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On the dynamics of a 4d local Cournot model

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Abstract: The aim of this paper is to analyze the dynamics of a four dimensional system described in [Appl. Math. Let., 23, 836-838, (2010)], which generalizes the classical Cournot competition in a local way. In particular, by considering an aggregation parameter of production costs, we are able to describe for some ranges of the parameter: the periodic structure, the asymptotic dynamics and the chaotic properties of the system via the study of its topological entropy and the Li and Yorke chaos.

Keywords: Cournot duopoly, fixed point, periodic point, topological entropy, Li and Yorke chaos

1 Introduction

The Cournot duopoly is a competitive game devised in the early part of last century and named after Augustin Cournot [6], who is considered one of the forerunners of modern microeconomics. This economic process is based on two firms which produce an identical product, good and compete for control of the market. In each step of the game the firms decide on the amount of the product to be introduced onto the market in the next step. In order to take this decision, both firms know the amount of the product introduced onto the market in the previous step by the rival firm. This economic interaction is deterministic and is thus modeled by the iteration of the following two dimensional system

$$F(x,y) = (g(y), f(x)), \tag{1}$$

where both f and g are continuous self–maps defined on a compact interval which can be considered, without loss of generality, by normalization to be defined on $[0,1]$. Here, the maps f and g are called the *reaction functions* and they force the decisions taken by the firms.

Note that if firm A releases α_0 product onto the market at the beginning of the game and firm B releases β_0 product, in the next step of the game firm A will produce $g(\beta_0)$, i.e. an amount of product which directly depends on the production level of firm B in the previous step, and firm B will produce $f(\alpha_0)$ and so on. Therefore, the whole process is governed by the dynamics of system (1), which strongly depends on the dynamics of the one dimensional maps f and g.

A duopoly is an intermediate situation between a monopoly and a perfect competition, and analytically it is a more complicated case. The reason for this is that an oligopolist must consider not only the behavior of the customers, but also those of the competitors and their reactions. This classical model has been extensively studied in the literature from different points of view, see for instance [2], [7], [8], [12], [13], [16], [17] and [18].

Given a model, economists want to make predictions on the asymptotic behavior of the system, i.e. how the model will behave in the future. To do this it is essential to have a tool to measure the dynamical complexity of the model. It is known from the literature that there are plenty of examples of seemingly simple systems which have very complicated dynamics, where it is very difficult to deduce reliable information on the future of the system (see for instance papers [18], [19], [20] where some models are analyzed from a numerical point of view).

In [5], a topological characterization for the dynamical complexity of (1) was given. The precise result was the generalization for two dimensional maps of the form $(x,y) \mapsto (g(y), f(x))$ on the one hand of the one

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dimensional Misiurewicz's theorem (see [15], (1) $\Leftrightarrow$ (2)) and on the other hand of certain results proved by Sharkovskiĭ in the nineteen–sixties (see [21], (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4)). Denote by $h_{top}(\cdot)$, $\mathrm{UR}(\cdot)$, $\mathrm{Rec}(\cdot)$ and $\mathrm{AP}(\cdot)$ the *topological entropy* and the sets of *uniformly recurrent*, *recurrent* and *almost periodic points*, respectively (for definitions see [4]).

Proposition 1(Misiurewicz, Sharkovskiĭ). *Let $\phi : [0,1] \to [0,1]$ be a continuous map. Then the following properties are equivalent:*

(1)$h_{top}([0,1],\phi) = 0$,
(2)the period of any periodic point for $([0,1],\phi)$ *is a power of two,*
(3)$\mathrm{UR}([0,1],\phi) = \mathrm{Rec}([0,1],\phi)$ *and*
(4)$\mathrm{AP}([0,1],\phi) = \left\{ x \in [0,1] : \lim_{n\to+\infty} \phi^{2^n}(x) = x \right\}$.

Note that the previous characterization of dynamical simplicity is given in terms of the property "to have zero topological entropy". From a dynamic point of view when a system has zero topological entropy its dynamics are simple, and therefore predictions on its future behavior can be done in some sense, see [4]. Moreover, from the viewpoint of dynamics this type of result is the best possible, as we can check if the behavior of the system is simple or not by confirming the validity of one of the properties (1)–(4).

While dynamical properties of duopolies have been extensively studied, adjustment dynamics in Cournot processes with more than two players have received much less attention as a consequence of the difficulties which appear for studying systems with more than two dimensions. The direct generalization of the Cournot duopoly situation is the Cournot oligopoly, i.e. consider n firms which produce an identical good and in each step of the process any firm knows the amount of product generated by the $n-1$ rival firms in the previous step. Thus, the system which models the situation is of the form $F(x_1,\cdots,x_n)$ equal to

$$\begin{aligned}(f_1(x_2,x_3,\cdots,x_n), f_2(x_1,x_3,\cdots,x_n),\cdots,\\ f_n(x_1,x_2,\cdots,x_{n-1}))\,,\end{aligned} \tag{2}$$

where each $f_i : [0,1]^{n-1} \to [0,1]$ is a continuous map. We note that the reaction function f_i depends on $n-1$ variables of indices $j \in \{1,\cdots,n\}$, $j \neq i$.

For a system like (2) a characterization of dynamical simplicity does not exist in the way of Theorem 1 and is far from being obtained due to ignoring of the topological dynamics of n dimensional systems with $n > 2$ (e.g. note that for these types of systems the possible ω–limit sets of the orbits are not even characterized).

Thus, if we want to measure its dynamical complexity, we need to simplify the system at the cost of losing information by the firms on the production level of the rivals. In [10] the following model called a Cournot–like system is introduced:

Definition 1.*A map ϕ from $[0,1]^n$ into itself is Cournot–like if it is of the form:*

$$\phi(x_1,\cdots,x_n) = \left(\phi_{\sigma(1)}(x_{\sigma(1)}),\cdots,\phi_{\sigma(n)}(x_{\sigma(n)})\right),$$

where each $\phi_i : [0,1] \to [0,1]$ is continuous and σ is a cyclic permutation of $\{1,\cdots,n\}$.

In the economic situation described by the iteration of this type of model the level of information of each player is quite limited because each firm only has information on the production level of one of the other firms in the previous step of the process. It is proved in [10] that Theorem 1 works similarly for these kind of systems.

From our point of view Cournot–like models do not represent a real economic situation since it is very difficult to explain the fact that each player firm can only have information on another firm and completely ignoring the rest of the other firms' behavior. For that reason Guirao et al. [11] introduced a new model where the information level is higher than that in Cournot–like ones and where it is possible to explain using a mathematical approach its dynamical complexity.

1.1 Our model: local competition "à la Cournot"

Let $N = \{1,\cdots,n\}$ be the set of firms (i.e. rival firms which produce an identical good) and assume that they are physically located around a circle or a line. We assume that the firms compete "à la Cournot" in a local way, i.e. each firm $i \in N$ competes with its closest neighbour in the right and left direction. Let $B_i^\alpha \subseteq N$ be the neighbor located at a distance less or equal to α of the firm i in the right and left direction. If we denote by $(x_1,\cdots,x_n)$ the production of the firms at some moment in time and by $(c_1,\cdots,c_n)$ their production costs (and so we assume $c_i > 0$ for each $i = 1,\cdots,n$), the best response function for the firm i will have the form

$$\sqrt{\frac{\sum_{k\in B_i^\alpha} x_k}{c_i}} - \sum_{k\in B_i^\alpha} x_k \text{ (denoted by } \phi_i(x_{B_i^\alpha})).$$

Therefore our model is governed by

$$F(x_1,\cdots,x_n) = \left(\phi_1(x_{B_1^\alpha}),\cdots,\phi_n(x_{B_n^\alpha})\right). \tag{3}$$

Note that in this model the dimension of the reaction functions depends on the number of firms and on the size of the influence of neighboring set B_i^α. For example, considering B_i^α composed of the left and right neighbor, i.e. $\alpha = 1$, in this case, if $n = 2$ then we have the classical situation of the Cournot duopoly. Note that the model considers a situation where a price discrimination exists in the sense that the same product could have different prices depending on the market. Recall that here the market is composed of the central firm plus the influence of its neighbors given by the size of α.

Compared with Cournot–like models, from an economic point of view these types of systems with local competition in a Cournot sense are more realistic, which presents a more paradoxical situation. [11] proposes to analyze these systems from a mathematical point of view with the methods coming from the topological dynamics theory to obtain a deep understanding of them. In this sense [9] studied some aspects of the dynamics of the model (3) in the case of $\alpha = 1$ and $n = 3$.

1.2 Statements of our main results

Observe that our model is governed by (3), which is defined from the viewpoint of mathematics on a subset of

$$\left\{(x_1,\cdots,x_n)\in\mathbb{R}^n : \sum_{k\in B_i^{\alpha}} x_k \geq 0 \text{ for each } i=1,\cdots,n\right\}. \tag{4}$$

Though in a real economic situation each variable $x_1,\cdots,x_n$ should be non-negative, we are willing to manage our model in a more general setting of allowing negative $x_1,\cdots,x_n$ under the assumption (4) from a purely mathematical viewpoint.

The aim of this paper is to present a complete study of the dynamics of the system $F_4(x,y,z,w)$ given by

$$\left(\sqrt{\tfrac{y+w}{c_1}}-(y+w),\sqrt{\tfrac{x+z}{c_2}}-(x+z),\right.$$
$$\left.\sqrt{\tfrac{y+w}{c_3}}-(y+w),\sqrt{\tfrac{x+z}{c_4}}-(x+z)\right),$$

defined on a subset of

$$\Omega_4^1 := \{(x,y,z,w)\in\mathbb{R}^4 : y+w\geq 0, x+z\geq 0\}, \tag{5}$$

which is obtained from (3) for $\alpha = 1$ and $n = 4$, for convenience we also denote it by

$$\begin{aligned}&(F_{4,1}(x,y,z,w),F_{4,2}(x,y,z,w),\\&F_{4,3}(x,y,z,w),F_{4,4}(x,y,z,w)).\end{aligned} \tag{6}$$

From the dynamical viewpoint, our model is well defined over

$$(0,0,0,0)\in\Omega_4 := \text{the closure of } \bigcap_{n=-\infty}^{+\infty} F_4^n\Omega_4^1 \text{ in } \mathbb{R}^4. \tag{7}$$

To simplify our study we consider the following aggregated parameters of the production costs in terms of which we shall discuss the dynamics of the model. Let $a_1 = 1/\sqrt{c_1}+1/\sqrt{c_3} > 0, a_2 = 1/\sqrt{c_2}+1/\sqrt{c_4} > 0$ and let $a > 0$ be such that $a_1 = aa_2$. In this setting we are going to distinguish in our study between two different cases:

1. $a \in \left[\frac{5\sqrt{2}}{8},\frac{8}{5\sqrt{2}}\right]$ and

2. $a \in \left\{\frac{\sqrt{2}}{2},\sqrt{2}\right\}$.

Our main results are stated as follows.

Theorem 1.(F_4,Ω_4) *generates a discrete dynamical system in the sense that $F_4(\Omega_4)\subseteq\Omega_4$ and Ω_4 is a compact subset of $\mathbb{R}^4$.*

For $5\sqrt{2}/8 \leq a \leq 8/5\sqrt{2}$ let us introduce $P_1,\cdots,P_6$ as at the beginning of Section 4. Note here that $P_1,\cdots,P_6$ need not be six different points. Then the asymptotic behavior of the model can be characterized as follows.

Theorem 2.*Assume $5\sqrt{2}/8 \leq a \leq 8/5\sqrt{2}$. Then the period of any periodic point of (Ω_4,F_4) is at most 2. In fact, $\{(0,0,0,0)\}\cup\{P_i : i=1,\cdots,6\}$ is just the set of all periodic points of (Ω_4,F_4) and $F_4(0,0,0,0)=(0,0,0,0)$,*

$$F_4(P_1)=P_4, F_4(P_4)=P_1, F_4(P_2)=P_5,$$
$$F_4(P_5)=P_2, F_4(P_3)=P_6, F_4(P_6)=P_3.$$

Theorem 3.*Assume $5\sqrt{2}/8 \leq a \leq 8/5\sqrt{2}$ and let $p=(x,y,z,w)\in\Omega_4$.*

1. If $x+z=0=y+w$ then $F_4^n(p)=(0,0,0,0)$ for each $n\in\mathbb{N}$.
2. If $x+z=0<y+w$ then $F_4^{2n}(p)\to P_6$ and $F_4^{2n+1}(p)\to P_3$ as $n\to+\infty$.
3. If $x+z>0=y+w$ then $F_4^{2n}(p)\to P_5$ and $F_4^{2n+1}(p)\to P_2$ as $n\to+\infty$.
4. If $x+z>0<y+w$ then $F_4^{2n}(p)\to P_4$ and $F_4^{2n+1}(p)\to P_1$ as $n\to+\infty$.

Let (X,d) be a compact metric space and $T : X\to X$ a continuous map. We say (X,T) is *Li-Yorke chaotic* [3], [14] if an uncountable subset $S\subseteq X$ exits such that, for each pair (x_1,x_2) from S^2 with $x_1\neq x_2$, one has

$$\liminf_{n\to+\infty} d(T^n x_1,T^n x_2)=0$$

whereas

$$\limsup_{n\to+\infty} d(T^n x_1,T^n x_2)>0.$$

Based on the above discussions, we have:

Theorem 4.*The system (Ω_4,F_4) has positive topological entropy and is Li-Yorke chaotic if $a=\sqrt{2}/2$ or $\sqrt{2}$, and has zero topological entropy and is not Li-Yorke chaotic if $5\sqrt{2}/8\leq a\leq 8/5\sqrt{2}$.*

The structure of the paper is organized as follows.

In Section 2 we present the proof of Theorem 1 jointly with some generic results on the model. Related to our model in Section 3 we analyze the dynamics of f_a with the parameter $\sqrt{2}/2 \leq a \leq \sqrt{2}$ in preparation, for the definition of f_a see (10). Then with the help of discussions in Section 3 we prove Theorem 2, Theorem 3 and Theorem 4 in Section 4 and Section 5, respectively.

2 Rewriting the model F_4

In this section, we shall give some generic analysis about the model F_4.

First, let us prove Theorem 1.

Proof(Proof of Theorem 1). Recalling Ω_4^1 from (5), we have directly

$$F_4(\Omega_4^1) \subseteq \left(-\infty, \tfrac{1}{4c_1}\right] \times \left(-\infty, \tfrac{1}{4c_2}\right] \times \times \left(-\infty, \tfrac{1}{4c_3}\right] \times \left(-\infty, \tfrac{1}{4c_4}\right],$$

and so $\Omega_4^1 \cap F_4(\Omega_4^1)$ is a bounded subset of $\mathbb{R}^4$.

To prove the conclusion, as the map F_4 is continuous on Ω_4^1, by the construction (7), $\Omega_4 \subseteq \Omega_4^1$ is the closure of Ω_4^2 in $\mathbb{R}^4$, where

$$(0,0,0,0) \in \Omega_4^2 = \bigcap_{n=-\infty}^{+\infty} F_4^n \Omega_4^1, \tag{8}$$

we only need prove that $F_4(\Omega_4^2) \subseteq \Omega_4^2$ and Ω_4^2 is a bounded subset of $\mathbb{R}^4$. It is simple to check $F_4(\Omega_4^2) \subseteq \Omega_4^2$, and the boundedness of Ω_4^2 follows directly from that of $\Omega_4^1 \cap F_4(\Omega_4^1)$, as $\Omega_4^2 \subseteq \Omega_4^1 \cap F_4(\Omega_4^1)$. This completes the proof.

Related to F_4, now let us consider the following model

$$F_4^* : (X,Y) \mapsto \left(a_1\sqrt{Y} - 2Y, a_2\sqrt{X} - 2X\right).$$

By direct calculations $G(X,Y) := (F_4^*)^2(X,Y) =$

$$\left(a_1\sqrt{a_2\sqrt{X} - 2X} - 2\left(a_2\sqrt{X} - 2X\right),\right. \left. a_2\sqrt{a_1\sqrt{Y} - 2Y} - 2\left(a_1\sqrt{Y} - 2Y\right)\right).$$

Recalling $a = a_1/a_2 > 0$, to simplify the calculations, it is not hard to obtain:

Lemma 1. *The model G can be normalized by the map ϕ : $(X,Y) \mapsto (X/a_1^2, Y/a_2^2)$ as*

$$F_a : (x,y) \mapsto \left(\sqrt{a^{-1}\sqrt{x} - 2x} - 2\left(a^{-1}\sqrt{x} - 2x\right),\right. \left. \sqrt{a\sqrt{y} - 2y} - 2\left(a\sqrt{y} - 2y\right)\right), \tag{9}$$

that is, the models G and F_a are equivalent in the sense of $\phi \circ G = F_a \circ \phi$.

Observe that, for each $a > 0$ from (9) one has $F_a : (x,y) \mapsto (f_{a^{-1}}(x), f_a(y))$, where

$$f_a : x \mapsto \sqrt{a\sqrt{x} - 2x} - 2(a\sqrt{x} - 2x). \tag{10}$$

To understand the dynamics of G, by Lemma 1 we only need to study the map F_a for all $a > 0$, equivalently, the map f_a for all $a > 0$.

When $\sqrt{2}/2 \leq a \leq \sqrt{2}$, it is not hard to check that f_a is well defined on $[0,1/8]$, and once $f_a(x)$ is well defined for some $x \in \mathbb{R}$ then $f_a(x) \in [0,1/8]$, as

$$f_a(x) = -2\left(\sqrt{-2\left(\sqrt{x} - \frac{a}{4}\right)^2 + \frac{a^2}{8}} - \frac{1}{4}\right)^2 + \frac{1}{8},$$

which implies

$$\Omega_4 \subseteq \left\{(x,y,z,w) \in \mathbb{R}^4 : 0 \leq x+z \leq \frac{a_1^2}{8}, 0 \leq y+w \leq \frac{a_2^2}{8}\right\}. \tag{11}$$

Thus in the following, under the assumption of $\sqrt{2}/2 \leq a \leq \sqrt{2}$, we are to study the dynamics of f_a on $[0,1/8]$, which contains all the information about the dynamics of (Ω_4, F_4) via the normalization by Proposition 1.

In Figure 1 we present the morphology of f_a for different values of the parameter a.

3 The dynamics of $f_a, \sqrt{2}/2 \leq a \leq \sqrt{2}$

In this section, we shall discuss the dynamics of f_a (especially its asymptotic behavior) when the parameter $\sqrt{2}/2 \leq a \leq \sqrt{2}$ in preparation for the following sections.

First, we have:

Lemma 2. *When $a = \sqrt{2}$ or $\sqrt{2}/2$, $([0,1/8], f_a)$ has positive topological entropy.*

Proof. As the proof is similar, we only discuss the case of $a = \sqrt{2}/2$. When $a = \sqrt{2}/2$, by direct calculations it is easy to check that

(i) $f_a(0) = 0 = f_a(1/8), f_a(1/32) = 1/8$ and

(ii) $f_a(x)$ increases on $[0,1/32]$ and decreases on $[1/32,1/8]$.

From this it is well known in one dimensional dynamics that the system $([0,1/8], f_a)$ has positive topological entropy, see for example [15].

We also need the following result.

Lemma 3. *Assume $5\sqrt{2}/8 \leq a \leq 8/5\sqrt{2}$. Then $([0,1/8], f_a)$ has exactly two fixed points 0 and $\lambda_a \in [a^2/16, 1/8]$ such that $\lim_{n\to+\infty} f_a^n(x) = \lambda_a$ whenever $x \in (0,1/8]$.*

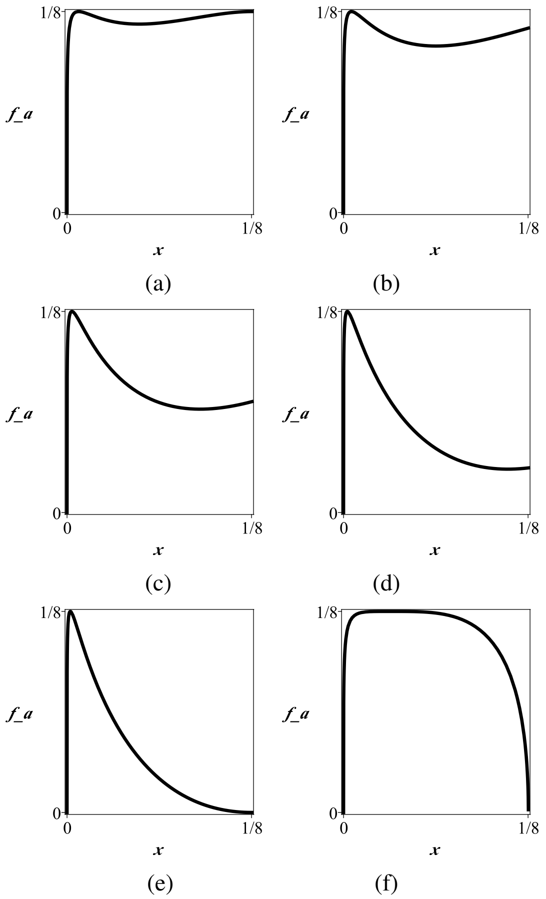

Fig. 1: Morphology of f_a for some values of the parameter. Case (a) corresponds to $a = 5\sqrt{2}/8$, case (b) corresponds to $a = 1$, case (c) corresponds to $a = 6/5$, case (d) corresponds to $a = 4/3$, case (e) corresponds to $a = \sqrt{2}$ and case (f) corresponds to $a = \sqrt{2}/2$.

In order to prove Lemma 3, we need the following instrumental results.

Lemma 4. *Assume $\sqrt{2}/2 \le a \le 8/5\sqrt{2}$. Then*

1. *in $[0,1/8]$, f_a has exactly two fixed points 0 and $\lambda_a \in [a^2/16, 1/8]$, moreover, $\lambda_a = a^2/16$ if and only if $a = 8/5\sqrt{2}$ and $\lambda_a = 1/8$ if and only if $a = 5\sqrt{2}/8$, $\lambda_1 = 1/9$.*
2. *for each $0 < x < \lambda_a$, either $f_a^n(x) \ge \lambda_a$ for some $n \in \mathbb{N}$ or $\lim_{n\to+\infty} f_a^n(x) = \lambda_a$.*

Proof. Let $0 \le x \le 1/8$ and say $X = \sqrt{x} \in [0, \sqrt{2}/4]$. It is easy to check that $f_a(x) = x$ if and only if $f(X) := 9X^3 + (4a^2+2)X - 12aX^2 - a = 0$ or $x = 0$. Whereas,

(i) $f(0) = -a < 0$, $f(a/4) = a(25a^2-32)/64 \le 0$, $f\left(\sqrt{2}/4\right) = \sqrt{2}\left(a - 5\sqrt{2}/8\right)^2 \ge 0$,
(ii) $f(Y)$ tends to $+\infty$ as Y tends to $+\infty$ and
(iii) $f'(Y) = 27Y^2 + (4a^2+2) - 24aY = 27(Y - 12a/27)^2 + \left(2 - 4a^2/3\right) > 0$ for all Y.

Thus, in $[0,1/8]$ a unique λ_a exists such that $f(\sqrt{\lambda_a}) = 0$ (and so $f_a(\lambda_a) = \lambda_a$), moreover, $\lambda_a \in [a^2/16, 1/8]$. From this we know that, in $[0,1/8]$, f_a has exactly two fixed points (and they are just 0 and λ_a).

Moreover, by direct calculations, it is not hard to check that $\lambda_a = a^2/16$ if and only if $a = 8/5\sqrt{2}$, $\lambda_a = 1/8$ if and only if $a = 5\sqrt{2}/8$ and $\lambda_1 = 1/9$. This proves (1).

In fact, the above discussions also tell us that, for $0 < x < \lambda_a$, $f(\sqrt{x}) < 0$ which is equivalent to $f_a(x) > x$. Now let $0 < x < \lambda_a$. If $f_a^n(x) < \lambda_a$ for each $n \in \mathbb{N}$ then the sequence $f_a^m(x) \nearrow z > 0$ for some $z \le \lambda_a$ as $m \nearrow +\infty$, and so $f_a(z) = z$, thus $z = \lambda_a$ follows from (1), this proves (2).

Lemma 5. *Assume $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$. Then $f_a(x) \ge a^2/16$ for each $a^2/16 \le x \le 1/8$.*

Proof. It suffices to prove that f_a is increasing on $[a^2/16, 1/8]$ and $f_a\left(a^2/16\right) \ge a^2/16$.

Note that $a \le 8/5\sqrt{2}$. So, it is easy to check

$$f_a\left(\frac{a^2}{16}\right) = \frac{\sqrt{2}a - a^2}{4} \ge \frac{a^2}{16}.$$

Now for $x \in [a^2/16, 1/8]$, $a\sqrt{x} - 2x \in [(\sqrt{2}a - 1)/4, a^2/8]$ and so (recall $5\sqrt{2}/8 \le a$)

$$\frac{a}{2\sqrt{x}} - 2 \le 0 \text{ and } \frac{1}{2\sqrt{a\sqrt{x} - 2x}} - 2 \le \frac{1}{\sqrt{\sqrt{2}a - 1}} - 2 \le 0,$$

thus

$$f_a'(x) = \left(\frac{a}{2\sqrt{x}} - 2\right)\left(\frac{1}{2\sqrt{a\sqrt{x} - 2x}} - 2\right) \ge 0.$$

That is, f_a is increasing on $[a^2/16, 1/8]$. This completes the proof.

Lemma 6. *Assume $5\sqrt{2}/8 \le a < \sqrt{2}$. Then f_a is a contraction on $[a^2/16, 1/8]$, that is, there exists $\gamma_a \in (0,1)$ such that $|f_a(x_1) - f_a(x_2)| \le \gamma_a |x_1 - x_2|$ whenever $a^2/16 \le x_1, x_2 \le 1/8$.*

Proof. The proof is completed by estimating $|f_a'(x)|$ on $[a^2/16, 1/8]$.

Recall that $\sqrt{2}/2 < a < \sqrt{2}$, for $x \in [a^2/16, 1/8]$, $a\sqrt{x} - 2x \in [(\sqrt{2}a - 1)/4, a^2/8]$ and

$$-1 < \sqrt{2}a - 2 \le \frac{a}{2\sqrt{x}} - 2 \le 0, \tag{12}$$

and so

$$\frac{\sqrt{2}}{a} - 2 \le \frac{1}{2\sqrt{a\sqrt{x} - 2x}} - 2 \le \frac{1}{\sqrt{\sqrt{2}a - 1}} - 2, \tag{13}$$

thus

$$|f_a'(x)| = \left|\frac{a}{2\sqrt{x}} - 2\right| \cdot \left|\frac{1}{2\sqrt{a\sqrt{x}-2x}} - 2\right| \le \left(2-\sqrt{2}a\right)\max\left\{\left|\frac{\sqrt{2}}{a}-2\right|, \left|\frac{1}{\sqrt{\sqrt{2}a-1}}-2\right|\right\}$$

(using (12) and (13)). Denote by γ_a the constant in the last inequality. Recall $5\sqrt{2}/8 \le a < \sqrt{2}$, then

$$0 < \left(2-\sqrt{2}a\right)\left|\frac{\sqrt{2}}{a}-2\right| < 1$$

and

$$-1 < \frac{1}{\sqrt{\sqrt{2}a-1}} - 2 \le 0,$$

and so $\gamma_a \in (0,1)$. This completes our proof.

Now we are ready to present the proof of Lemma 3.

Proof(Proof of Lemma 3). Observe $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$. By Lemma 4, $([0,1/8], f_a)$ has exactly two fixed points 0 and $\lambda_a \in [a^2/16, 1/8]$. Now let $x \in (0,1/8]$.

Possibility 1: $x \in [a^2/16, 1/8]$**.** By Lemma 5 and Lemma 6, $f_a : [a^2/16,1/8] \to [a^2/16,1/8]$ is a contraction, that is, there exists $\gamma_a \in (0,1)$ such that $|f_a(x_1) - f_a(x_2)| \le \gamma_a|x_1 - x_2|$ whenever $a^2/16 \le x_1, x_2 \le 1/8$, recalling $\lambda_a \in [a^2/16,1/8]$, then $|f_a^n(x) - \lambda_a| = |f_a^n(x) - f_a^n(\lambda_a)| \le \gamma_a^n|x-\lambda_a|$, and so $\lim\limits_{n\to+\infty} f_a^n(x) = \lambda_a$.

Possibility 2: $x \in \left(0, a^2/16\right)$**.** As $x < \lambda_a$, by Lemma 4 either $f_a^N(x) \ge \lambda_a$ for some $N \in \mathbb{N}$ or $\lim\limits_{n\to+\infty} f_a^n(x) = \lambda_a$. If $\lim\limits_{n\to+\infty} f_a^n(x) = \lambda_a$ then we are done, if $f_a^N(x) \ge \lambda_a \ge a^2/16$ for some $N \in \mathbb{N}$ then $\lim\limits_{n\to+\infty} f_a^n(x) = \lim\limits_{n\to+\infty} f_a^{N+n}(x) = \lambda_a$ by Possibility 1.

Summing up, $\lim\limits_{n\to+\infty} f_a^n(x) = \lambda_a$, completing the proof of the conclusion.

4 Proofs of Theorem 2 and Theorem 3

In this section, we aim to prove Theorem 2 and Theorem 3, and so we shall assume $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$ throughout the whole section.

As shown by Lemma 3, $f_a : [0,1/8] \to [0,1/8]$ has exactly two fixed points 0 and $\lambda_a \in [a^2/16, 1/8]$. Let us introduce:

$$P_1 = \left(a_2\sqrt{\frac{\lambda_a}{c_1}} - a_2^2\lambda_a, a_1\sqrt{\frac{\lambda_{a^{-1}}}{c_2}} - a_1^2\lambda_{a^{-1}}, a_2\sqrt{\frac{\lambda_a}{c_3}} - a_2^2\lambda_a, a_1\sqrt{\frac{\lambda_{a^{-1}}}{c_4}} - a_1^2\lambda_{a^{-1}}\right),$$

$$P_2 = \left(0, a_1\sqrt{\frac{\lambda_{a^{-1}}}{c_2}} - a_1^2\lambda_{a^{-1}}, 0, a_1\sqrt{\frac{\lambda_{a^{-1}}}{c_4}} - a_1^2\lambda_{a^{-1}}\right),$$

$$P_3 = \left(a_2\sqrt{\frac{\lambda_a}{c_1}} - a_2^2\lambda_a, 0, a_2\sqrt{\frac{\lambda_a}{c_3}} - a_2^2\lambda_a, 0\right),$$

$$P_4 = \left(a_1\sqrt{\frac{a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}}{c_1}} - a_1^2\left(a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}\right), a_2\sqrt{\frac{a\sqrt{\lambda_a}-2\lambda_a}{c_2}} - a_2^2\left(a\sqrt{\lambda_a}-2\lambda_a\right), a_1\sqrt{\frac{a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}}{c_3}} - a_1^2\left(a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}\right), a_2\sqrt{\frac{a\sqrt{\lambda_a}-2\lambda_a}{c_4}} - a_2^2\left(a\sqrt{\lambda_a}-2\lambda_a\right)\right),$$

$$P_5 = \left(a_1\sqrt{\frac{a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}}{c_1}} - a_1^2\left(a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}\right), 0, a_1\sqrt{\frac{a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}}{c_3}} - a_1^2\left(a^{-1}\sqrt{\lambda_{a^{-1}}}-2\lambda_{a^{-1}}\right), 0\right),$$

$$P_6 = \left(0, a_2\sqrt{\frac{a\sqrt{\lambda_a}-2\lambda_a}{c_2}} - a_2^2\left(a\sqrt{\lambda_a}-2\lambda_a\right), 0, a_2\sqrt{\frac{a\sqrt{\lambda_a}-2\lambda_a}{c_4}} - a_2^2\left(a\sqrt{\lambda_a}-2\lambda_a\right)\right).$$

Observe that the introduced $P_1, \cdots, P_6$ need not to be pairwise different. In fact, we have proved in Lemma 4 that $\lambda_1 = 1/9$, and so in the case of $a = 1$ one has $a_1 = a_2$ and hence

$$P_1 = P_4 = \left(\frac{a_1}{9}\left(\frac{2}{\sqrt{c_1}} - \frac{1}{\sqrt{c_3}}\right), \frac{a_2}{9}\left(\frac{2}{\sqrt{c_2}} - \frac{1}{\sqrt{c_4}}\right), \frac{a_1}{9}\left(\frac{2}{\sqrt{c_3}} - \frac{1}{\sqrt{c_1}}\right), \frac{a_2}{9}\left(\frac{2}{\sqrt{c_4}} - \frac{1}{\sqrt{c_2}}\right)\right),$$

$$P_2 = P_6 = \left(0, \frac{a_2}{9}\left(\frac{2}{\sqrt{c_2}} - \frac{1}{\sqrt{c_4}}\right), 0, \frac{a_2}{9}\left(\frac{2}{\sqrt{c_4}} - \frac{1}{\sqrt{c_2}}\right)\right),$$

$$P_3 = P_5 = \left(\frac{a_1}{9}\left(\frac{2}{\sqrt{c_1}} - \frac{1}{\sqrt{c_3}}\right), 0, \frac{a_1}{9}\left(\frac{2}{\sqrt{c_3}} - \frac{1}{\sqrt{c_1}}\right), 0\right).$$

Now let's present the proofs of Theorem 2 and Theorem 3 stated at the beginning of the paper, which explore the asymptotic behavior of the model.

Proof(Proof of Theorem 2). Denote by $\mathrm{Per}(\Omega_4, F_4)$ the set of all periodic points of (Ω_4, F_4).

Note that $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$ and $f_a(\lambda_a) = \lambda_a, f_{a^{-1}}(\lambda_{a^{-1}}) = \lambda_{a^{-1}}$, by direct calculations it is not hard to obtain

1. $\mathrm{Per}(\Omega_4, F_4) \supseteq \{(0,0,0,0)\} \cup \{P_i : i = 1, \cdots, 6\}$ (denoted by $\mathscr{P}$) and
2. $F_4(P_1) = P_4, F_4(P_4) = P_1, F_4(P_2) = P_5, F_4(P_5) = P_2, F_4(P_3) = P_6, F_4(P_6) = P_3$ and $F_4(0,0,0,0) = (0,0,0,0)$.

Now we aim to complete our proof by claiming $\mathrm{Per}(\Omega_4, F_4) \subseteq \mathscr{P}$.

In fact, say $(x,y,z,w) \in \mathrm{Per}(\Omega_4, F_4)$, then $(x+z, y+w)$ is a periodic point of F_4^* and hence a periodic point of G (by the constructions of F_4^* and G), thus $\left((x+z)/a_1^2, (y+w)/a_2^2\right)$ is a periodic point of F_a (using Lemma 1). As $F_a : (x', y') \mapsto (f_{a^{-1}}(x'), f_a(y'))$, $(x+z)/a_1^2$ and $(y+w)/a_2^2$ are periodic points of $f_{a^{-1}}$ and f_a, respectively. Recalling $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$, by (11) and Lemma 3 we have $(x+z)/a_1^2 \in \{0, \lambda_{a^{-1}}\}$ and $(y+w)/a_2^2 \in \{0, \lambda_a\}$, and so by direct calculations (combined with the fact that (x,y,z,w) is a periodic point of (Ω_4, F_4)) it is not hard to obtain $(x,y,z,w) \in \mathscr{P}$, which completes the proof.

Proof(Proof of Theorem 3). As the proof is similar, we only need prove the second item.

Let $\pi : (x,y,z,w) \mapsto (x+z, y+w)$. Then $\pi \circ F_4 = F_4^* \circ \pi$ from the construction of F_4^*. Recall $\phi \circ G = F_a \circ \phi$ from Lemma 1, we have $\phi \circ \pi \circ (F_4^2)^n = F_a^n \circ \phi \circ \pi$ for each $n \in \mathbb{N}$. Now recall $F_{4,1}, F_{4,2}, F_{4,3}, F_{4,4}$ from (6), one has

$$F_{4,2}(F_4^{2n+1}(x,y,z,w)) + F_{4,4}(F_4^{2n+1}(x,y,z,w)) = \\ = a_2^2 f_a^{n+1}\left(\frac{y+w}{a_2^2}\right)$$

and

$$F_{4,1}(F_4^{2n+1}(x,y,z,w)) + F_{4,3}(F_4^{2n+1}(x,y,z,w)) = \\ = a_1^2 f_{a^{-1}}^{n+1}\left(\frac{x+z}{a_1^2}\right)$$

for each $n \in \mathbb{N}$. As $x+z = 0 < y+w$, using (11) and Lemma 3 we obtain

$$F_{4,1}(F_4^{2n+1}(x,y,z,w)) + F_{4,3}(F_4^{2n+1}(x,y,z,w)) = 0$$

for each $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \left[F_{4,2}(F_4^{2n+1}(x,y,z,w)) + F_{4,4}(F_4^{2n+1}(x,y,z,w))\right] = \\ = a_2^2 \lambda_a,$$

and hence

$$\lim_{n\to+\infty} F_4^{2n+1}(x,y,z,w) = P_3 \text{ and } \lim_{n\to+\infty} F_4^{2n}(x,y,z,w) = P_6.$$

This completes the proof.

5 Proof of Theorem 4

In this section, we aim to discuss the topological entropy of the model with the parameter $a = \sqrt{2}$ or $\sqrt{2}/2$ or $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$.

The following result is well known, see for example [3], [22].

Proposition 2. *Let (X,d) be a compact metric space and $T : X \to X$ a continuous map. Assume $h_{top}(X,T) > 0$. Then (X,T) is Li-Yorke chaotic.*

We also need the following result with Ω_4^2 introduced by (8) in Section 2.

Lemma 7. *Assume $\sqrt{2}/2 \le a \le \sqrt{2}$ and $0 \le A_1 \le a_1^2/8, 0 \le A_2 \le a_2^2/8$. Then there exists $(x,y,z,w) \in \Omega_4^2$ such that $x+z = A_1$ and $y+w = A_2$.*

To prove Lemma 7, we need the following easy observation.

Lemma 8. *Assume $\sqrt{2}/2 \le a \le \sqrt{2}$. Then $\emptyset \neq \Omega_4^3 \subseteq F_4(\Omega_4^3)$, where Ω_4^3 denotes the set of all $(x,y,z,w) \in \Omega_4^1$ satisfying*

$$0 \le x+z \le \frac{a_1^2}{8}$$

and

$$x - z = \left(\frac{1}{\sqrt{c_1}} - \frac{1}{\sqrt{c_3}}\right)\left(\frac{a_1}{4} - \sqrt{\frac{a_1^2}{16} - \frac{x+z}{2}}\right),$$

$$0 \le y+w \le \frac{a_2^2}{8}$$

and

$$y - w = \left(\frac{1}{\sqrt{c_2}} - \frac{1}{\sqrt{c_4}}\right)\left(\frac{a_2}{4} - \sqrt{\frac{a_2^2}{16} - \frac{y+w}{2}}\right).$$

Proof. Obviously $\Omega_4^3 \neq \emptyset$. Recall $\sqrt{2}/2 \le a_1/a_2 \le \sqrt{2}$ from the assumption. Now let $(x,y,z,w) \in \Omega_4^3$, it is easy to see that there exists $(x',y',z',w') \in \Omega_4^3$ satisfying

$$y' + w' = \left(\frac{a_1}{4} - \sqrt{\frac{a_1^2}{16} - \frac{x+z}{2}}\right)^2$$

and

$$x' + z' = \left(\frac{a_2}{4} - \sqrt{\frac{a_2^2}{16} - \frac{y+w}{2}}\right)^2.$$

By direct calculation $F_4(x',y',z',w') = (x,y,z,w)$, which completes the proof.

Proof(Proof of Proposition 7). Using (8), we only need to show that there exists $(x,y,z,w) \in \mathbb{R}^4$ such that $x+z = A_1, y+w = A_2$ and $(x,y,z,w) \in F_4^n \Omega_4^1$ for all $n \in \mathbb{Z}$.

As $\sqrt{2}/2 \le a \le \sqrt{2}$ and $0 \le A_1 \le a_1^2/8, 0 \le A_2 \le a_2^2/8$, it is easy to check

$$\{(x,y,z,w) \in \mathbb{R}^4 : x+z = A_1, y+w = A_2\} \subseteq \bigcap_{n=-\infty}^{0} F_4^n(\Omega_4^1). \tag{14}$$

Let Ω_4^3 be the subset introduced by Lemma 8. Then we have

$$\Omega_4^3 \subseteq \bigcap_{n=1}^{+\infty} F_4^n(\Omega_4^1).$$

Combined with (14), $\emptyset \neq \Omega_4^3 \cap \{(x,y,z,w) \in \mathbb{R}^4 : x+z = A_1, y+w = A_2\} \subseteq \Omega_4^2$.

With the above preparation, now we can present the proof of Theorem 4.

Proof(Proof of Theorem 4). First we consider the case of $a = \sqrt{2}$ or $\sqrt{2}/2$.

From the construction of F_4^* (and combined with (11)), we have that the system $\left([0,a_1^2/8] \times [0,a_2^2/8], F_4^*\right)$ is a factor of (Ω_4, F_4) in the sense of $\pi \circ F_4 = F_4^* \circ \pi$, where

$$\pi : \Omega_4 \to \left[0, \frac{a_1^2}{8}\right] \times \left[0, \frac{a_2^2}{8}\right], (x,y,z,w) \mapsto (x+z, y+w)$$

is a continuous surjection (using Lemma 7), by [1, Theorem 5],

$$h_{\text{top}}(\Omega_4, F_4) \ge h_{\text{top}}\left(\left[0, \frac{a_1^2}{8}\right] \times \left[0, \frac{a_2^2}{8}\right], F_4^*\right). \tag{15}$$

Whereas, using Lemma 1 the system $\left([0,a_1^2/8] \times [0,a_2^2/8], G\right)$ allows a factor $([0,1/8] \times [0,1/8], f_{a^{-1}} \times f_a)$ via the normalization map $(X,Y) \mapsto (X/a_1^2, Y/a_2^2)$, thus

$$\begin{aligned}
0 &< h_{\text{top}}\left(\left[0, \frac{1}{8}\right] \times \left[0, \frac{1}{8}\right], f_{a^{-1}} \times f_a\right) \\
&\quad \text{(using Lemma 2 and [1, Theorem 3])} \\
&\le h_{\text{top}}\left(\left[0, \frac{a_1^2}{8}\right] \times \left[0, \frac{a_2^2}{8}\right], G\right) \\
&\quad \text{(applying [1, Theorem 5] again)} \\
&= 2h_{\text{top}}\left(\left[0, \frac{a_1^2}{8}\right] \times \left[0, \frac{a_2^2}{8}\right], F_4^*\right) \\
&\quad \text{(by [1, Theorem 2]).}
\end{aligned}$$

We obtain that $h_{\text{top}}(\Omega_4, F_4) > 0$ by (15) and (Ω_4, F_4) is Li-Yorke chaotic by Proposition 2.

Now let us consider the case of $5\sqrt{2}/8 \le a \le 8/5\sqrt{2}$.

Denote by A_j the set of all points in Ω_4 satisfying (j) of Theorem 3, $j = 1,2,3,4$. Let d be the Euclidean metric on Ω_4. Then, for any five points $(x_i,y_i,z_i,w_i), i = 1,2,3,4,5$ from Ω_4, there exist $j \in \{1,2,3,4\}$ and $1 \le k < l \le 5$ such that both (x_k,y_k,z_k,w_k) and (x_l,y_l,z_l,w_l) are contained in A_j and hence (using Theorem 3)

$$\lim_{n\to+\infty} d(F_4^n(x_k,y_k,z_k,w_k), F_4^n(x_l,y_l,z_l,w_l)) = 0.$$

In other words, for the compact metric space Ω_4, it is simple to check that $F_4 : \Omega_4 \to \Omega_4$ is continuous. From the above discussions, there is no uncountable subset $S \subseteq \Omega_4$ such that, for each pair $((x_1,y_1,z_1,w_1),(x_2,y_2,z_2,w_2))$ from S^2 with $(x_1,y_1,z_1,w_1) \neq (x_2,y_2,z_2,w_2)$, one has

$$\liminf_{n\to+\infty} d(F_4^n(x_1,y_1,z_1,w_1), F_4^n(x_2,y_2,z_2,w_2)) = 0 \text{ whereas}$$

$$\limsup_{n\to+\infty} d(F_4^n(x_1,y_1,z_1,w_1), F_4^n(x_2,y_2,z_2,w_2)) > 0.$$

This implies that (Ω_4, F_4) is not Li–Yorke chaotic and $h_{\text{top}}(\Omega_4, F_4) = 0$ thus ending the proof.

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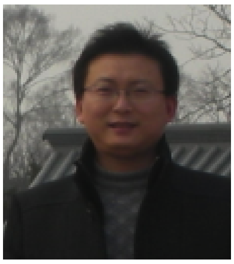

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