, a_2, b_1, b_2, \alpha, \beta) L(x, y / a_1, a_2, b_1, b_2 \alpha\,, \beta) da_1\, db_1 da_2 db_2\, d\alpha\, d\beta,$$

Therefore, the Bayesian estimator of R under squared error loss function is given by:

$$\hat{R}_{BS3} = E\,(R | x, y) = \int_0^1 R\, \pi(R | x, y)\, dR. \quad (56)$$

The Bayes estimate of R under squared error loss cannot be computed analytically. Alternatively, numerical solution based on MATHCAD15 program is employed to evaluate $\hat{R}_{BS3}$ for different values of the parameters.

6 Simulation Results

In this section, Monte Carlo simulation is performed to test the behavior of the proposed estimators for different sample sizes and for different parameter values.

The Performances of the maximum likelihood estimates and the Bayes estimates are compared in terms of biases and mean squares errors (MSEs). (BE) are computed based non-informative prior distribution, where we have three cases for ELED.

Case 1: when the parameters (a, b) are fixed where $a_1 = a_2 = a = 1$ and $b_1 = b_2 = b = 2$ for X and Y , respectively.

Case 2: when the parameters (a, b) are common unknown where $a_1 = a_2 = a$ and $b_1 = b_2 = b$ for X and Y, respectively.

Case 3: when the parameters are unknown and different for X andY.

We will obtain the MLE of the unknown parameters of the ELED to obtain the MSE of the reliability function and the Bayes estimators of the reliability function of ELED distribution will be obtained by the same way. The following steps will be considered to obtain the estimators:

Step (1): Generate random samples $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ from ELED with sample sizes 5, 10, 15, 25 and 50, then we have three cases:

Case 1, the parameters (a, b) are fixed where $a = 1$ and $b = 2$, where parameters $\alpha = 0.01$, $\beta = 0.5$, $\alpha = 0.02$, $\beta = 0.3$ and $\alpha = 0.01$, $\beta = 0.2$ are unknown for X, Y , respectively.Where their results shown in tables 1 and table 2.

Table 1: MLE of R in the Case of Fixed Parameters $(a$, b) where **α**, **β** are Unknown

(n, m)	a	b	α	β	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	R	$Bias(R)$	MSE
(5 , 5)	1	2	0.01	0.5	0.911	2.166	0.01	0.408	0.36	-0.133	0.018
	1	2	0.02	0.3	0.97	1.881	$1.99*10^{-2}$	0.289	0.41	0.207	0.043
	1	2	0.01	0.2	1.039	2.021	$1.1*10^{-2}$	0.201	0.438	0.012	$1.488*10^{-4}$
(5 , 10)	1	2	0.01	0.5	1.227	1.842	0.01	0.761	0.36	0.479	0.23
	1	2	0.02	0.3	1.069	2.1	0.19	0.437	0.41	0.553	0.305
	1	2	0.01	0.2	1.144	1.965	$1.002*10^{-2}$	0.187	0.438	0.384	0.147
(10,10)	1	2	0.01	0.5	1.108	1.348	0.01	0.522	0.365	$5.504*10^{-3}$	$3.029*10^{-5}$
	1	2	0.02	0.3	1.159	1.933	$1.98*10^{-2}$	0.488	0.41	-0.043	$1.819*10^{-3}$
	1	2	0.01	0.2	1.042	2.02	$1.004*10^{-2}$	0.195	0.438	-0.105	0.011

Table 2: BE of R in the Case of Fixed $(a$, b) Parameters where (**α**, **β**) are Unknown.

(n, m)	a	b	α	β	R	$Bias(R)$	MSE
(5 , 5)	1	2	0.01	0.5	0.36	-0.22	0.129
	1	2	0.02	0.3	0.41	-0.32	0.168
	1	2	0.01	0.2	0.438	0.38	0.192
(5 , 10)	1	2	0.01	0.5	0.36	0.16	0.12
	1	2	0.02	0.3	0.41	0.31	0.11
	1	2	0.01	0.2	0.438	0.39	0.102

(10,10)	1	2	0.01	0.5	0.365	0.22	0.11
	1	2	0.02	0.3	0.41	-0.21	0.09
	1	2	0.01	0.2	0.438	0.38	0.99

Case 2, the parameters (a, b) are unknown also, the parameters α, β are unknown for X, Y , respectively where the initial values can be taken as follow:

$$a = 0.5, \quad b = 0.5, \qquad \alpha = 1, \quad \beta = 1$$
$$a = 0.5, \quad b = 0.2, \qquad \alpha = 3, \quad \beta = 4$$
$$a = 0.3, \quad b = 0.6, \qquad \alpha = 2, \quad \beta = 3$$

Their results for MLE, BE results also the asymptotic confidence interval shown in tables 3 and 4.

Case 3, where the parameters are unknown and different $(a_1, b_1, a_2, b_2, \alpha, \beta)$ are unknown for X, Y. So we take different initial values such as:

$$a_1 = 1, \quad a_2 = 1.5, \quad b_1 = 0.5, \quad b_2 = 0.5, \quad \alpha = 1.5, \quad \beta = 1$$
$$a_1 = 1, \quad a_2 = 1.5, \quad b_1 = 0.5, \quad b_2 = 0.5, \quad \alpha = 1.5, \quad \beta = 1.5$$
$$a_1 = 0.5, \quad a_2 = 2, \quad b_1 = 0.7, \quad b_2 = 2, \quad \alpha = 1.5, \quad \beta = 1.5$$

For each values of the sample size and $(a_1, a_2, b_1, b_2, \alpha, \beta)$ we will generate 1000 random samples from ELED.

Step (2): Using the Eq. (8) to find the MLE of R and use Eq. (14) to find Bayes estimators of R by using non-informative prior distribution for the first case. Also , Using the Eq. (21) to find the MLE of R and use Eq.(43) to find Bayes estimators of R using non-informative prior for the second case.

Table 3: MLE of R in the Case of Common Unknown Parameter where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are unknown and Asymptotic Confidence Intervals of R Based on MLE at Significance Level 0.05.

(n, m)	a	b	α	β	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	R	CI_{AS}	$Bias(R)$	MSE
(15 , 15)	0.5	0.5	1	1	0.484	0.572	1.161	1.08	0.164	(0.214 , 0.244)	0.06	$9.298*10^{-1}$
	0.5	0.2	3	4	0.555	0.2	2.212	3.424	0.03	(0.171 , 0.199)	0.07	$2.142*10^{-3}$
	0.3	0.6	2	3	0.315	0.64	2.019	2.463	0.087	(0.113 , 0.133)	-0.088	$1.32*10^{-3}$
(15 , 25)	0.5	0.5	1	1	0.533	0.547	1.179	1.318	0.164	(0.205 , 0.184)	0.061	$3.39*10^{-3}$
	0.5	0.2	3	4	0.537	0.215	1.215	1.9	0.03	(0.116, 0.135)	-0.058	$2.62*10^{-3}$
	0.3	0.6	2	3	0.377	0.639	2.206	3.095	0.087	(0.115 , 0.132)	-0.063	$1.88*10^{-3}$
(25, 25)	0.5	0.5	1	1	0.519	0.538	0.904	0.921	0.164	(0.106 , 0.126)	0.031	$1.116*10^{-3}$
	0.5	0.2	3	4	0.414	0.22	2.606	3.666	0.03	(0.076 , 0.091)	-0.012	$1.021*10^{-3}$
	0.3	0.6	2	3	0.294	0.68	2.533	2.867	0.045	(0.014 , 0.026)	-0.033	$1.033*10^{-3}$

Table 4: BE of R in the Case of Common Unknown Parameter where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are Unknown.

(n, m)	a	b	α	β	R	Bias(R)	MSE
(25 , 25)	0.5	0.5	1	1	0.216	-0.211	$4.604*10^{-3}$
	0.5	0.2	3	4	0.64	-0.064	$4.104*10^{-8}$
	0.3	0.6	2	3	0.087	-0.088	$4.093*10^{-5}$
(25 , 50)	0.5	0.5	1	1	0.216	-0.16	$4.304*10^{-4}$
	0.5	0.2	3	4	0.64	-0.044	$4.404*10^{-6}$

	0.3	0.6	2	3	0.087	-0.187	$3.593*10^{-6}$
(50,50)	0.5	0.5	1	1	0.216	-0.05	$2.204*10^{-4}$
	0.5	0.2	3	4	0.64	-0.034	$3.144*10^{-6}$
	0.3	0.6	2	3	0.087	-0.037	$2.791*10^{-6}$

Table 5: MLE of R in the General Case where $\boldsymbol{a_1, a_2, b_1, b_2, \alpha, \beta}$ are Unknown

(n,m)	a_1	b_1	a_2	b_2	α	β	$\hat{a}_1$	$\hat{b}_1$	$\hat{a}_2$	$\hat{b}_2$	$\hat{\alpha}$	$\hat{\beta}$	R	$Bias(R)$	MSE
(25, 25)	1	0.5	1.5	0.5	1.5	1	1.023	0.418	1.549	0.668	1.672	1.116	0.229	$3.10*10^{-2}$	$9.87*10^{-4}$
	1	0.5	1.5	0.5	1.5	1.5	1.057	0.759	1.517	0.495	1.482	1.884	0.117	$-3.30*10^{-2}$	$1.22*10^{-3}$
	0.5	0.7	2	2	1.5	1.5	0.953	0.867	1.795	2.654	1.438	1.419	0.208	$-5.72*10^{-3}$	$3.27*10^{-5}$
(25 ,50)	1	0.5	1.5	0.5	1.5	1	1.039	0.968	1.553	0.658	1.106	1.378	0.229	$2.50*10^{-2}$	$6.214*10^{-4}$
	1	0.5	1.5	0.5	1.5	1.5	0.925	0.654	1.385	0.48	1.713	1.282	0.177	$-3.30*10^{-2}$	$1.121*10^{-3}$
	0.5	0.7	2	2	1.5	1.5	0.563	0.751	2.294	1.894	1.586	1.498	0.208	$-1.00*10^{-2}$	$1.056*10^{-4}$
(50,50)	1	0.5	1.5	0.5	1.5	1	1.196	0.516	1.744	0.418	1.484	1.047	0.229	$-2.219*10^{2}$	$4.925*10^{-3}$
	1	0.5	1.5	0.5	1.5	1.5	1.204	0.512	1.602	0.898	1.477	1.529	0.177	$-3.843*10^{2}$	$1.477*10^{-5}$
	0.5	0.7	2	2	1.5	1.5	0.615	0.897	1.921	1.745	1.481	1.531	0.208	0.022	$4.851*10^{-4}$

Finally, Using the Eq. (52) to find the MLE of R and use Eq.(56) to find BE of R using non-informative prior for the third case.

Step (3): we take the average of their 1000 values then calculate the bias and the mean square error of R in different cases where $\hat{R}_{ML1}, \hat{R}_{BL1}$ for first case, $\hat{R}_{ML2}, \hat{R}_{BL2}$ for second case and $\hat{R}_{ML3}$, $\hat{R}_{BL3}$ for the last case.

It can be noted that for large sample sizes, the performance of the MLE are better than the BE of R in terms of biases and MSEs. It is also observed that when (n, m) increases, the MSE and biases decrease for MLE and increase for BE except in some points in large sample only.

In addition, it is noted that for the small samples the MSE of both MLE and BE of R increase little bit than it recorded for large samples. The confidence intervals CI_{AS}, performs quite well as the sample sizes increases, have large interval length.

Table 6: Bayes Estimation of R in the General Case where $\boldsymbol{a_1, a_2, b_1, b_2, \alpha, \beta}$ are Unknown

(n, m)	a_1	b_1	a_2	b_2	α	β	R	Bias(R)	MSE
(25 , 25)	1	0.5	1.5	0.5	1.5	1	0.229	-0.219	0.066
	1	0.5	1.5	0.5	1.5	1.5	0.177	-0.206	0.054
	0.5	0.7	2	2	1.5	1.5	0.208	-0.202	0.05
(25 , 50)	1	0.5	1.5	0.5	1.5	1	0.229	-0.22	0.053
	1	0.5	1.5	0.5	1.5	1.5	0.177	-0.227	0.031
	0.5	0.7	2	2	1.5	1.5	0.208	-0.108	0.033
(50,50)	1	0.5	1.5	0.5	1.5	1	0.229	-0.022	0.051
	1	0.5	1.5	0.5	1.5	1.5	0.177	-0.006	0.003
	0.5	0.7	2	2	1.5	1.5	0.208	-0.004	0.001

7 Conclusion

In this paper, the problem of estimating P(Y <X) for ELED has been addressed. The asymptotic distribution of the maximum likelihood estimator has been used to construct confidence intervals whose function is good except for small sample sizes. It has been observed that the BE behave quite converge to zero. Moreover, the MSE of the estimates of R by two ways of estimators decrease as the sample sizes increase. The performance of the MLE estimators is also quite well and the MSEs of MLE estimators are smaller than the MSEs of Bayes. Finally, the average lengths of all intervals decrease rapidly as (n, m)increase.

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