Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/030105

Upper and Lower Solutions Method for Higher Order Boundary Value Problems

Rabah Khaldi and Assia Guezane-Lakoud*

Laboratory of Advanced Materials, Departement of Mathematics, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria

Received: 30 Apr. 2016, Revised: 27 Sep. 2016, Accepted: 29 Sep. 2016
Published online: 1 Jan. 2017

Abstract: This paper is devoted to the study of existence of solutions for two point boundary value problems (P1) for fractional differential equations of arbitrary order $q \geq 2$, by applying upper and lower solutions method together with Schauder's fixed point Theorem. First, we transform the posed problem to an ordinary first order initial value problem, that we modify to prove the existence of solutions for the problem (P1), moreover we give the explicit expression of the upper and lower solutions of problem (P1). The obtained results are illustrated by some examples.

Keywords: Fractional boundary value problem, fractional derivative, upper and lower solutions method, existence of solution.

1 Introduction

In this paper we investigate the existence of solutions for the following two point boundary value problem with higher order fractional derivative (P1)

$$\begin{aligned} D^q u(t) + f\left(t, u(t), D^{q-2}u(t)\right) &= 0, 0 < t < 1, \\ u^{(k)}(0) &= 0, k = 0, 1, \ldots n-3, \\ D^{(q-2)}u(1) = D^{q-1}u(0) &= 0, \end{aligned}$$

D^q denotes Riemann-Liouville fractional derivative, $n-1 < q < n$, $n = [q]+1$, $q \geq 2$, u is the unknown function and f is a real continuous function on $[0,1] \times \mathbb{R}^2$.

A variety of techniques are applied to obtain the existence of solutions for fractional boundary value problems, such the method of upper and lower solutions, Mawhin theory, fixed point theorems. The main idea of the upper and lower solutions method, is to modify the given problem and prove the existence results for the modified problem, then establish the existence of solutions for the given problem. The remarkable point in this method is that we don't prove only the existence of solution but also we obtain its location between what is called the lower and the upper solutions. This method was introduced by Picard in 1893 and has been developed later by Dragoni. Recently, we find a large number of papers on ordinary differential equations devoted to this theory [1,2,3,4,5,6,7,8], but few of them are on differential equations with fractional order [9,10,11,12,13,14,15].

The organization of this paper is as follows. In Section 2, we introduce some definitions on fractional calculus and lemmas that will be used later, then we define the upper and lower solutions. In Section 3, we solve the corresponding problem of order $(q-1)$, then we reduce the problem (P1) to an equivalent first order initial value problem that we modify to conclude the existence of solutions for problem (P1). Furthermore we construct the upper and lower solutions for (P1). Finally, we give two examples illustrating the obtained results.

2 Preliminaries

We recall the definitions of Riemann-Liouville fractional integral and derivative, we can find their properties in [16,13].

* Corresponding author e-mail: rkhadi@yahoo.fr

Definition 1. *Let $\alpha > 0$, then the Riemann-Liouville fractional integral of a function g is defined by*

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2. *The Riemann-Liouville fractional derivative of order q of g is defined by*

$$D_{a^+}^{q} g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{g(s)}{(t-s)^{q-n+1}} ds,$$

where $n = [q]+1.([q]$ is the integer part of $q)$.

Lemma 1. *The homogenous fractional differential equation $D_{a^+}^{q} g(t) = 0$ has a solution*

$$g(t) = c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_n t^{q-n},$$

where, $c_i \in \mathbb{R}$, $i = 1, \dots, n$ and $n = [q]+1$.

Lemma 2. *Let $p, q \geq 0$, $f \in L_1[a,b]$. Then:*
1- $I_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^{q} I_{0^+}^{p} f(t)$ and ${}^{c}D_{0^+}^{q} I_{0^+}^{q} f(t) = f(t)$, for all $t \in [a,b]$.
2- If $p > q > 0$, then the formula ${}^{c}D_{0^+}^{q} I_{0^+}^{p} f(t) = I_{0^+}^{p-q} f(t)$, holds almost everywhere on $t \in [a,b]$, for $f \in L_1[a,b]$ and it is valid at any point $t \in [a,b]$ if $f \in C[a,b]$.
3- If $q \geq 0$ and $p > 0$ then $D_{a^+}^{q} (t-a)^{p-1}(x) = \frac{\Gamma(p)}{\Gamma(p-q)} (x-a)^{p-q-1}$, $D_{a^+}^{q} (t-a)^{q-j}(x) = 0$, $j = 1, 2, \dots, n$.

Definition 3. *The functions α, $\beta \in AC^n[0,1]$ are called lower and upper solutions of problem (P1) respectively, if*

$$\begin{aligned} D^q \alpha(t) + f\left(t, \alpha(t), D^{q-2}\alpha(t)\right) &\geq 0, \quad 0 < t < 1, \\ \alpha^{(k)}(0) &= 0, \ k = 0, 1, \dots n-3, \\ D^{q-1}\alpha(0) \leq 0, &\quad D^{(q-2)}\alpha(1) \geq 0, \end{aligned}$$

and

$$\begin{aligned} D^q \beta(t) + f\left(t, \beta(t), D^{q-2}\beta(t)\right) &\leq 0, \quad 0 < t < 1, \\ \beta^{(k)}(0) &= 0, \ k = 0, 1, \dots n-3, \\ D^{q-1}\beta(0) \geq 0, &\quad D^{(q-2)}\beta(1) \leq 0, \end{aligned}$$

where $AC^n[0,1] = \left\{u \in C^{n-1}[0,1], u^{(n-1)} \text{absolutely continuous function on } [0,1]\right\}$.

3 Main Results

First we solve the corresponding boundary value problem of order $(q-1)$. Let (P2) denotes the following problem

$$\begin{aligned} D^{q-1}u(t) &= -v(t), \quad 0 < t < 1, \\ u^{(k)}(0) &= 0, \ k = 0, 1, \dots n-3. \ D^{(q-2)}u(1) = 0. \end{aligned}$$

Lemma 3. *The solution of problem (P2) is given by*

$$u(t) = \int_0^1 G(t,s) v(s) ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(q-1)} \begin{cases} -(t-s)^{q-2} + t^{q-2}, s \leq t \\ t^{q-2}, s \geq t. \end{cases}$$

Proof. Using the properties of the fractional calculus, we get

$$u(t) = -I^{q-1}v(t) + \sum_{k=1}^{n-1} c_k t^{q-1-k}.$$

Applying the boundary conditions it yields

$$u(t) = -I^{q-1}v(t) + \frac{t^{q-2}}{\Gamma(q-1)} \int_0^1 v(s)\,ds = \int_0^1 G(t,s)\,v(s)\,ds.$$

The proof is completed.

The properties of the Green function are given in the following Lemma.

Lemma 4. *The function G is continuous, nonnegative and satisfies* $G(t,s) \leq \frac{1}{\Gamma(q-1)}, \forall s,t \in [0,1]$.

Proof. The proof is direct,therefore we omit it.

Define the operators T and K by

$$Tv(t) = \int_0^1 G(t,s)\,v(s)\,ds, \;\; Kv(t) = \int_t^1 v(s)\,ds.$$

From Lemma 3, we get $u(t) = Tv(t)$. The boundary condition $D^{(q-2)}u(1) = 0$, implies $Kv(t) = \int_t^1 v(s)\,ds = D^{(q-2)}u(t)$. From here and taking into account that $D^{q-1}u(0) = 0$, we see that problem (P1) is equivalent to the following first order initial value problem that we denote by (P3)

$v'(t) = f(t, Tv(t), Kv(t)),$
$v(0) = 0.$

We have the following Lemmas.

Lemma 5. *Assume that there exists a constant* $A \geq 0$, *such that* $f(t,x,y) \leq A$, *for* $0 \leq t \leq 1$, $0 \leq x \leq \frac{A}{\Gamma(q-1)}$ *and* $0 \leq y \leq A$. *Then problem (P1) has an upper solution.*

Proof. Let $\varphi(t) = At$, then $T\varphi(t) = A\int_0^1 G(t,s)\,s ds \leq \frac{A}{\Gamma(q-1)}$ and $K\varphi(t) \leq A$, then

$\varphi'(t) - f(t, T\varphi(t), K\varphi(t)) \geq 0$ and $\varphi(0) \geq 0$. This implies that $\beta(t) = T\varphi(t)$ is an upper solution of problem (P1).

Lemma 6. *Assume that there exists a constant* $B \leq 0$, *such that* $f(t,x,y) \geq B$, *for* $0 \leq t \leq 1$, $\frac{B}{2\Gamma(q-1)} \leq x \leq 0$ *and* $\frac{B}{2} \leq y \leq 0$, *then problem (P1) has a lower solution.*

Proof. Let $\psi(t) = Bt$, so, $T\psi(t) = B\int_0^1 G(t,s)\,s ds \geq \frac{B}{2\Gamma(q-1)}$ and $K\psi(t) \geq \frac{B}{2}$, then

$\psi'(t) - f(t, T\psi(t), K\psi(t)) \leq 0$ and $\psi(0) \leq 0$. This implies that $\alpha(t) = T\psi(t)$ is a lower solution of problem (P1).

Lemma 7. *Under the assumptions of Lemmas 5 and 6, the upper and lower solutions of problem (P1) satisfy*

$$\alpha(t) \leq \beta(t),\; D^{q-1}\beta(t) \leq D^{q-1}\alpha(t),\; 0 \leq t \leq 1.$$

Proof. From Lemmas 5 and 6 and their proofs, we know that $\beta(t) = T\varphi(t)$ and $\alpha(t) = T\psi(t)$ are upper and lower solutions of problem (P1) respectively. Simple computations give

$$\beta(t) = \frac{At^{q-2}}{2\Gamma(q-1)}\left(1 - \frac{2t^2}{q(q-1)}\right) \geq 0,$$
$$\alpha(t) = \frac{Bt^{q-2}}{2\Gamma(q-1)}\left(1 - \frac{2t^2}{q(q-1)}\right) \leq 0,$$

$$D^{q-1}\beta(t) = -\varphi(t) = -At \leq -Bt = \psi(t) = D^{q-1}\alpha(t).$$

This completes the proof.

Define the operator $F : C[0,1] \to C[0,1]$,

$$(Fv)(t) = f(t, T(\min[\varphi, (\max(v,\psi))]), K(\min[\varphi, (\max(v,\psi))]))$$

and consider the modified problem (P4) of the problem (P3)

$$v'(t) = (Fv)(t), 0 \le t \le 1,$$
$$v(0) = 0.$$

Next we give the relation between the solutions of problem (P4) and those of problem (P1).

Lemma 8. *If v is a solution of problem (P4) then $u = Tv$ is a solution of problem (P1) satisfying $\alpha(t) \le u(t) \le \beta(t)$ and $D^{q-1}\beta(t) \le D^{q-1}u(t) \le D^{q-1}\alpha(t)$, $0 \le t \le 1$.*

Proof. Let v be a solution of problem (P4), then $\psi(t) \le v(t) \le \varphi(t), t \in [0,1]$. In fact, suppose that there exists $t_1 \in (0,1]$ such that $v(t_1) > \varphi(t_1)$, set $\omega(t) = v(t) - \varphi(t)$, from the initial condition we have $\omega(0) = 0$. From the continuity of ω we conclude that there exist $t_2 \in [0,t_1)$ and $t_3 \in [t_1,1]$ such that $\omega(t_2) = 0$ and $\omega(t) \ge 0$, $t \in [t_2,t_3]$. Moreover we have $\omega'(t) = v'(t) - \varphi'(t) = f(t, T(\min[\varphi, (\max(v,\psi))]), K(\min[\varphi, (\max(v,\psi))])) + D^q\beta(t) \le 0$, $t \in [t_2,t_3]$, this with $\omega(t_2) = 0$ implies that ω is decreasing on $[t_2,t_3]$, and then $v(t) \le \varphi(t), t \in [t_2,t_3]$, this contradict the fact $v(t_1) > \varphi(t_1)$. Similarly we prove that $\psi(t) \le v(t), t \in [0,1]$. From the above discussion, it yields $v'(t) = f(t, Tv(t), Kv(t))$ that implies that v is solution of (P3) and therefor $u = Tv$ is a solution of (P1). Finally, in view of the positivity of the function G and the monotony of the operator T, we obtain $T\psi(t) \le Tv(t) \le T\varphi(t), t \in [0,1]$ that is $\alpha(t) \le u(t) \le \beta(t)$ and $D^{q-1}\beta(t) \le D^{q-1}u(t) \le D^{q-1}\alpha(t)$, $t \in [0,1]$.

Now we give the existence theorem for the problem (P1):

Theorem 1. *Assume that there exist two constants A and B such that $A \ge 0$, $B \le 0$ and $A \ge |B|$ and the following hypotheses hold*

$(H1)$- $f(t,x,y) \le A$, for $0 \le t \le 1$, $0 \le x \le \frac{A}{\Gamma(q-1)}$ and $0 \le y \le A$,

$(H2)$- $f(t,x,y) \ge B$, for $0 \le t \le 1$, $\frac{B}{2\Gamma(q-1)} \le x \le 0$ and $\frac{B}{2} \le y \le 0$,

then the problem (P1) has at least one solution u such that

$$\alpha(t) \le u(t) \le \beta(t),$$
$$D^{q-1}\beta(t) \le D^{q-1}u(t) \le D^{q-1}\alpha(t),\ 0 \le t \le 1.$$

Proof. Define the operator $R : C[0,1] \to C[0,1]$,

$$Rv(t) = \int_0^t (Fv)(s)\, ds.$$

Let us remark that any fixed point v of R then v is a solution of (P4). Put $\Omega = \{v \in C[0,1], \|v\| \le A\}$. Since φ and $\psi \in \Omega$ then by condition (H1) it yields

$$|Rv(t)| = \left| \int_0^t (Fv)(s)\, ds \right|$$

$$\le \int_0^t |f(s, T(\min[\varphi, (\max(v,\psi))])(s), K(\min[\varphi, (\max(v,\psi))(s)]))|\, ds \le A,$$

thus $R(\Omega)$ is uniformly bounded and $R(\Omega) \subset \Omega$. For $0 \le t_1 < t_2 \le 1$, we have

$$\begin{aligned} & |Rv(t_1) - Rv(t_2)| \\ \le & \int_{t_1}^{t_2} |f(s, T(\min[\varphi, (\max(v,\psi))])(s), K(\min[\varphi, (\max(v,\psi))(s)]))|\, ds \\ \le & A(t_2 - t_1). \end{aligned}$$

So, $R(\Omega)$ is equicontinuous. From Arzela-Ascoli Theorem, we conclude that R is completely continuous. Thanks to Schauder fixed point theorem we get that R has a fixed point $u \in \Omega$, that is a solution of (P1) satisfying from Lemma 8 $\alpha(t) \le u(t) \le \beta(t)$ and $D^{q-1}\beta(t) \le D^{q-1}u(t) \le D^{q-1}\alpha(t)$, $0 \le t \le 1$. The proof is completed.

Example 1. Consider the problem (P1) with

$$f\left(t,u(t),D^{q-2}u(t)\right)=-t+\frac{\Gamma(q-1)}{4}u(t)+\frac{1}{4}D^{q-2}u(t),$$

then $f(t,x,y)=-t+\Gamma(q-1)\frac{x}{4}+\frac{y}{4}$. For $A=2$ and $B=-2$, f satisfies the assumptions of Theorem 1. Consequently problem (P1) has a solution u such that $\frac{-t^{q-2}}{\Gamma(q-1)}\left(1-\frac{2t^2}{q(q-1)}\right)=\alpha(t)\leq u(t)\leq\beta(t)=\frac{t^{q-2}}{\Gamma(q-1)}\left(1-\frac{2t^2}{q(q-1)}\right)$.

Example 2. Now if we choose

$$f\left(t,u(t),D^{q-2}u(t)\right)=t+\frac{\Gamma(q-1)}{4}u(t)+\frac{1}{4}D^{q-2}u(t),$$

then $f(t,x,y)=t+\Gamma(q-1)\frac{x}{4}+\frac{y}{4}$. For $A=2$ and $B=0$, f satisfies the assumptions of Theorem 1. Consequently problem (P1) has at least one solution u, moreover this solution is positive and satisfies $0=\alpha(t)\leq u(t)\leq\beta(t)=\frac{t^{q-2}}{\Gamma(q-1)}\left(1-\frac{2t^2}{q(q-1)}\right)$.

4 Conclusions

In the present work, the existence of solutions for a fractional boundary value problem of arbitrary order (P1) is investigated by the help of upper and lower solutions method with Schauder's fixed point Theorem. To solve the higher order fractional boundary value problem we transform it to an equivalent first order ordinary initial value problem. We conclude that this method is efficient since we obtain the localization of the solution between the upper and lower solutions that we gave explicitly their expressions.

References

[1] A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, *Bound. Value Probl.* **2011**, Article ID 893753, 18 pages, (2011).

[2] C. De Coster and P. Habets, *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, **205**, Series Editor: C. K. Chui, Elsevier, 2006.

[3] C. De Coster and P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results,*Chapter, Nonlinear analysis and boundary value problems for ordinary differential equations*, **371** of the series International Centre for Mechanical Sciences, 1–78 (1996).

[4] A. Elhaffaf and M. Naceri, Existence theorems for a fifth-order boundary value problem, *J. Math. Syst. Sci.* **4**, 1–5 (2014).

[5] W. Feng and J. R. L. Webb, Solvability of m-point boundary value problems with nonlinear growth, *J. Math. Anal. Appl.* **212**, 467–480 (1997).

[6] F. Li, M. Jia, X. Liu, C. Li and G. Li, Existence and uniqueness of solutions of second-order three-point boundary value problems with upper and lower solutions in the reversed order, *Nonlin. Anal. Theor. Meth. Appl.* **68**, 2381–2388 (2008).

[7] Z. Wei and C. Pang, The method of lower and upper solutions for fourth order singular m-point boundary value problems, *J. Math. Anal. Appl.* **322**, 675–692 (2006).

[8] Q. Zhang and D. Jiang, Upper and lower solutions method and a second order three-point singular boundary value problem, *Comput. Math. Appl.* **56**, 1059–1070 (2008).

[9] A. Chidouh, A. Guezane-Lakoud and R. Bebbouchi, Positive solutions of the fractional relaxation equation using lower and upper solutions, *Vietnam J. Math.* **44**, 739–748(2016).

[10] D. Jiang, Upper and lower solutions method and a superlinear singular boundary value problem, *Comput. Math. Appl.* **44**, 323–337 (2002).

[11] M. Jia and X. Liu, Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions, *Appl. Math. Comput.* **232**, 313–323 (2014).

[12] X. Liu and M. Jia, Multiple solutions for fractional differential equations with nonlinear boundary conditions, *Comput. Math.Appl.* **59**, 2880–2886 (2010).

[13] I. Podlubny, *Fractional differential equations*, Academic Press, Sain Diego, 1999.

[14] A. Shi and S. Zhang, Upper and lower solutions method and a fractional differential equation boundary value problem, *Electr. J. Qualit. Theor. Differ. Equ.* **30**, 1–13 (2009).

[15] S. Zhang and X. Su, The existence of a solution for a frational differential equation with nonlinear boundary conditions considered using upper and lower solutions in reversed order, *Comput. Math. Appl.* **62**, 1269–1274 (2011).

[16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science, Amsterdam, The Netherlands, 2006.