Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/070303

Higher Order Nonlinear Multi-Point Fractional Boundary Value Problems

İsmail Yaslan

Department of Mathematics, Pamukkale University, , 20070 Denizli, Turkey

Received: 26 Apr. 2018, Revised: 21 Feb. 2019, Accepted: 03 Mar. 2019
Published online: 1 Jul. 2021

Abstract: In this study, we investigate the conditions for the existence of at least one and three positive solutions to nonlinear higher order multi-point fractional boundary value problems using Krasnosel'skii fixed point theorem and the five functionals fixed point theorem, respectively.

Keywords: Boundary value problems, cone, fixed point theorems, positive solutions, Riemann-Liouville fractional derivative, integral boundary conditions.

1 Introduction

In this paper, we consider the m-point boundary value problem (BVP) for higher order fractional differential equation

$$\begin{cases} -D_{0^+}^{\eta-2}(u''(t)) + f(t,u(t)) = 0, \ t \in [0,1], \\ u''(0) = u'''(0) = \ldots = u^{(n-2)}(0) = 0, \ u'''(1) = 0, \\ \alpha u(0) - \beta u'(0) = \sum\limits_{p=1}^{m-2} a_p \int\limits_0^{\xi_p} u(s)ds, \\ \gamma u(1) + \delta u'(1) = \sum\limits_{p=1}^{m-2} b_p \int\limits_0^{\xi_p} u(s)ds, \end{cases} \tag{1}$$

where $D_{0^+}^{\eta-2}$ is the Riemann-Liouville fractional derivative of order $\eta-2$. Throughout the paper we suppose that $m,n \geq 3$, $n-1 < \eta \leq n$ where $n,m \in \mathbb{N}$ and α, β, γ, $\delta > 0$, a_p, $b_p \geq 0$ are given constants and $0 < \xi_1 < \ldots < \xi_{m-2} < 1$. We assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Fractional calculus is the extension of integer order calculus to arbitrary order calculus. Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as control theory, physics, chemistry, biology, economics, mechanics and electromagnetic, see [1,2,3,4]. Recently, several papers have addressed the existence and uniqueness of boundary value problems for nonlinear differential equations of fractional order. For examples and recent development of the topic, see [5,6,7,8,9,10] and references therein. Boundary value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems and arise in the study of various biological, physical and chemical processes [11,12,13,14] such as heat conduction, thermo-elasticity, chemical engineering, underground water flow and plasma physics. In [15,16,17,18, 19,20,21,22,23,24], some results on the existence of positive solutions of the boundary value problems for some specific fractional differential equations with integral boundary conditions have been obtained.

Bai and Lü [25] as well as Jiang and Yuan [26] considered the Dirichlet-type fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}(u(t)) + f(t,u(t)) = 0, \ 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

* Corresponding author e-mail: iyaslan@pau.edu.tr

where $1 < \alpha \leq 2$ and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Liang and Song [27] investigated the following nonlinear fractional three-point boundary-value problem:

$$\begin{cases} D_{0^+}^{\alpha}(u(t)) + f(t,u(t)) = 0, \ 0 < t < 1, \ 2 < \alpha \leq 3, \\ u(0) = u'(0) = 0, \ u'(1) = \int\limits_0^{\eta} u(s)ds. \end{cases}$$

Zhang and Han [28] are concerned with the existence and uniqueness of positive solutions for the following singular nonlinear fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}(x(t)) + f(t,x(t)) = 0, \ 0 < t < 1, \ n-1 < \alpha \leq n, \ \alpha \geq 2, \\ x^{(k)}(0) = 0, \ 0 \leq k \leq n-2, \ x(1) = \int\limits_0^1 x(s)dA(s). \end{cases}$$

Wang and Zhang [29] explored the existence of one and two positive solutions of the nonlinear higher order fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}(u(t)) + h(t)f(t,u(t)) = 0, \ 0 < t < 1, \ \alpha \in (n-1,n], \ \alpha > 2, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \ u^{(i)}(1) = \lambda \int\limits_0^{\eta} u(s)ds, \end{cases}$$

where $\eta \in (0,1]$, $i \in \mathbb{N}$ and $0 \leq i \leq n-2$.

Jleli et al. [30] considered the fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}(u(t)) + q(t)u(t) = 0, \ a < t < b, \ n-1 < \alpha \leq n, \ n \geq 2, \\ u(a) = u'(a) = \ldots = u^{(n-2)}(a) = 0, \ u(b) = I_a^{\alpha}(hu)(b). \end{cases}$$

Yaslan and Günendi [31] investigated the existence of positive solutions to multi-point boundary value problems for higher order fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \ t \in [0,1], \ n-1 < \alpha \leq n, \ n \geq 3, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \\ u(1) = \sum\limits_{p=1}^{m-2} a_p \int\limits_0^{\xi_p} u(s)ds, \end{cases}$$

where $a_p \geq 0$ are given constants and $0 < \xi_1 < \ldots < \xi_{m-2} < 1$.

Jin et al. [32] addressed the existence of a positive solution for the fractional boundary value problem

$$\begin{cases} {}^{C}D^{p}(u(t)) = \lambda h(t)f(t,u(t)), \ t \in (0,1), \\ u(0) - \alpha u(1) = \int\limits_0^1 g_0(s)u(s)ds, \\ u'(0) - b^{C}D^{q}u(1) = \int\limits_0^1 g_1(s)u(s)ds, \\ u''(0) = u'''(0) = \ldots = u^{(n-1)}(0) = 0, \end{cases}$$

where ${}^{C}D$ is the standard Caputo derivative, $n \geq 3$ is an integer, $p \in (n-1,n)$, $0 < q < 1$, $0 < a < 1$, $0 < b < \Gamma(2-q)$ are real numbers.

In [33], Günendi and Yaslan investigated the conditions for the existence of at least one, two and three positive solutions for the BVP (1) using four functionals fixed point theorem, Avery-Henderson fixed point theorem and Leggett-Williams fixed point theorem, respectively.

In this paper, conditions for the existence of at least one positive solutions to the BVP ((1)) are first discussed by using the Krasnosel'skii fixed point theorem. Then, we apply the five functionals fixed point theorem to prove the existence of at least three positive solutions to the BVP ((1)).

2 Some lemmas

We give some notations and prove several lemmas which are needed later.

Definition 1. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $u:(0,\infty)\to\mathbb{R}$ is defined by*

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds$$

where $n = [\alpha]+1$.

Definition 2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u:(0,\infty)\to\mathbb{R}$ is given by*

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 1. *([1]) The equality $D^{\gamma}_{0+}I^{\gamma}_{0+}f(t) = f(t)$, $\gamma > 0$, holds for $f \in L(0,1)$.*

Lemma 2. *([1]) Let $\alpha > 0$. Then the differential equation $D^{\alpha}_{0+}u = 0$ has a unique solution $u(t) = c_1t^{\alpha-1}+c_2t^{\alpha-2}+\ldots+c_nt^{\alpha-n}$, $c_i\in\mathbb{R}$, $i = 1,\ldots,n$, where $n-1<\alpha\le n$.*

Lemma 3. *([1]) Let $\alpha > 0$. Then the following equality holds for $u\in L(0,1)$, $D^{\alpha}_{0+}u\in L(0,1)$;*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\ldots+c_nt^{\alpha-n},$$

$c_i\in\mathbb{R}$, $i = 1,\ldots,n$, where $n-1<\alpha\le n$.

If we take $-u''(t) = y(t)$, the BVP

$$\begin{cases} -D^{\eta-2}_{0+}(u''(t))+f(t,u(t)) = 0,\ t\in[0,1], \\ u''(0) = u'''(0) = \ldots = u^{(n-2)}(0) = 0,\ u'''(1) = 0 \end{cases}$$

becomes

$$\begin{cases} D^{\eta-2}_{0+}y(t)+f(t,u(t)) = 0,\ t\in[0,1], \\ y(0) = y'(0) = \ldots = y^{(n-4)}(0) = 0,\ y'(1) = 0. \end{cases} \tag{2}$$

We denote by $AC[0,1]$ the space of real valued and absolutely continuous functions on $[0,1]$. Also, we denote by $AC^n[0,1]$ the space of real valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[0,1]$ with $f^{(n-1)}\in AC[0,1]$.

Lemma 4. *([33]) Let $u\in C^{(n-2)}[0,1]\cap AC^{\eta}[0,1]$. If $y\in C^{(n-4)}[0,1]\cap AC^{\eta-2}[0,1]$, then the boundary value problem (2) has a unique solution*

$$y(t) = \int_0^1 H(t,s)f(s,u(s))ds$$

where

$$H(t,s) = \begin{cases} \frac{(1-s)^{\eta-4}t^{\eta-3}}{\Gamma(\eta-2)}, & t\le s, \\ \frac{(1-s)^{\eta-4}t^{\eta-3}-(t-s)^{\eta-3}}{\Gamma(\eta-2)}, & t\ge s. \end{cases}$$

Now, we find the solution of the BVP

$$\begin{cases} -u''(t) = y(t),\ t\in[0,1], \\ \alpha u(0)-\beta u'(0) = \sum\limits_{p=1}^{m-2} a_p\int\limits_0^{\xi_p} u(s)ds \\ \gamma u(1)+\delta u'(1) = \sum\limits_{p=1}^{m-2} b_p\int\limits_0^{\xi_p} u(s)ds. \end{cases} \tag{3}$$

Let us define $\theta(t)$ and $\varphi(t)$ be the solutions of the corresponding homogeneous equation

$$u''(t) = 0 \tag{4}$$

under the initial conditions

$$\begin{aligned} \theta(0) = \beta,\, \theta'(0) = \alpha, \\ \varphi(1) = \delta,\, \varphi'(1) = -\gamma. \end{aligned} \tag{5}$$

From (4) and (5), we can obtain

$$\theta(t) = \alpha t + \beta,\ \ \varphi(t) = \gamma + \delta - \gamma t.$$

If we define $D := \alpha\gamma + \alpha\delta + \beta\gamma$, then the Green's function for the BVP (3) is

$$G(t,s) = \frac{1}{D}\begin{cases} \theta(t)\varphi(s),\, 0 \le t \le s \le 1 \\ \theta(s)\varphi(t),\, 0 \le s \le t \le 1. \end{cases} \tag{6}$$

Lemma 5._([33]) The solution of the BVP (3) is_

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\theta(t)}{D}\sum_{p=1}^{m-2} b_p \int_0^{\xi_p} u(s)ds + \frac{\varphi(t)}{D}\sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s)ds$$

where $G(t,s)$ is given by (6).

Lemma 6._([33]) The Green's function $G(t,s)$ in (6), $\theta(t)$ and $\varphi(t)$ satisfy_

$0 < G(t,s) \le G(s,s),\ \ 0 \le \theta(t) \le \theta(1),\ \ 0 \le \varphi(t) \le \varphi(0)$

for $(t,s) \in [0,1] \times [0,1]$.

Lemma 7._([33])_

The Green's function $G(t,s)$ in (6), $\theta(t)$ and $\varphi(t)$ satisfy

$G(t,s) \ge zG(s,s),\ \ \theta(t) \ge z\theta(1),\ \ \varphi(t) \ge z\varphi(0)$

where

$$z = \min\left\{\frac{\beta}{\alpha+\beta}, \frac{\delta}{\gamma+\delta}\right\} \in (0,1) \tag{7}$$

for $(t,s) \in [0,1] \times [0,1]$.

Lemma 8._([33]) For $t,s \in [0,1]$, we have $0 \le H(t,s) \le H(1,s)$._

Lemma 9._([33])_ $\min_{t\in[\xi_{m-2},1]} H(t,s) \ge k^{\eta-3}H(1,s)$ _for $0 \le t,s \le 1$, where $k \in (0,\xi_{m-2})$ is a constant._

From Lemma 4 and Lemma 5, we know that $u(t)$ is a solution of the problem (1) if and only if

$$u(t) = \int_0^1 G(t,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds + \frac{\theta(t)}{D}\sum_{p=1}^{m-2} b_p \int_0^{\xi_p} u(s)ds + \frac{\varphi(t)}{D}\sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s)ds. \tag{8}$$

Let $\mathbb{B}$ denote the Banach space $C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the cone $P \subset \mathbb{B}$ by

$$P = \{u \in \mathbb{B} : u(t) \ge 0 \text{ for } \forall t \in [0,1],\ \min_{t\in[0,1]} u(t) \ge z\|u\|\}, \tag{9}$$

where z is given in (7).

BVP (1) is equivalent to the nonlinear integral equation (8). We can define the operator $A : P \to \mathbb{B}$ by

$$Au(t) = \int_0^1 G(t,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds + \frac{\theta(t)}{D}\sum_{p=1}^{m-2} b_p \int_0^{\xi_p} u(s)ds + \frac{\varphi(t)}{D}\sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s)ds,$$

for $u \in P$. Hence, solving (8) in P is equivalent to finding fixed points of the operator A.

Lemma 10.*([33]) $A : P \to P$.*

We state the fixed point theorems to prove the main results of this paper.

Theorem 1.*[34] (Krasnosel'skii Fixed Point Theorem)Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either
(i) $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$;
or
(ii) $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$
hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let φ, η, θ be nonnegative continuous convex functionals on the cone P, and γ, Ψ nonnegative continuous concave functionals on the cone P. For nonnegative numbers h, p, q, d and r, define the following convex sets:

$$\begin{cases} P(\varphi, r) = \{x \in P : \varphi(x) < r\}, \\ P(\varphi, \gamma, p, r) = \{x \in P : p \leq \gamma(x), \varphi(x) \leq r\}, \\ Q(\varphi, \eta, d, r) = \{x \in P : \eta(x) \leq d, \varphi(x) \leq r\}, \\ P(\varphi, \theta, \gamma, p, q, r) = \{x \in P : p \leq \gamma(x), \theta(x) \leq q, \varphi(x) \leq r\}, \\ Q(\varphi, \eta, \Psi, h, d, r) = \{x \in P : h \leq \Psi(x), \eta(x) \leq d, \varphi(x) \leq r\}. \end{cases} \tag{10}$$

Now, we give the five functionals fixed point theorem found in [35].

Theorem 2.*(Five Functionals Fixed Point Theorem) Let P be a cone in a real Banach space E. Suppose that there exist nonnegative numbers r and M, nonnegative continuous concave functionals γ and Ψ on P, and nonnegative continuous convex functionals φ, η and θ on P, with*

$$\gamma(x) \leq \eta(x), \|x\| \leq M\varphi(x), \forall x \in \overline{P(\varphi, r)}.$$

Suppose that $A : \overline{P(\varphi, r)} \to \overline{P(\varphi, r)}$ is a completely continuous and there exist nonnegative numbers h, p, k, q, with $0 < p < q$ such that

(i)$\{x \in P(\varphi, \theta, \gamma, q, k, r) : \gamma(x) > q\} \neq \emptyset$ and $\gamma(Ax) > q$ for $x \in P(\varphi, \theta, \gamma, q, k, r)$,
(ii)$\{x \in Q(\varphi, \eta, \Psi, h, p, r) : \eta(x) < p\} \neq \emptyset$ and $\eta(Ax) < p$ for $x \in Q(\varphi, \eta, \Psi, h, p, r)$,
(iii)$\gamma(Ax) > q$, for $x \in P(\varphi, \gamma, q, r)$, with $\theta(Ax) > k$,
(iv)$\eta(Ax) < p$, for $x \in Q(\varphi, \eta, p, r)$, with $\Psi(Ax) < h$,

then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, r)}$ such that

$$\eta(x_1) < p, \gamma(x_2) > q, \eta(x_3) > p \text{ with } \gamma(x_3) < q.$$

3 Existence of positive solutions

We define

$$M = \int_0^1 H(1, \tau) d\tau, \tag{11}$$

$$L = \int_0^1 G(s, s) ds, \tag{12}$$

$$I = \int_{\xi_{m-2}}^1 G(s, s) ds \tag{13}$$

and

$$K = \frac{1}{D}\left((\alpha+\beta)\sum_{p=1}^{m-2} b_p\xi_p + (\gamma+\delta)\sum_{p=1}^{m-2} a_p\xi_p\right). \tag{14}$$

To prove the existence of at least one positive solution for the BVP (1), we apply the Krasnosel'skii Fixed Point Theorem.

Theorem 3. *Assume $u \in C^{(n-2)}[0,1] \cap AC^{\eta}[0,1]$. Let there exist numbers $0 < r < R < \infty$ such that*

$$f(s,u(s)) \leq \frac{(1-K)u(s)}{ML}, \textit{ for } (s,u(s)) \in [0,1]\times[0,r],$$

and

$$f(s,u(s)) \geq \frac{u(s)}{z^2 IM}, \textit{ for } (s,u(s)) \in [0,\xi_{m-2}]\times[R,\infty).$$

Then the BVP (1) has at least one positive solution.

Proof. The operator $A : P \to P$ is completely continuous by a standard application of the Arzelà-Ascoli theorem. If we let

$$\Omega_1 := \{u \in P : \|u\| < r\},$$

then for $u \in P \cap \partial\Omega_1$, we get

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]}\left(\int_0^1 G(t,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds\right.\\
&\quad \left. + \frac{\theta(t)}{D}\sum_{p=1}^{m-2} b_p\int_0^{\xi_p} u(s)ds + \frac{\varphi(t)}{D}\sum_{p=1}^{m-2} a_p\int_0^{\xi_p} u(s)ds\right)\\
&\leq \int_0^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds\\
&\quad + \frac{\theta(1)}{D}\sum_{p=1}^{m-2} b_p\int_0^{\xi_p} u(s)ds + \frac{\varphi(0)}{D}\sum_{p=1}^{m-2} a_p\int_0^{\xi_p} u(s)ds\\
&\leq \frac{(1-K)\|u\|}{ML}ML + K\|u\|\\
&= \|u\|.
\end{aligned}$$

Thus, $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$.
Let us now set

$$\Omega_2 := \{u \in P : \|u\| < \frac{1}{z}R\}.$$

Then $u \in P \cap \partial\Omega_2$ implies

$$u(t) \geq z\|u\| = R, \ \ t \in [0,1].$$

Thus,

$$\|Au\| \geq z\int_0^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds$$
$$\geq z\int_{\xi_{m-2}}^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds$$
$$\geq k^{\eta-3}z\int_{\xi_{m-2}}^1 G(s,s)\int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau ds$$
$$\geq k^{\eta-3}z\frac{z\|u\|}{k^{\eta-3}z^2IM}IM$$
$$= \|u\|.$$

Hence, $\|Au\| \geq \|u\|$ for $u \in P\cap\partial\Omega_2$. By the first part of Theorem 1, A has a fixed point in $P\cap(\overline{\Omega_2}\setminus\Omega_1)$, such that $r \leq \|u\| \leq \frac{1}{z}R$. Therefore, the BVP (1) has at least one positive solution.

Now, we will apply the five functionals fixed point theorem to investigate the existence of at least three positive solutions for the BVP (1).

Theorem 4. *Let $u \in C^{(n-2)}[0,1]\cap AC^{\eta}[0,1]$. Assume that there exist constants a,b,c with $0<a<b<\frac{b}{z}<c$ such that the function f satisfies the following conditions:*

(i) $f(t,u(t)) \leq \frac{(1-K)c}{ML}$ *for* $(t,u(t)) \in [0,1]\times[0,c]$,
(ii) $f(t,u(t)) > \frac{b}{k^{\eta-3}zIM}$ *for* $(t,u(t)) \in [\xi_{m-2},1]\times[b,\frac{b}{z}]$,
(iii) $f(t,u(t)) < \frac{(1-K)a}{ML}$ *for* $(t,u(t)) \in [0,1]\times[0,a]$,

where M,L,I,K are as defined in (11), (12), (13), (14), respectively. Then the BVP (1) has at least three positive solutions u_1,u_2 and u_3 such that

$$\max_{t\in[0,1]} u_1(t) < a < \max_{t\in[0,1]} u_3(t),$$
$$\min_{t\in[\xi_{m-2},1]} u_3(t) < b < \min_{[\xi_{m-2},1]} u_2(t).$$

Proof. Define the cone P as in (9) and define these maps $\zeta(u) = \Psi(u) = \min_{t\in[\xi_{m-2},1]} u(t)$, $\nu(u) = \max_{t\in[\xi_{m-2},1]} u(t)$, and $\phi(u) = \omega(u) = \max_{t\in[0,1]} u(t)$. Then ζ and Ψ are nonnegative continuous concave functionals on P, and ϕ,ω and ν are nonnegative continuous convex functionals on P. Let $P(\phi,c)$, $P(\phi,\zeta,a,c)$, $Q(\phi,\omega,d,c)$, $P(\phi,\nu,\zeta,a,b,c)$ and $Q(\phi,\omega,\Psi,h,d,c)$ be defined by (10). It is clear that

$$\zeta(u) \leq \omega(u),\ \|u\| = \phi(u),\ \forall u \in \overline{P(\phi,c)}.$$

If $u \in \overline{P(\phi,c)}$, then we have $u(t) \in [0,c]$ for all $t \in [0,1]$. By hypothesis (i), we get

$$\phi(Au) \leq \int_0^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds + \frac{\theta(1)}{D}\sum_{p=1}^{m-2} b_p\int_0^{\xi_p} u(s)ds + \frac{\varphi(0)}{D}\sum_{p=1}^{m-2} a_p\int_0^{\xi_p} u(s)ds$$
$$\leq \frac{(1-K)c}{ML}ML + Kc$$
$$= c.$$

This proves that $A:\overline{P(\phi,c)} \to \overline{P(\phi,c)}$.

Now we verify that the remaining conditions of Theorem 2.

Let $u_1 = b+\varepsilon_1$ such that $0<\varepsilon_1<(\frac{1}{z}-1)b$. Since $\zeta(u_1) = b+\varepsilon_1 > b$, $\nu(u_1) = b+\varepsilon_1 < \frac{b}{z}$ and $\phi(u_1) = b+\varepsilon_1 < \frac{b}{z} < c$, we obtain $\{u \in P(\phi,\nu,\zeta,b,\frac{b}{z},c) : \zeta(u) > b\} \neq \emptyset$.

If $u \in P(\phi,\nu,\zeta,b,\frac{b}{z},c)$, we have $b \leq u(t) \leq \frac{b}{z}$ for all $t \in [\xi_{m-2},1]$. Using the hypothesis (ii), we get

$$
\begin{aligned}
\zeta(Au) &\geq z\int_0^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds \geq z\int_{\xi_{m-2}}^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds\\
&\geq k^{\eta-3}z\int_{\xi_{m-2}}^1 G(s,s)\int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau ds\\
&\geq k^{\eta-3}z\frac{b}{k^{\eta-3}zIM}IM\\
&= b.
\end{aligned}
$$

Thus, the condition (i) of Theorem 2 holds.

Let $u_2 = a-\varepsilon_2$ such that $0<\varepsilon_2<(1-z)a$. Since $\omega(u_2)=a-\varepsilon_2<a$, $\Psi(u_2)=a-\varepsilon_2>za$ and $\phi(u_2)=a-\varepsilon_2<c$, we find $\{u\in Q(\phi,\omega,\Psi,za,a,c):\omega(u)<a\}\neq\emptyset$. If $u\in Q(\phi,\omega,\Psi,za,a,c)$, then we obtain $0\leq u(t)\leq a$, for $t\in[0,1]$. Hence,

$$
\begin{aligned}
\omega(Au) &\leq \int_0^1 G(s,s)\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau ds\\
&+\frac{\theta(1)}{D}\sum_{p=1}^{m-2}b_p\int_0^{\xi_p}u(s)ds+\frac{\varphi(0)}{D}\sum_{p=1}^{m-2}a_p\int_0^{\xi_p}u(s)ds\\
&<\frac{(1-K)a}{ML}ML+Ka\\
&=a
\end{aligned}
$$

by hypothesis (iii). It follows that condition (ii) of Theorem 2 is fulfilled.

The conditions (iii) and (iv) of Theorem 2 are clear.

This completes the proof.

Example 1. Taking $n=m=4$, $\xi_1=\frac{1}{3}$, $\xi_2=\frac{1}{2}$, $\alpha=\gamma=1$, $\beta=\delta=2$, $a_1=b_1=1$, $a_2=b_2=\frac{1}{2}$, $k=\frac{1}{4}$ and $\eta=\frac{7}{2}$, we consider the boundary value problem

$$
\begin{cases}
-D_{0^+}^{\frac{3}{2}}(u''(t))+\frac{10u^2}{u^2+1}=0,\ t\in[0,1],\\
u''(0)=0,\ u'''(1)=0,\\
u(0)-2u'(0)=\int_0^{\frac{1}{3}}u(s)ds+\frac{1}{2}\int_0^{\frac{1}{2}}u(s)ds,\\
u(1)+2u'(1)=\int_0^{\frac{1}{3}}u(s)ds+\frac{1}{2}\int_0^{\frac{1}{2}}u(s)ds.
\end{cases}
$$

Then, we have $M=\frac{8}{3\sqrt{\pi}}$, $L=\frac{37}{30}$, $I=\frac{37}{60}$, $z=\frac{2}{3}$ and $D=5$. If we take $a=0.015$, $b=1$ and $c=60$, all the conditions in Theorem 4 are satisfied. Thus the boundary value problem has at least three positive solutions u_1,u_2 and u_3 such that

$$
\max_{t\in[0,1]}u_1(t)<0.015<\max_{t\in[0,1]}u_3(t),\quad \min_{t\in[\frac{1}{2},1]}u_3(t)<1<\min_{[\frac{1}{2},1]}u_2(t).
$$

4 Conclusion

In the present work, the nonlinear higher order multi-point fractional boundary value problems were studied. First, we obtained the criteria for the existence of at least one positive solution of the BVP ((1)) as a result of the Krasnosel'skii fixed point theorem. Then, by using the five functionals fixed-point theorem, the existence results of at least three positive solutions of the BVP ((1)) were established.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
[2] K. B. Oldham and J. Spanier, *Fractional calculus: theory and applications, differentiation and integration to arbitrary order*, Academic Press, New York, NY, USA, 1974.
[3] O. P. Sabatier, J. A. Agrawal and J. A. T. Machado, *Advances in fractional calculus*, Springer, Dordrecht, The Netherlands, 2007.
[4] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivatives: theory and applications*, Gordon and Breach, Yverdon, Switzerland, 1993.
[5] B. Ahmad and S. K. Ntouyas, Existence of solutions for Riemann-Liouville multi-valued fractional boundary value problems, *Georgian Math. J.* **24**, 479–488 (2017).
[6] C. Bai, Existence and uniqueness of solutions for fractional boundary value problems with p-Laplacian operator, *Adv. Differ. Equ.* **4**, 1–12, (2018).
[7] F. T. Fen, I. Y. Karaca and O. B. Ozen, Positive solutions of boundary value problems for p-Laplacian fractional differential equations, *Filomat* **31**, 1265-1277 (2017).
[8] P. Kalamani, D. Baleanu and M. M. Arjunan, Local existence for an impulsive fractional neutral integro-differential system with Riemann-Liouville fractional derivatives in a Banach space, *Adv. Differ. Equ.*, Paper No. 416, 1-26 (2018).
[9] H. Khan, Y. Li, W. Chen, D. Baleanu and A. Khan, Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator, *Bound. Value Probl.* Paper No. 157, 1-16 (2017).
[10] E. Uğurlu, D. Baleanu and K. Taş, On the solutions of a fractional boundary value problem, *Turkish J. Math.* **42**, 1307-1311 (2018).
[11] J. R. Cannon, The solution of the heat equation subject to the specification of energy, *Q. Appl. Math.* **21**, 155-160 (1963).
[12] N. I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differ. Uravn.* **13**, 294-304 (1977).
[13] R. Yu. Chegis, Numerical solution of a heat conduction problem with an integral condition, *Litov. Mat. Sb.* **24**, 209-215 (1984).
[14] W. M. Whyburn, Differential equations with general boundary conditions, *Bull. Am. Math. Soc.* **48**, 692-704 (1942).
[15] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with non-linear fractional differential equations, *Applic. Anal.* **87**, 851–863 (2008).
[16] W. Benhamida, J. R. Graef and S. Hamani, Boundary value problems for fractional differential equations with integral and anti-periodic conditions in a Banach Space, *Progr. Fract. Differ. Appl.* **4**, 65–70 (2018).
[17] A. Cabada and G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *J. Math. Anal. Appl.* **389**, 403–411 (2012).
[18] J. R. Graef, L. Kong, Q. Kong and M. Wang, Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions, *Fract. Calc. Appl. Anal.* **15**, 509–528 (2012).
[19] M. Jia, X. Liu and X. Gu, Uniqueness and asymptotic behavior of positive solutions for a fractional-order integral boundary value problem, *Abstr. Appl. Anal.* **2012**, 1–21 (2012).
[20] C. S. Sin, G. I. Ri and M. C. Kim, Existence of smooth solutions of multi-term Caputo-type fractional differential equations, *Progr. Fract. Differ. Appl.* **4**, 211–218 (2018).
[21] W. Sun and Y. Wang, Multiple positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *Fract. Calc. Appl. Anal.* **17**, 605-616 (2014).
[22] S. Vong, Positive solutions of singular fractional differential equations with integral boundary conditions, *Math. Comput. Model.* **57**, 1053–1059 (2013).
[23] W. Yang, Positive solutions for nonlinear Caputo fractional differential equations with integral boundary conditions, *J. Appl. Math. Comput.* **44**, 39–59 (2014).
[24] C. Zhao, Existence and uniqueness of positive solutions to higher-order nonlinear fractional differential equation with integral boundary conditions, *Electr. J. Differ. Equ.* **2012**, 1–11 (2012).
[25] Z. B. Bai and H.S. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311**, 495-505 (2005).
[26] D. Q. Jiang and C. J. Yuan, The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, *Nonlin. Anal.* **72**, 710-719 (2010).
[27] S. Liang and Y. Song, Existence and uniqueness of positive solutions to nonlinear fractional differential equation with integral boundary conditions, *Lithuanian Math. J.* **52**, 62–76 (2012).
[28] X. Zhang and Y. Han, Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations, *Appl. Math. Lett.* **25**, 555-560 (2012).
[29] L. Wang and X. Zhang, Existence of positive solutions for a class of higher-order nonlinear fractional differential equations with integral boundary conditions and a parameter, *J. Appl. Math. Comput.* **44**, 293–316 (2014).

[30] M. Jleli, J. J. Nieto and B. Samet, Lyapunov-type inequalities for a higher order fractional differential equation with fractional integral boundary conditions, *Electr. J. Qual. Theor. Differ. Equ.* **16**, 1–17 (2017).
[31] İ. Yaslan and M. Günendi, Higher order multi-point fractional boundary value problems with integral boundary conditions, *Int. J. Nonlin. Anal. Appl.* **9**, 247-260 (2018).
[32] J. Jin, X. Liu and M. Jia, Existence of positive solutions for singular fractional differential equations with integral boundary conditions, *Electr. J. Differ. Equ.* **2012**, 1-14 (2012).
[33] M. Günendi and İ. Yaslan, Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions, *Fract. Calc. Appl. Anal.* **19**, 989–1009 (2016).
[34] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, San Diego, 1988.
[35] R. I. Avery, A generalization of the Legget-Williams fixed point theorem, *Math. Sci. Res. Hot-Line* **3**, 9–14 (1999).