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Extended Spectral Method for Fractional order Three-dimensional Heat Conduction Problem

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Abstract: This paper presents new operational matrices of fractional integral and derivatives for shifted Legendre polynomials. These operational matrices are employed to design a new spectral method for solving three-dimensional heat conduction problem. The main advantage of the proposed method is to reduce this complicated problem with its initial and boundary conditions into a system of easily solvable algebraic equations. The efficiency of the proposed method is shown with some test problems. The results are displayed graphically.

Keywords: Orthogonal Polynomials; Approximation theory; Spectral Methods; Numerical Simulation

1 Introduction

Diffusion equation is one of the most important partial differential equation frequently used to model many engineering and biomedical phenomena. Some examples are the cyclic heating of the cylinder surface of internal combustion engines, heating and cooling of building structures, heating lakes and water reservoirs by radiation, the heating of solid surfaces in material processing, the cyclic heating of laminated steel during pickling, heating and cooling of vials contained DNA for polimerase-chain-reaction activation, the heating of electronics and many more see for example [1,2,3,4,5,6].

In literature various techniques are used to obtain the exact and approximate solution of fractional order diffusion equation. Very recently Kulish and Lage[20] studied the fractional diffusion equation and provide analytic solution using laplace transform method. Akbarzade and Langari [21] proposed the homotopy method to approximate the solution of integer order three dimensional transient state heat conduction. Ning and Jiang [22] proposed the method of variable seperation for analytically solving time-fractional heat conduction equation in spherical coordinate system. More ever Wu and Lee [25,26] applied the variational iteration technique to fractional diffusion equation and provided good approximation to the solution.

In this paper we consider the fractional order partial equation (FPDEs) of the form

$$\begin{aligned}
&\chi_t \frac{\partial^{\sigma} U}{\partial t^{\sigma}} = \lambda_x \frac{\partial^{\beta_1} U}{\partial x^{\beta_1}} + \lambda_y \frac{\partial^{\beta_2} U}{\partial y^{\beta_2}} + \lambda_z \frac{\partial^{\beta_3} U}{\partial z^{\beta_3}} + I(x,y,t,z)\\
&U(0,x,y,z) = f(x,y,z), \qquad U(t,0,y,z) = g_1(t,y,z) \qquad \frac{\partial}{\partial x}U(t,0,y,z) = g_2(t,y,z),\\
&\qquad U(t,x,0,z) = h_1(t,x,z) \qquad \frac{\partial}{\partial y}U(t,x,0,z) = h_2(t,x,z),\\
&\qquad U(t,x,y,0) = l_1(t,x,y) \qquad U(t,x,y,c) = l_2(t,x,y),
\end{aligned} \tag{1}$$

where χ_t is the volumetric heat capacity$(J/(m^3K))$, $\lambda_x, \lambda_y, \lambda_z$ are thermal conductivities $(W/m.K)$ in x,y and z directions respectively, $0 < \alpha \leq 1$, $1 < \beta_1, \beta_2, \beta_3 \leq 2$, $t \in [0, \tau]$, $x \in [0,a]$, $y \in [0,b]$ and $z \in [0,c]$. $I(x,y,t,z)$ is the internal source term.

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It is some time impossible to obtain the exact analytic solution of transient diffusion problems because of the mathematical intricacies involved in solving the differential equations governing the phenomenon. This paper is devoted to the study of a numerical scheme for the approximate solution of the above problem. Operational matrix technique is one of the extensively used method and is applied by many authors to approximate the solution of different kind of problems like integral, differential and partial differential equations [27,28,29,30]. The motivation of the high applicability of the method is its simplicity and ease of application. Different orthogonal polynomials are used in the construction of operational matrices like Sine-Cosine [28,29], Legendre polynomials [27,30], Jacobi [35,39], Muntz polynomials [34] (see for instance [31,32,33,34,35]).

To the best of our knowledge these matrices are used to approximate FPDEs only up to two variables . In [36] we derived and developed operational matrix of differentiation for a column function vector of two variables. These operational matrices have the ability to approximate solution of partial differential equations (PDEs) with three variables. Also we approximate solution of coupled system of FDEs and FPDEs using operational matrices see for example [37]. The matrices derived in [36] have the ability to obtain the fractional order partial derivative of a function of three variables . The matrices developed in this paper have two advantages over the previously derived matrices. Firstly they can calculate the fractional order derivative of function of four variables and secondly they can calculate the derivatives on any finite domain.

We organize the paper as: in section 2 we provide some basic properties of fractional calculus and orthogonal polynomials, in section 3 we provide some results on Legendre approximation of a function of four variables and its absolute error of approximation, in section 4 we derive some operational matrices of integration and differentiation, in section 5 the operational matrices are used to convert the corresponding equation to a system of algebraic equations, in section 6 we study the structure and performance of the operational matrices with some test functions, and the proposed algorithm is applied to several test problems and finally in section 7 a conclusion about the method is made.

2 Preliminaries

For convenience, this section summarizes some definitions and basic results from fractional calculus.

Definition 2.1.

[37,36] Given an interval $[0,\tau]\subset\mathbf{R}$, the Riemann-Liouville fractional order integral of order $\sigma\in\mathbf{R}_+$ of a function $\varphi\in(L^1[0,\tau],\mathbf{R})$ is defined by

$$I^{\sigma}_{0+}\phi(t)=\frac{1}{\Gamma(\sigma)}\int_0^t(t-s)^{\sigma-1}\varphi(s)ds.$$

Definition 2.2. For a given function $\varphi(t)\in C^n[0,a]$, the fractional order derivative of order σ in Caputo sense is defined as

$$D^{\sigma}\varphi(t)=\frac{1}{\Gamma(n-\sigma)}\int_0^x\frac{\varphi^{(n)}(t)}{(x-t)^{\sigma+1-n}}dt,\ \sigma\in[n-1,n),\ n\in N,$$

provided that the right side is pointwise defined on $(0,\infty)$, where $n=[\sigma]+1$.

Hence, it follows that

$$D^{\sigma}t^s=\frac{\Gamma(1+k)}{\Gamma(1+s-\sigma)}t^{s-\sigma},I^{\sigma}t^s=\frac{\Gamma(1+k)}{\Gamma(1+s+\sigma)}t^{s+\sigma}. \tag{2}$$

Also it is clear that the $D^{\sigma}C$ is zero for a constant C.

2.1 Shifted Legendre Polynomials

The analytical expression for the shifted Legendre polynomials on $[0,\tau]$ are given by

$$P_i^{\tau}(t)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!t^k}{(i-k)!(k!)^2(\tau^k)},\ i=0,1,2,3...,\tag{3}$$

These polynomials are orthogonal and orthogonality condition is

$$\int_0^{\tau}P_i^{\tau}(t)P_j^{\tau}(t)dt=\begin{cases}\frac{\tau}{2i+1}, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases}\tag{4}$$

The orthogonality relation relation allows us to write any $u(t) \in C[0,\tau]$ as a series expansion of Shifted Legendre Polynomials. In practice we are interested in the finite series therefore we may write

$$u(t) \approx \sum_{a=0}^{m} C_a P_a^{\tau}(t), \text{ where } C_a = \frac{(2a+1)}{\tau} \int_0^{\tau} u(t) P_a^{\tau}(t) dt. \tag{5}$$

In vector notation, we write

$$u(t) = C_M^T \Psi_M(t). \tag{6}$$

where $M = m+1$, C_M and $\Psi_M(t)$ are the coefficient vector and column vector, each of order M.

We extend the notation to three-dimensional space and define three-dimensional Legendre polynomials on the space $[0,a] \times [0,b] \times [0,c]$ of order M as a product of three Legendre polynomials.

$$P_n^{(a,b,c)}(x,y,z) = P_q^a(x) P_r^b(y) P_s^b(x), \ n = M^2 q + Mr + s + 1, \ q,r,s = 0,1,2,...,m. \tag{7}$$

These polynomial are orthogonal. The orthogonality condition of $P_n^{(a,b,c)}(x,y,z)$ is defined as

$$\int_0^c \int_0^b \int_0^a P_q^a(x) P_r^b(y) P_s^c(z) P_{q'}^a(x) P_{r'}^b(y) P_{s'}^c(z) dxdydz = \begin{cases} \frac{abc}{(2q+1)(2r+1)(2s+1)} & \text{if } q = q', r = r', s = s'; \\ 0 & \text{otherwise.} \end{cases}$$

Any $u(x,y,z) \in C([0,a] \times [0,b] \times [0,c])$ can be written as a truncated series of three dimensional Shifted Legendre polynomials $P_n^{(a,b,c)}(x,y,z)$.

$$u(x,y,z) = \sum_{q=0}^{m} \sum_{r=0}^{m} \sum_{s=0}^{m} c_{qrs} P_q^a(x) P_r^b(y) P_s^c(z). \tag{8}$$

Where c_{qrs} can be obtained by the relation

$$c_{qrs} = \frac{(2q+1)(2r+1)(2s+1)}{abc} \int_0^c \int_0^b \int_0^a u(x,y,y) P_q^a(x) P_r^b(y) P_s^c(z) dxdydz. \tag{9}$$

By using the notation $c_n = c_{qrs}$ where $n = M^2 q + Mr + s + 1$, and we can write (8) as follows

$$u(x,y,z) = \sum_{n=1}^{M^3} c_n P_n^{(a,b,c)}(x,y,z) = C_{M^3}^T \Psi^{(a,b,c)}(x,y,z). \tag{10}$$

Where C_{M^3} is coefficient column vector of order M^3 and $\Psi^{(a,b,c)}(x,y,z)$ is $M^3 \times 1$ is column vector of Shifted Legendre polynomials defined as

$$\Psi^{(a,b,c)}(x,y,z) = \left[P_1^{(a,b,c)}(x,y,z) \ P_2^{(a,b,c)}(x,y,z) \ \cdots \ P_{M^3}^{(a,b,c)}(x,y,z) \right]^T \tag{11}$$

2.2 Function Approximation

We generalize four-dimensional Legendre polynomials on the space $[0,\tau] \times [0,a] \times [0,b] \times [0,c]$ of order M by the product function of Legendre polynomials as

$$P_{rsuv}^{(\tau,a,b,c)}(t,x,y,z) = P_r^{\tau}(t) P_s^a(x) P_u^b(y) P_v^c(z), \qquad r,s,u,v = 0,1,2..m. \tag{12}$$

The orthogonality relation for $P_{rsuv}^{(\tau,a,b,c)}(t,x,y,z)$ is found to be

$$(\int_0^c \int_0^b \int_0^a \int_0^{\tau} P_{rsuv}^{(\tau,a,b,c)} P_{r's'u'v'}^{(\tau,a,b,c)} dtdxdydz) = \begin{cases} \frac{\tau abc}{(2r+1)(2s+1)(2u+1)(2v+1)} & \text{if } r = r', s = s', u = u', v = v'; \\ 0 & \text{otherwise.} \end{cases}$$

2.2.1 Function approximation with Four-dimensional Legendre polynomials

Any $u(t,x,y,z) \in C([0,\tau]\times[0,a]\times[0,b]\times[0,c])$ can easily be approximated with $P_{rsuv}^{(\tau,a,b,c)}(t,x,y,z)$ in the form

$$u(t,x,y,z) = \sum_{r=0}^{m}\sum_{s=0}^{m}\sum_{u=0}^{m}\sum_{v=0}^{m} c_{(rsuv)} P_r^{\tau}(t) P_s^{a}(x) P_u^{b}(y) P_v^{c}(z), \tag{13}$$

where $c_{(rsuv)}$ can be obtained by the relation

$$c_{(rsuv)} = \frac{(2r+1)(2s+1)(2u+1)(2v+1)}{\tau abc} \int_0^c \int_0^b \int_0^a \int_0^\tau u(t,x,y,z) P_{rsuv}^{(\tau,a,b,c)}(t,x,y,z)\,dt\,dx\,dy\,dz. \tag{14}$$

For simplicity we change the notation as

$$C_{(rn)} = C_{(rsuv)},$$

where $n = M^2 s + Mu + v + 1$. Using this simplified notation we can write equation (13) as

$$u(t,x,y,z) = \sum_{r=1}^{M}\sum_{n=1}^{M^3} C_{(rn)} P_r^{(\tau)}(t) P_n^{(a,b,c)}(x,y,z). \tag{15}$$

Or in matrix form we can write them as

$$u(t,x,y,z) = \Psi^{(\tau)}(t)^T K_{M\times M^3} \Psi^{(a,b,c)}(x,y,z). \tag{16}$$

Where $K_{M\times M^3}$ is coefficient matrix and $\Psi^{(\tau)}(t)$ is a function vector related to t and $\Psi^{(a,b,c)}(x,y,z)$ is function vector related to the variable x, y and z.

3 Error Bounds for Shifted Legendre Polynomials

In this section, we derive analytic relation for the absolute error of function of four variable. Consider $\prod_M(t,x,y,z)$ be the space span by M terms four dimensional Shifted Legendre polynomials. Then for a smooth function $u(t,x,y,z) \in \Delta$, where $\Delta = C([0,\tau]\times[0,a]\times[0,b]\times[0,c])$, assume that $U_{(M)}(t,x,y,z)$ is its best Legendre approximation in $\prod_{(M)}(t,x,y,z)$. Then, for any function $\hat{F}_{(M)}(t,x,y,z)$ of degree $\leq M$ in variable t, x, y and z it follows that

$$\|u(t,x,y,z) - U_{(M)}(t,x,y,z)\|_2 \leq \|u(t,x,y,z) - \hat{F}_{(M)}(t,x,y,z)\|_2. \tag{17}$$

The inequality (17) also holds if $\hat{F}_{(M,M,M,M)}(t,x,y,z)$ is interpolating polynomial of the function u at point (t_i, x_j, y_k, t_l) where $t_i = i\frac{\tau}{M}$, $x_j = j\frac{a}{M}$, $y_k = k\frac{b}{M}$ and $z_l = l\frac{c}{M}$. Then from the same arguments in [36] we can write

$$\begin{aligned} u(t,x,y,z) - \hat{F}_{(M,M,M,M)}(t,x,y,x) &= \frac{\partial^{M+1}}{\partial t^{M+1}(M+1)!} u(\xi,x,y,z) \prod_{i=0}^{M}(t-t_i) \\ &+ \frac{\partial^{M+1}}{\partial x^{M+1}(M+1)!} u(t,\sigma,y,z) \prod_{j=0}^{M}(x-x_j) + \frac{\partial^{M+1}}{\partial y^{M+1}(M+1)!} u(t,x,\omega,z) \prod_{k=0}^{M}(y-y_k) \\ &+ \frac{\partial^{M+1}}{\partial z^{M+1}(M+1)!} u(t,x,y,\mu) \prod_{k=0}^{M}(z-z_k) \\ &- \frac{\partial^{4M+4}}{\partial t^{M+1}\partial x^{M+1}\partial y^{M+1}\partial z^{M+1} 4(M+1)!} u(\xi',\sigma',\omega',\mu') \prod_{i=0}^{M}(t-t_i) \prod_{j=0}^{M}(x-x_j) \prod_{k=0}^{M}(y-y_k) \prod_{k=0}^{M}(z-z_l), \end{aligned} \tag{18}$$

such that $\xi, \xi' \in [0,\tau]$, $\sigma, \sigma' \in [0,a]$, $\omega, \omega' \in [0,b]$ and $\mu, \mu' \in [0,c]$. Therefore

$$\begin{aligned} |u(t,x,y,z) - \hat{F}_{(M,M,M,M)}(t,x,y,x)| \leq & \frac{\Upsilon_t}{(M+1)!)} \prod_{i=0}^{M} |(t-t_i)| + \frac{\Upsilon_x}{(M+1)!} \prod_{j=0}^{M} |(x-x_j)| \\ & + \frac{\Upsilon_y}{(M+1)!} \prod_{k=0}^{M} |(y-y_k)| + \frac{\Upsilon_z}{(M+1)!} \prod_{l=0}^{M} |(z-z_l)| \\ & - \frac{\Upsilon_o}{4(M+1)!} \prod_{i=0}^{M} |(t-t_i)| \prod_{j=0}^{M} |(x-x_j)| \prod_{k=0}^{M} |(y-y_k)| \prod_{k=0}^{M} |(z-z_l)|. \end{aligned} \tag{19}$$

Where

$$\Upsilon_t = max_{(t,x,y,z)\in\Delta}|\frac{\partial^{M+1}}{\partial t^{M+1}}u(t,x,y,z)|, \qquad \Upsilon_x = max_{(t,x,y,z)\in\Delta}|\frac{\partial^{M+1}}{\partial x^{M+1}}u(t,x,y,z)|,$$

$$\Upsilon_y = max_{(t,x,y,z)\in\Delta}|\frac{\partial^{M+1}}{\partial y^{M+1}}u(t,x,y,z)|, \qquad \Upsilon_z = max_{(t,x,y,z)\in\Delta}|\frac{\partial^{M+1}}{\partial z^{M+1}}u(t,x,y,z)|,$$

and

$$\Upsilon_o = max_{(t,x,y,z)\in\Delta}|\frac{\partial^{4M+4}u(t,x,y,z)}{\partial t^{M+1}\partial x^{M+1}\partial y^{M+1}z^{M+1}}|.$$

To derive a bound for the terms like $\prod_{i=0}^{M}|(t-t_i)|$, $\prod_{j=0}^{M}|(x-x_j)|$, $\prod_{k=0}^{M}|(y-y_k)|$, and $\prod_{k=0}^{M}|(z-z_l)|$. We make the change of variables as

$$t=\theta\frac{\tau}{M},\ x=\phi\frac{a}{M},\ y=\varphi\frac{b}{M},\ z=\nu\frac{c}{M}. \tag{20}$$

We obtain

$$\begin{aligned}&\prod_{i=0}^{M}|(t-t_i)| = (\frac{\tau}{M})^{M+1}\prod_{i=0}^{M}|\theta-i|, \qquad \prod_{j=0}^{M}|(x-x_j)| = (\frac{(a)}{M})^{M+1}\prod_{j=0}^{M}|\phi-j|,\\ &\prod_{k=0}^{M}|(y-y_k)| = (\frac{(b)}{M})^{M+1}\prod_{k=0}^{M}|\varphi-k|, \qquad \prod_{l=0}^{M}|(z-z_l)| = (\frac{(c)}{M})^{M+1}\prod_{l=0}^{M}|\nu-k|.\end{aligned} \tag{21}$$

Suppose that $\rho_1,\ \rho_2,\ \rho_3$ and ρ_4 are integers such that

$$\rho_1<\theta<\rho_1+1, \rho_2<\phi<\rho_2+1, \rho_3<\varphi<\rho_3+1, \rho_4<\nu<\rho_4+1.$$

.

Then we can write

$$\begin{aligned}&\prod_{i=0}^{M}|\theta-i| = |(\theta-\rho_1)(\theta-\rho_1-1)|\prod_{i=0}^{\rho_1-1}|(\theta-i)|\prod_{i=\rho_1-2}^{M}|(\theta-i)|,\\ &\prod_{j=0}^{M}|\phi-j| = |(\phi-\rho_2)(\phi-\rho_2-1)|\prod_{j=0}^{\phi-1}|(\phi-j)|\prod_{j=\rho_2-2}^{M}|(\phi-j)|,\\ &\prod_{k=0}^{M}|\varphi-k| = |(\varphi-\rho_3)(\varphi-\rho_3-1)|\prod_{k=0}^{\rho_3-1}|(\varphi-k)|\prod_{k=\rho_3-2}^{M}|(\varphi-k)|,\\ &\prod_{l=0}^{M}|\nu-k| = |(\nu-\rho_4)(\nu-\rho_4-1)|\prod_{l=0}^{\rho_4-1}|(\nu-k)|\prod_{k=\rho_4-2}^{M}|(\nu-k)|.\end{aligned} \tag{22}$$

The terms $|(\theta-\rho_1)(\theta-\rho_1-1)|$, $|(\theta-\rho_1)(\theta-\rho_1-1)|$, $|(\varphi-\rho_3)(\varphi-\rho_3-1)|$ and $|(\nu-\rho_4)(\nu-\rho_4-1)|$ gives there maximum value at points $\theta+\frac{1}{2}$, $\phi+\frac{1}{2}$, $\varphi+\frac{1}{2}$ and $\nu+\frac{1}{2}$ respectively. Therefore we can write

$$\begin{aligned}&|(\theta-\rho_1)(\theta-\rho_1-1)|\le\frac{1}{4}, \qquad |(\phi-\rho_2)(\phi-\rho_2-1)|\le\frac{1}{4},\\ &|(\varphi-\rho_3)(\varphi-\rho_3-1)|\le\frac{1}{4}, \qquad |(\nu-\rho_4)(\nu-\rho_4-1)|\le\frac{1}{4}.\end{aligned} \tag{23}$$

By the application of (20) we can write

$$\begin{aligned}
&\prod_{i=0}^{\rho_1-1}|(\theta-i)|\leq\prod_{i=0}^{\rho_1-1}(\rho_1+1-i)\leq(\rho_1+1)!, && \prod_{j=0}^{\rho_2-1}|(\phi-j)|\leq\prod_{j=0}^{\rho_2-1}(\rho_2+1-j)\leq(\rho_2+1)!,\\
&\prod_{k=0}^{\rho_3-1}|(\varphi-k)|\leq\prod_{k=0}^{\rho_3-1}(\rho_3+1-k)\leq(\rho_3+1)!, && \prod_{l=0}^{\rho_4-1}|(\nu-j)|\leq\prod_{j=0}^{\rho_4-1}(\rho_4+1-j)\leq(\rho_4+1)!,\\
&\prod_{i=\rho_1-2}^{M}|(\theta-i)|\leq\prod_{i=\rho_1-2}^{M}|(i-\rho_1)|\leq(M-\rho_1)!, && \prod_{j=\rho_2-2}^{M}|(\phi-j)|\leq\prod_{j=\rho_2-2}^{M}|(j-\rho_2)|\leq(M-\rho_2)!,\\
&\prod_{k=\rho_3-2}^{M}|(\varphi-k)|\leq\prod_{k=\rho_3-2}^{M}|(k-\rho_3)|\leq(M-\rho_3)!, && \prod_{l=\rho_3-2}^{M}|(\nu-l)|\leq\prod_{l=\rho_3-2}^{M}|(l-\rho_3)|\leq(l-\rho_3)!.
\end{aligned}\tag{24}$$

Using (24) and (23) in (22), we get the following relation.

$$\prod_{i=0}^{M}|\theta-i|\leq\frac{1}{4}(M+1)!,\quad \prod_{j=0}^{M}|\phi-j|\leq\frac{1}{4}(M+1)!,\quad \prod_{k=0}^{M}|\varphi-k|\leq\frac{1}{4}(M+1)!,\quad \prod_{l=0}^{M}|\nu-k|\leq\frac{1}{4}(M+1)!.\tag{25}$$

Now using the bounds (25) in (21) we get

$$\begin{aligned}
&\prod_{i=0}^{M}|(t-t_i)|=(\frac{\tau}{M})^{M+1}\frac{1}{4}(M+1)!, && \prod_{j=0}^{M}|(x-x_j)|=(\frac{a}{M})^{M+1}\frac{1}{4}(M+1)!,\\
&\prod_{k=0}^{M}|(y-y_k)|=(\frac{b}{M})^{M+1}\frac{1}{4}(M+1)!, && \prod_{l=0}^{M}|(z-z_k)|=(\frac{c}{M})^{M+1}\frac{1}{4}(M+1)!.
\end{aligned}\tag{26}$$

Using (26) in (27) we get the bound for the absolute error as

$$\begin{aligned}
&|u(t,x,y,z)-\hat{F}_{(M)}(t,x,y,x)|\leq\frac{\Upsilon_t}{4}(\frac{\tau}{M})^{M+1}+\frac{\Upsilon_x}{4}(\frac{a}{M})^{M+1}+\frac{\Upsilon_y}{4}(\frac{b}{M})^{M+1}\\
&+\frac{\Upsilon_z}{4}(\frac{c}{M})^{M+1}-\frac{\Upsilon_o}{256}(\frac{\tau}{M})^{M+1}(\frac{a}{M})^{M+1}(\frac{b}{M})^{M+1}(\frac{c}{M})^{M+1}.
\end{aligned}\tag{27}$$

Now using (17), we can easily get the upper bound of the absolute error of approximation

$$\begin{aligned}
&\|u(t,x,y,z)-U_{(M,M,M,M)}(t,x,y,z)\|_2\leq\sqrt[4]{(\tau abc)}\{\frac{\Upsilon_t}{4}(\frac{\tau}{M})^{M+1}+\frac{\Upsilon_x}{4}(\frac{a}{M})^{M+1}\\
&+\frac{\Upsilon_y}{4}(\frac{b}{M})^{M+1}+\frac{\Upsilon_z}{4}(\frac{c}{M})^{M+1}-\frac{\Upsilon_o}{256}(\frac{\tau}{M})^{M+1}(\frac{a}{M})^{M+1}(\frac{b}{M})^{M+1}(\frac{c}{M})^{M+1}\}.
\end{aligned}\tag{28}$$

4 Operational matrices of Integration and Differentiations

The operational matrices of derivatives and integration are the frequently used in literature. In [36] we derive the operational matrices of fractional order integration and derivatives for the two dimensional function vector of Legendre polynomial defined on $[0,1]\times[0,1]$. Here we define the notion on the three dimensional space $[0,a]\times[0,b]\times[0,c]$.

Theorem 4.0.1: Consider the function vector $\Psi^{(a,b,c)}(x,y,z)$ as defined in (11), then the operational matrix for fractional derivative of order σ of $\Psi^{(a,b,c)}(x,y,z)$ w.r.t x is generalized as

$$D_x^{\sigma}\Psi^{(a,b,c)}(x,y,z)={}^{x}W^{(\sigma,a,b,c)}_{M^3\times M^3}\Psi^{(a,b,c)}(x,y,z).\tag{29}$$

Where ${}^{x}W^{(\sigma,a,b,c)}_{M^3\times M^3}$ is defined as

$$
{}^{x}W^{(\sigma,a,b,c)}_{M^3\times M^3}=\begin{bmatrix}\Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,n'} & \cdots & \Omega_{1,M^3}\\ \Omega_{2,1} & \Omega_{2,2} & \cdots & \Omega_{2,n'} & \cdots & \Omega_{2,M^3}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Omega_{n,1} & \Omega_{n,2} & \cdots & \Omega_{n,n'} & \cdots & \Omega_{n,M^3}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Omega_{M^3,1} & \Omega_{M^3,2} & \cdots & \Omega_{M^3,n'} & \cdots & \Omega_{M^3,M^3}\end{bmatrix}, \tag{30}
$$

where $n'=M^2h+Mi+j+1, n=M^2q+Mr+s+1, h,i,j,q,r,s=0,1,2,...,m$ and

$$
\Omega_{n,n'}=C^{qrs}_{hij}=\frac{\delta_{(r,i)}\delta_{(s,j)}(2h+1)}{a}\sum_{l=0}^{h}\sum_{k=\lceil\sigma\rceil}^{q}\frac{(-1)^{q+k+h+l}(q+k)!(h+l)!}{(q-k)!(k!)\Gamma(k-\sigma+1)(a^k)(h-l)!(l!)^2(k+l-\sigma+1)}, \tag{31}
$$

With $C^{qrs}_{hij}=0$ if $q<\sigma$.

Proof: Consider the general element $P^{(a,b,c)}_n(x,y,z)$ as defined by (7), then by application of fractional derivative of order σ of $P^{(a,b,c)}_n(x,y,z)$ w.r.t x is given by relation

$$
D^{\sigma}_x(P^{(a,b,c)}_n(x,y,z))=P^b_r(y)P^c_s(z)\sum_{k=0}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)^2(a^k)}D^{\sigma}_x x^k.
$$

By definition 2.1 we may write

$$
D^{\sigma}_x(P^{(a,b,c)}_n(x,y,z))=\sum_{k=\lceil\sigma\rceil}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)\Gamma(k-\sigma+1)(a^k)}P^b_r(y)P^c_s(z)x^{k-\sigma}, q=\lceil\sigma\rceil,..,M. \tag{32}
$$

Approximating $P^b_r(y)P^c_s(z)x^{k-\sigma}$ by M terms of Legendre polynomials in three variables, we get

$$
P^b_r(y)P^c_s(z)x^{k-\sigma}\approx\sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}C_{hij}P^a_h(x)P^b_i(y)P^c_j(z), \tag{33}
$$

where $C_{hij}=\frac{(2h+1)(2i+1)(2j+1)}{abc}\int_0^c\int_0^b\int_0^a P^b_r(y)P^c_s(z)x^{k-\sigma}P^a_h(x)P^b_i(y)P^c_j(z)dxdydz$, After simplification we may write

$$
C_{hij}=\frac{\delta_{(r,i)}\delta_{(s,j)}(2h+1)}{a}\sum_{l=0}^{h}\frac{(-1)^{h+l}(h+l)!}{(h-l)!(l!)^2(k+l-\sigma+1)}, \tag{34}
$$

where

$$
\delta_{(r,i)}=\begin{cases}1, \text{ if } r=i\\ 0, \text{ if } i\neq u.\end{cases}
$$

Hence it follows that

$$
\begin{aligned}
D^{\sigma}_x P^a_q(x)P^b_r(y)P^c_s(z) &\approx \sum_{k=\lceil\sigma\rceil}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)\Gamma(k-\sigma+1)(a^k)}\sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}C_{hij}P^a_h(x)P^b_i(y)P^c_j(z),\\
&\approx \sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=\lceil\sigma\rceil}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)\Gamma(k-\sigma+1)(a^k)}C_{hij}P^a_h(x)P^b_i(y)P^c_j(z),\\
&\approx \sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}C^{qrs}_{hij}P^a_h(x)P^b_i(y)P^c_j(z).
\end{aligned} \tag{35}
$$

Where

$$
C^{qrs}_{hij}=\frac{\delta_{(r,i)}\delta_{(s,j)}(2h+1)}{a}\sum_{l=0}^{h}\sum_{k=\lceil\sigma\rceil}^{q}\frac{(-1)^{q+k+h+l}(q+k)!(h+l)!}{(q-k)!(k!)\Gamma(k-\sigma+1)(a^k)(h-l)!(l!)^2(k+l-\sigma+1)}. \tag{36}
$$

Using the notations, $n' = M^2h + Mi + j + 1$, $n = M^2q + Mr + s + 1$ and $\Omega_{n,n'} = C^{qrs}_{hij}$ for $h,i,j,q,r,s = 0,1,2,3,..m$, we get the desired proof.□

Theorem 4.0.2 Consider the function vector $\Psi^{(a,b,c)}(x,y,z)$ as defined in (11), then the operational matrix of fractional integral of order σ of $\Psi^{(a,b,c)}(x,y,z)$ w.r.t x is generalized as

$$I^{\sigma}_{x}\Psi^{(a,b,c)}(x,y,z) = {}^{x}V^{(\sigma,a,b,c)}_{M^3\times M^3}\Psi^{(a,b,c)}(x,y,z). \tag{37}$$

Where ${}^{x}V^{(\sigma,a,b,c)}_{M^3\times M^3}$ is defined as

$${}^{x}V^{(\sigma,a,b,c)}_{M^3\times M^3} = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,n'} & \cdots & \Omega_{1,M^3} \\ \Omega_{2,1} & \Omega_{2,2} & \cdots & \Omega_{2,n'} & \cdots & \Omega_{2,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{n,1} & \Omega_{n,2} & \cdots & \Omega_{n,n'} & \cdots & \Omega_{n,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{M^3,1} & \Omega_{M^3,2} & \cdots & \Omega_{M^3,n'} & \cdots & \Omega_{M^3,M^3} \end{bmatrix}, \tag{38}$$

where $n' = M^2h + Mi + j + 1$, $n = M^2q + Mr + s + 1$, $h,i,j,q,r,s = 0,1,2,...,m$ and

$$\Omega_{n,n'} = C^{qrs}_{hij} = \frac{\delta_{(r,i)}\delta_{(s,j)}(2h+1)}{a}\sum_{l=0}^{h}\sum_{k=0}^{q}\frac{(-1)^{q+k+h+l}(q+k)!(h+l)!}{(q-k)!(k!)\Gamma(k+\sigma+1)(a^k)(h-l)!(l!)^2(k+l+\sigma+1)}. \tag{39}$$

Proof: Consider the general element $P^{(a,b,c)}_{n}(x,y,z)$ as defined by (7), then the fractional order integral of order σ of $P^{(a,b,c)}_{n}(x,y,z)$ w.r.t x is given by relation

$$I^{\sigma}_{x}(P^{(a,b,c)}_{n}(x,y,z)) = P^{b}_{r}(y)P^{c}_{s}(z)\sum_{k=0}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)^2(a^k)}I^{\sigma}_{x}x^k.$$

Using the definition 2.1 we may write

$$I^{\sigma}_{x}(P^{(a,b,c)}_{n}(x,y,z)) = \sum_{k=0}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)\Gamma(k+\sigma+1)(a^k)}P^{b}_{r}(y)P^{c}_{s}(z)x^{k+\sigma}. \tag{40}$$

Approximating $P^{b}_{r}(y)P^{c}_{s}(z)x^{k+\sigma}$ by M terms of Legendre polynomials in three variables, we get

$$P^{b}_{r}(y)P^{c}_{s}(z)x^{k+\sigma} \approx \sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}C_{hij}P^{a}_{h}(x)P^{b}_{i}(y)P^{c}_{j}(z), \tag{41}$$

where $C_{hij} = \frac{(2h+1)(2i+1)(2j+1)}{abc}\int_0^c\int_0^b\int_0^a P^{b}_{r}(y)P^{c}_{s}(z)x^{k+\sigma}P^{a}_{h}(x)P^{b}_{i}(y)P^{c}_{j}(z)dxdydz$, Using the orthogonality condition we may write

$$C_{hij} = \frac{\delta_{(r,i)}\delta_{(s,j)}(2h+1)}{a}\sum_{l=0}^{h}\frac{(-1)^{h+l}(h+l)!}{(h-l)!(l!)^2(k+l+\sigma+1)}, \tag{42}$$

where

$$\delta_{(r,i)} = \begin{cases} 1, \text{ if } r = i \\ 0, \text{ if } i \neq u. \end{cases}$$

Hence it follows that

$$\begin{aligned} D^{\sigma}_{x}P^{a}_{q}(x)P^{b}_{r}(y)P^{c}_{s}(z) &\approx \sum_{k=0}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)\Gamma(k+\sigma+1)(a^k)}\sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}C_{hij}P^{a}_{h}(x)P^{b}_{i}(y)P^{c}_{j}(z), \\ &\approx \sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{q}\frac{(-1)^{q+k}(q+k)!}{(q-k)!(k!)\Gamma(k+\sigma+1)(a^k)}C_{hij}P^{a}_{h}(x)P^{b}_{i}(y)P^{c}_{j}(z). \\ &\approx \sum_{h=0}^{m}\sum_{i=0}^{m}\sum_{j=0}^{m}C^{qrs}_{hij}P^{a}_{h}(x)P^{b}_{i}(y)P^{c}_{j}(z). \end{aligned} \tag{43}$$

Where

$$C_{hij}^{qrs}=\frac{\delta_{(r,i)}\delta_{(s,j)}(2h+1)}{a}\sum_{l=0}^{h}\sum_{k=0}^{q}\frac{(-1)^{q+k+h+l}(q+k)!(h+l)!}{(q-k)!(k!)\Gamma(k+\sigma+1)(a^k)(h-l)!(l!)^2(k+l+\sigma+1)}. \tag{44}$$

Using the notations, $n'=M^2h+Mi+j+1, n=M^2q+Mr+s+1$ and $\Omega_{n,n'}=C_{hij}^{qrs}$ for $h,i,j,q,r,s=0,1,2,3,..m$, completes the proof.□

Theorem 4.0.3: Let $\Psi^{(a,b,c)}(x,y,z)$ be the function vector as defined in (11), then the fractional derivative of order σ of $\Psi^{(a,b,c)}(x,y,z)$ w.r.t y is given by

$$D_y^{\sigma}\Psi^{(a,b,c)}(x,y,z)={}^{y}W_{M^3\times M^3}^{(\sigma,a,b,c)}\Psi^{(a,b,c)}(x,y,z). \tag{45}$$

Where ${}^{y}W_{M^3\times M^3}^{(\sigma,a,b,c)}$ is the operational matrix of differentiation of order σ, and is defined as

$${}^{y}W_{M^3\times M^3}^{(\sigma,a,b,c)}=\left[\Omega_{n,n'}\right], \tag{46}$$

where $n'=M^2h+Mi+j+1, n=M^2q+Mr+s+1, h,i,j,q,r,s=0,1,2,...,m$ and

$$\Omega_{n,n'}=C_{hij}^{qrs}=\frac{\delta_{(h,q)}\delta_{(s,j)}(2i+1)}{b}\sum_{l=0}^{i}\sum_{k=\lceil\sigma\rceil}^{r}\frac{(-1)^{r+k+i+l}(r+k)!(i+l)!}{(r-k)!(k!)\Gamma(k-\sigma+1)(b^k)(i-l)!(l!)^2(k+l-\sigma+1)}, \tag{47}$$

With $C_{hij}^{qrs}=0$ if $r<\sigma$.

Proof: The proof of this Lemma is similar as theorem (4.0.1).

Theorem 4.0.4: Consider the function vector $\Psi^{(a,b,c)}(x,y,z)$ as defined in (11), then the operational matrix of order σ of $\Psi^{(a,b,c)}(x,y,z)$ w.r.t z is generalized as

$$D_z^{\sigma}\Psi^{(a,b,c)}(x,y,z)={}^{z}W_{M^3\times M^3}^{(\sigma,a,b,c)}\Psi^{(a,b,c)}(x,y,z). \tag{48}$$

Where ${}^{z}W_{M^3\times M^3}^{(\sigma,a,b,c)}$ is defined as

$${}^{z}W_{M^3\times M^3}^{(\sigma,a,b,c)}=\left[\Omega_{n,n'}\right], \tag{49}$$

where $n'=M^2h+Mi+j+1, n=M^2q+Mr+s+1, h,i,j,q,r,s=0,1,2,...,m$ and

$$\Omega_{n,n'}=C_{hij}^{qrs}=\frac{\delta_{(h,q)}\delta_{(i,r)}(2i+1)}{c}\sum_{l=0}^{j}\sum_{k=\lceil\sigma\rceil}^{s}\frac{(-1)^{s+k+j+l}(s+k)!(j+l)!}{(s-k)!(k!)\Gamma(k-\sigma+1)(c^k)(j-l)!(l!)^2(k+l-\sigma+1)}, \tag{50}$$

With $C_{hij}^{qrs}=0$ for $s<\sigma$.

Proof: By similar arguments as in Theorem (4.0.1) we may easily prove this Theorem.□

Theorem 4.0.5 Consider the function vector $\Psi^{(a,b,c)}(x,y,z)$ as defined in (11), then operational matrix of order σ of $\Psi^{(a,b,c)}(x,y,z)$ w.r.t y is generalized as

$$I_y^{\sigma}\Psi^{(a,b,c)}(x,y,z)={}^{y}V_{M^3\times M^3}^{(\sigma,a,b,c)}\Psi^{(a,b,c)}(x,y,z). \tag{51}$$

Where ${}^{y}V_{M^3\times M^3}^{(\sigma,a,b,c)}$ is defined as

$${}^{y}V_{M^3\times M^3}^{(\sigma,a,b,c)}=\left[\Omega_{n,n'}\right], \tag{52}$$

where $n'=M^2h+Mi+j+1, n=M^2q+Mr+s+1, h,i,j,q,r,s=0,1,2,...,m$ and

$$\Omega_{n,n'}=C_{hij}^{qrs}=\frac{\delta_{(h,q)}\delta_{(s,j)}(2i+1)}{b}\sum_{l=0}^{i}\sum_{k=0}^{r}\frac{(-1)^{r+k+i+l}(r+k)!(i+l)!}{(r-k)!(k!)\Gamma(k+\sigma+1)(b^k)(i-l)!(l!)^2(k+l+\sigma+1)}, \tag{53}$$

Proof: Using the arguments as in Theorem (4.0.2) we may prove the Theorem.

Theorem 4.0.6: Consider the function vector $\Psi^{(a,b,c)}(x,y,z)$ as defined in (11), then the operational matrix of fractional integral of order σ of $\Psi^{(a,b,c)}(x,y,z)$ w.r.t z is given by

$$I_z^\sigma \Psi^{(a,b,c)}(x,y,z) = {}^zW^{(\sigma,a,b,c)}_{M^3\times M^3}\Psi^{(a,b,c)}(x,y,z). \tag{54}$$

Where ${}^zW^{(\sigma,a,b,c)}_{M^3\times M^3}$ is defined as

$$^zW^{(\sigma,a,b,c)}_{M^3\times M^3} = \left[\Omega_{n,n'}\right], \tag{55}$$

where $n' = M^2h+Mi+j+1, n = M^2q+Mr+s+1$, $h,i,j,q,r,s = 0,1,2,...,m$ and

$$\Omega_{n,n'} = C^{qrs}_{hij} = \frac{\delta_{(h,q)}\delta_{(i,r)}(2i+1)}{c}\sum_{l=0}^{j}\sum_{k=0}^{s}\frac{(-1)^{s+k+j+l}(s+k)!(j+l)!}{(s-k)!(k!)\Gamma(k+\sigma+1)(c^k)(j-l)!(l!)^2(k+l+\sigma+1)}, \tag{56}$$

Proof: The proof of this Theorem is similar as Theorem (4.0.2).□

Theorem 4.0.7: [36] The operational matrix of integration of $\Psi^\tau(t)$,as defined in (6) is generalized as

$$I^\alpha(\Psi(t)\tau) \simeq P^\alpha\Psi^\tau(t), \tag{57}$$

where P^α is defined by

$$P^\alpha = \begin{pmatrix} \sum_{k=0}^{0}\Theta_{0,0,k} & \sum_{k=0}^{0}\Theta_{0,1,k} & \cdots & \sum_{k=0}^{0}\Theta_{0,j,k} & \cdots & \sum_{k=0}^{0}\Theta_{0,m,j} \\ \sum_{k=0}^{1}\Theta_{1,0,k} & \sum_{k=0}^{1}\Theta_{1,1,k} & \cdots & \sum_{k=0}^{1}\Theta_{1,j,k} & \cdots & \sum_{k=0}^{1}\Theta_{1,m,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{i}\Theta_{i,0,k} & \sum_{k=0}^{i}\Theta_{i,1,k} & \cdots & \sum_{k=0}^{i}\Theta_{i,j,k} & \cdots & \sum_{k=0}^{i}\Theta_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{m}\Theta_{m,0,k} & \sum_{k=0}^{m}\Theta_{m,1,k} & \cdots & \sum_{k=0}^{m}\Theta_{m,j,k} & \cdots & \sum_{k=0}^{m}\Theta_{m,m,k} \end{pmatrix}, \tag{58}$$

where

$$\Theta_{i,j,k} = \frac{(2j+1)}{\tau}\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k+\alpha+1)(j-l)!(l!)^2(k+l+\alpha+1)}. \tag{59}$$

5 Application of the new matrices to Fractional order Partial differential equations

The operational matrices derived in the previous section play important role in approximating the solution of fractional order partial differential equations. Consider a typical fractional order heat equation on cubic structure. For simplicity of notation we use simplified notation for the operational matrices. We use $\mathbf{P_x^\sigma}$,$\mathbf{P_y^\sigma}$ and $\mathbf{P_z^\sigma}$ to represent operational matrices of integration of order σ in x,y and z direction respectively. Similarly we use $\mathbf{D_x^\sigma}$,$\mathbf{D_y^\sigma}$ and $\mathbf{D_z^\sigma}$ to represent operational matrices for derivative of order σ.
The governing is

$$\begin{gathered} \chi_t\frac{\partial^\sigma U}{\partial t^\sigma} = \lambda_x\frac{\partial^{\beta_1}U}{\partial x^{\beta_1}} + \lambda_y\frac{\partial^{\beta_2}U}{\partial y^{\beta_2}} + \lambda_z\frac{\partial^{\beta_3}U}{\partial z^{\beta_3}} + I(t,x,y,z), \\ U(0,x,y,z) = f(x,y,z), \qquad U(t,0,y,z) = g_1(t,y,z), \qquad \frac{\partial}{\partial x}U(t,0,y,z) = g_2(t,y,z), \\ U(t,x,0,z) = h_1(t,x,z), \qquad \frac{\partial}{\partial y}U(t,x,0,z) = h_2(t,x,z), \\ U(t,x,y,0) = l_1(t,x,y), \qquad U(t,x,y,c) = l_2(t,x,y), \end{gathered} \tag{60}$$

where χ_t is the volumetric heat capacity$(J/(m^3K))$, $\lambda_x,\lambda_y,\lambda_z$ are thermal conductivities $(W/m.K)$ in x, y and z directions respectively, $0<\alpha\le 1$, $1<\beta_i\le 2$, $t\in[0,\tau]$, $x\in[0,a]$, $y\in[0,b]$ and $z\in[0,c]$. By making the following substitution in (60)

$$\hat{U} = U - \frac{z}{c}l_2(t,x,y) - \frac{(c-z)}{c}l_1(t,x,y), \tag{61}$$

we may write

$$
\begin{aligned}
&\chi_t \frac{\partial^{\sigma}\hat{U}}{\partial t^{\sigma}} = \lambda_x \frac{\partial^{\beta_1}\hat{U}}{\partial x^{\beta_1}} + \lambda_y \frac{\partial^{\beta_2}\hat{U}}{\partial y^{\beta_2}} + \lambda_z \frac{\partial^{\beta_3}\hat{U}}{\partial z^{\beta_3}} + \hat{I}(t,x,y,z),\\
&\hat{U}(0,x,y,z) = \hat{f}(x,y,z), \qquad \hat{U}(t,0,y,z) = \hat{g}_1(t,y,z) \qquad \frac{\partial}{\partial x}\hat{U}(t,0,y,z) = \hat{g}_2(t,y,z),\\
&\hat{U}(t,x,0,z) = \hat{h}_1(t,x,z) \qquad \frac{\partial}{\partial y}\hat{U}(t,x,0,z) = \hat{h}_2(t,x,z),\\
&\hat{U}(t,x,y,0) = 0 \qquad \hat{U}(t,x,y,c) = 0,
\end{aligned} \tag{62}
$$

Where $\hat{f},\hat{h_1},\hat{h_2},\hat{g_1},\hat{g_2},\hat{I}$ can be obtained using relation (61).

We seek the solution of the above problem in terms of shifted Legendre polynomials such that the following holds.

$$
\frac{\partial^{\beta_1+\beta_2}}{\partial x^{\beta_1}\partial y^{\beta_2}}\hat{U} = \Psi^{(\tau)}(t)^T K_{M\times M^3}\Psi^{(a,b,c)}(x,y,z) = \widehat{K_{M\times M^3}}. \tag{63}
$$

(Note that $\widehat{A} = \Psi^{(\tau)}(t)^T A \Psi^{(a,b,c)}(x,y,z)$, and is used through out only for the simplicity of notation.) Applying fractional integral of order β_1 w.r.t x on (63) we get

$$
\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\hat{U} = \widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{x}}^{\beta_1}} + \widehat{G}. \tag{64}
$$

where $\widehat{G} = \frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(\hat{g}_1(t,y,z) + x\hat{g}_2(t,y,z)\right)$. Similarly application of fractional integral of order β_2 w.r.t y on (63) we get

$$
\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\hat{U} = \widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{y}}^{\beta_2}} + \widehat{H}, \tag{65}
$$

where $\widehat{H} = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(\hat{h}_1(t,x,z) + y\hat{h}_2(t,x,z)\right)$. Now applying fractional order integral of order β_2 w.r.t y on (64) we get

$$
\hat{U} = \widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{x}}^{\beta_1}\mathbf{P}_{\mathbf{y}}^{\beta_2}} + \widehat{G\mathbf{P}_{\mathbf{y}}^{\beta_2}} + \widehat{H_1} \tag{66}
$$

Where $\widehat{H_1} = \hat{h}_1(t,x,z) + y\hat{h}_2(t,x,z)$. Using (66) we may write

$$
\frac{\partial^{\beta_3}}{\partial z^{\beta_3}}\hat{U} = \widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{x}}^{\beta_1}\mathbf{P}_{\mathbf{y}}^{\beta_2}\mathbf{D}_{\mathbf{z}}^{\beta_3}} + \widehat{G\mathbf{P}_{\mathbf{y}}^{\beta_2}\mathbf{D}_{\mathbf{z}}^{\beta_3}} + \widehat{H_1\mathbf{D}_{\mathbf{z}}^{\beta_3}}. \tag{67}
$$

Using (67), (65) and (64) in (62), we get

$$
\begin{aligned}
\chi_t \frac{\partial^{\sigma}\hat{U}}{\partial t^{\sigma}} =& \lambda_x \left(\widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{y}}^{\beta_2}} + \widehat{H}\right) + \lambda_z \left(\widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{x}}^{\beta_1}\mathbf{P}_{\mathbf{y}}^{\beta_2}\mathbf{D}_{\mathbf{z}}^{\beta_3}} + \widehat{G\mathbf{P}_{\mathbf{y}}^{\beta_2}\mathbf{D}_{\mathbf{z}}^{\beta_3}} + \widehat{H_1\mathbf{D}_{\mathbf{z}}^{\beta_3}}\right)\\
&+ \lambda_y \left(\widehat{K_{M\times M^3}\mathbf{P}_{\mathbf{x}}^{\beta_1}} + \widehat{G}\right) + \widehat{\mathbf{I}}.
\end{aligned} \tag{68}
$$

Where $\widehat{\mathbf{I}} = \hat{I}(t,x,y,z)$.On further simplification and using modified notation we get

$$
\frac{\partial^{\sigma}\hat{U}}{\partial t^{\sigma}} = \widehat{K_{M\times M^3}\mathbf{B}^{(\beta_1,\beta_2,\beta_3)}} + \widehat{\mathbf{Z}}. \tag{69}
$$

Where $\mathbf{B}^{(\beta_1,\beta_2,\beta_3)} = \frac{\lambda_x}{\chi_t}\mathbf{P}_{\mathbf{y}}^{\beta_2} + \frac{\lambda_z}{\chi_t}\mathbf{P}_{\mathbf{x}}^{\beta_1}\mathbf{P}_{\mathbf{y}}^{\beta_2}\mathbf{D}_{\mathbf{z}}^{\beta_3} + \frac{\lambda_y}{\chi_t}\mathbf{P}_{\mathbf{x}}^{\beta_1}$, and $\widehat{\mathbf{Z}} = \frac{\lambda_x}{\chi_t}\widehat{H} + \frac{\lambda_z}{\chi_t}\left(\widehat{G\mathbf{P}_{\mathbf{y}}^{\beta_2}\mathbf{D}_{\mathbf{z}}^{\beta_3}} + \widehat{H_1\mathbf{D}_{\mathbf{z}}^{\beta_3}}\right) + \frac{\lambda_y}{\chi_t}\left(\widehat{G}\right) + \frac{1}{\chi_t}\widehat{\mathbf{I}}$.

By the application of fractional integral of order σ w.r.t variable t we get

$$
\hat{U} = \widehat{\mathbf{P}^{\sigma T}K_{M\times M^3}\mathbf{B}^{(\beta_1,\beta_2,\beta_3)}} + \widehat{\mathbf{P}^{\sigma T}\mathbf{Z}} + \widehat{\mathbf{F_1}}. \tag{70}
$$

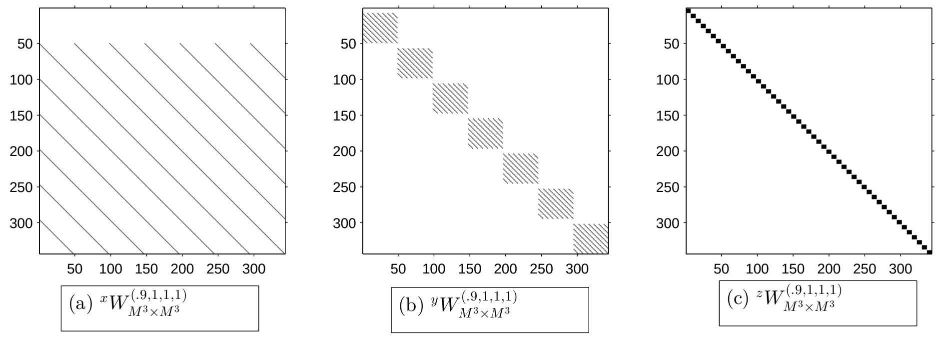

Fig. 1 The image display of operational matrices with selecting $\sigma = 0.9$ and $M = 7$.

Where $\overbrace{\mathbf{F_1}} = \hat{f}(x,y,z)$.

Now comparing (70) and (66) we get the following relation.

$$\overbrace{\mathbf{P}^{\sigma T} K_{M\times M^3} \mathbf{B}^{(\beta_1,\beta_2,\beta_3)}} + \overbrace{\mathbf{P}^{\sigma T}\mathbf{Z}} + \overbrace{\mathbf{F_1}} - \overbrace{K_{M\times M^3}\mathbf{P}_{\mathbf{x}}^{\beta_1}\mathbf{P}_{\mathbf{y}}^{\beta_2}} + \overbrace{G\mathbf{P}_{\mathbf{y}}^{\beta_2}} + \overbrace{H_1} = 0. \tag{71}$$

Equation (71) is generalized Sylvester type matrix equation and can be easily solvable for the unknown matrix $K_{M\times M^3}$. Using the value of K in (66) we get approximate solution $\hat{U}(t,x,y,z)$. Using the value of $\hat{U}$ in (61) will lead us to the desire solution of the problem.

6 Numerical Aspects of operational matrices

The operational matrices derived in the previous section are highly sparse in structure. And that is the reason that the resulting algebraic system of equations is easily solvable. For visualization purpose we show the image display of these matrices with the help of matlab command "imshow". Fig (1) show the matrices ${}^{x}W^{(0.9,a,b,c)}_{M^3\times M^3}$, ${}^{y}W^{(0.9,a,b,c)}_{M^3\times M^3}$ and ${}^{z}W^{(0.9,a,b,c)}_{M^3\times M^3}$ at scale level $M = 7$, while Fig (2) shows these matrices for $\alpha = 1.9$. One can easily observe the sparse structure of these matrices.

The operational matrices as shown above are very sparse. However they can approximate the fractional order partial derivatives very efficiently. To show the efficiency of the operational matrices we calculate fractional order partial derivatives of some test functions whose analytic form of the fractional derivatives are known.We select $f_1 = (xyzt)^5 - x^4y^3z^2t^5 + (xyzt)^3$, $f_2 = sin(x)cos(y) + sin(z) + cos(t)$ and $f_3 = (xyzt)^4 sin(x)sin(y) + cos(z)cos(t)$. In order to measure the accuracy we calculate the quantity $E_{f_i}$for every test functions at different scale level. Where E_{f_i} is defined by the relation

$$E_{f_1} = \frac{1}{\tau abc}\int_0^\tau\int_0^a\int_0^b\int_0^c |D_x^\alpha f(x,y,z,t) - f^{(\alpha)}_{approx}| dzdydxdt,$$

where $f^{(\alpha)}_{approx}$ is the fractional derivative of function f of order α calculated with the help of operational matrix. The results are displayed in Table(1). One can easily see that the accuracy increase with the increase of scale level.

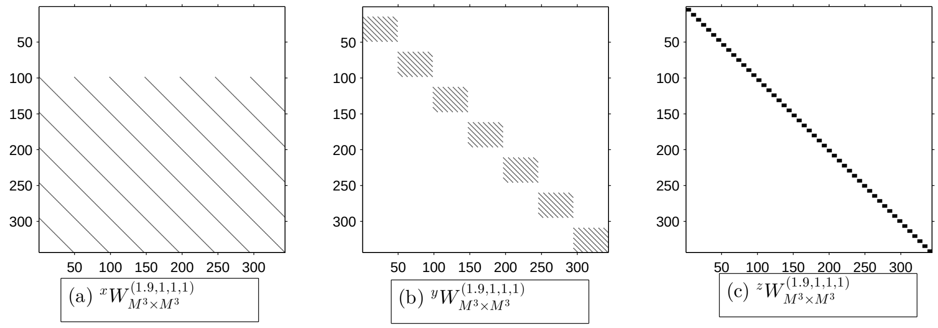

Fig. 2 The image display of operational matrices with selecting $\sigma = 1.9$ and $M = 7$.

Table 1: The approximated norm of calculating fractional derivatives

		$\alpha=0.7$			$\alpha=1.5$		
		$M=5$	$M=6$	$M=7$	$M=5$	$M=6$	$M=7$
E_{f_1}	D_x^{α}	0.0026	5.45×10^{-4}	3.41×10^{-5}	0.021	0.0093	1.4×10^{-4}
	D_y^{α}	0.0091	1.1×10^{-3}	0.9×10^{-4}	0.009	0.00012	1.5×10^{-5}
	D_z^{α}	0.0072	7.3×10^{-4}	6.9×10^{-5}	0.062	0.0013	4.6×10^{-4}
E_{f_2}	D_x^{α}	0.0142	0.0092	1.6×10^{-3}	0.00123	1.7411×10^{-4}	9.2045×10^{-5}
	D_y^{α}	0.07628	0.00163	4.9×10^{-3}	3.9×10^{-3}	7.613×10^{-4}	2.8×10^{-5}
	D_z^{α}	0.0071	2.49×10^{-3}	7.773×10^{-4}	0.0719	0.00182	2.65×10^{-3}
E_{f_3}	D_x^{α}	0.0038	1.4×10^{-3}	6.565×10^{-4}	0.0931	0.00192	2.39×10^{-4}
	D_y^{α}	0.0094	9.7×10^{-3}	1.947×10^{-4}	0.0751	7.91×10^{-3}	4.99×10^{-4}
	D_z^{α}	0.05932	0.00734	2.89×10^{-4}	0.0104	5.53×10^{-3}	4.052×10^{-4}

6.1 Illustrative Examples

To show the applicability of the method, we solve some test problems.

Example 1 As a First example consider the following integer order heat conduction equation.

$$\begin{aligned} &\chi_t\frac{\partial U}{\partial t} = \lambda_x\frac{\partial^2 U}{\partial x^2} + \lambda_y\frac{\partial^2 U}{\partial y^2} + \lambda_z\frac{\partial^2 U}{\partial z^2} + I, \\ &U(0,x,y,z) = (1-y)e^{(x+z)}, \quad U(t,0,y,z) = (1-y)e^{(z+2t)} \quad \frac{\partial}{\partial x}U(t,0,y,z) = (1-y)e^{(z+2t)}, \\ &U(t,x,0,z) = e^{(x+z+2t)} \quad \frac{\partial}{\partial y}U(t,x,0,z) = -e^{(x+z+2t)}, \\ &U(t,x,y,1) = (1-y)e^{(x+z+2)} \quad \frac{\partial}{\partial x}U(t,x,y,1) = (1-y)e^{(x+z+2)}, \end{aligned} \tag{72}$$

Also let $\chi_t = \lambda_x = \lambda_y = \lambda_z = 1$, $t \in [0,1]$, $x \in [0,1]$, $y \in [0,1]$ and $z \in [0,1]$. It can be easily verified that the exact solution of the problem is

$$U(t,x,y,z) = (1-y)e^{(x+z+2t)}.$$

We approximate the solution of this problem with proposed method, and as expected we found that the approximate solution matches very well with the exact solution. We display the exact and approximate solution of the problem at some

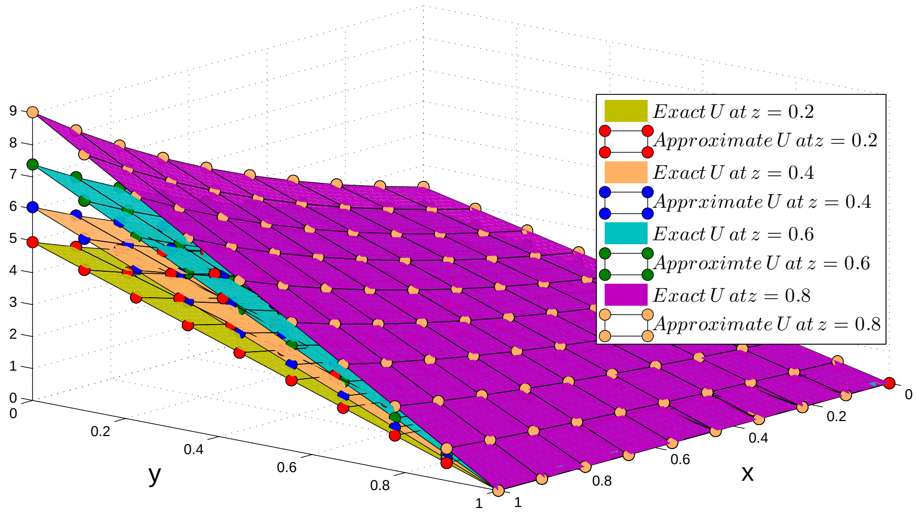

Fig. 3 Comparison of exact(surface) and approximate solution (dots) of example 1 at different value of z $M=13$, $t=0.3$.

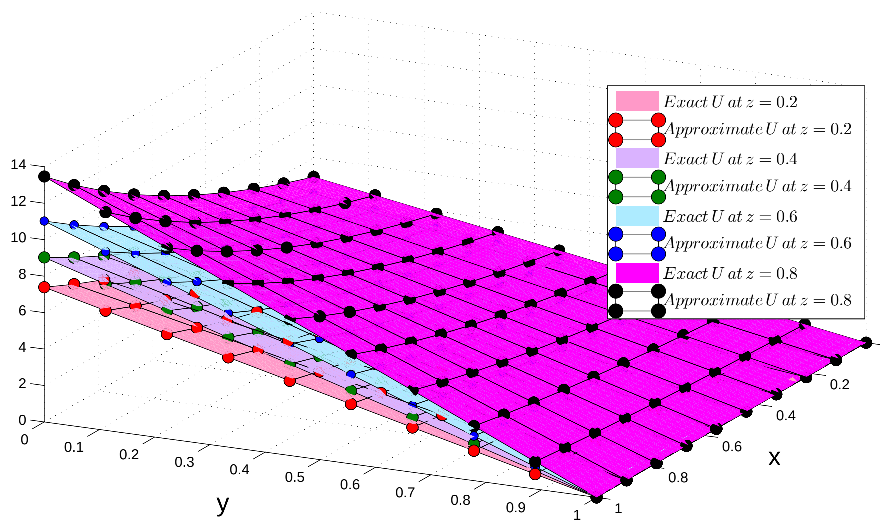

Fig. 4 Comparison of exact(surface) and approximate solution (dots) of example 1 at different value of z $M=13$, $t=0.6$.

fixed value of t,ie $t=0.3,0.6,0.9$ and at each value of t the solution is displayed at fix value of z, the results are displayed in $Fig(3)$,$Fig(4)$ and $Fig(5)$. Note that here we fix $M=10$. We observe that the method yields a very high accurate estimate of the solution.And the error of approximation (absolute error) decreases significantly by the increase of the scale level M. We approximate the absolute error at $M=7,8,9$ and we observe that as the scale level increases the error decreases.$Fig(6)$ and $Fig(7)$ shows amount of absolute error at point $(x,y,z)=(0.2,0.2,0.2)$ and point $(x,y,z)=(0.8,0.8,0.8)$ respectively.

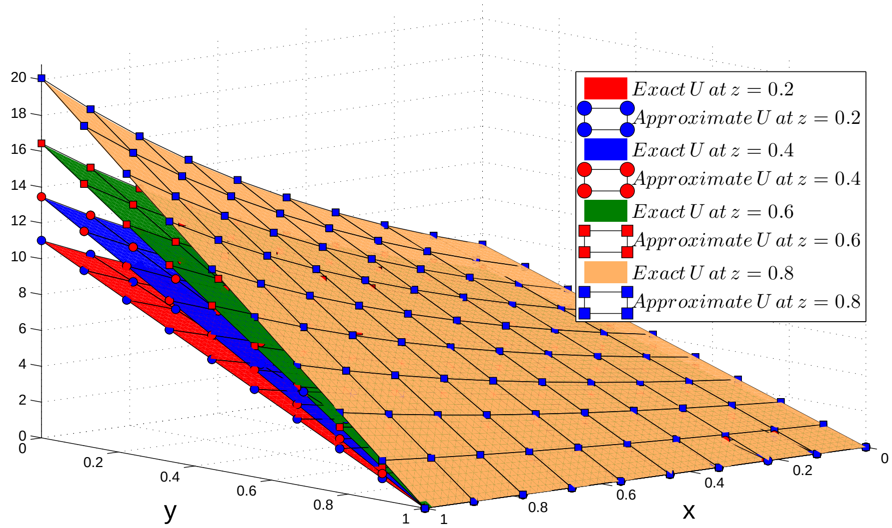

Fig. 5 Comparison of exact(surface) and approximate solution (dots) of example 1 at different value of z $M = 13$, $t = 0.9$.

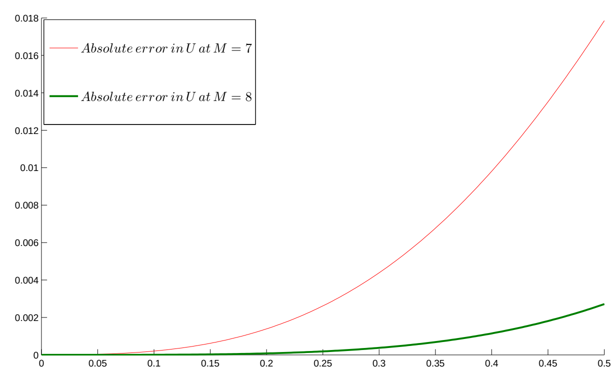

Fig. 6 The absolute error of example 1 at different value of M.Here we fix $(x,y,z) = (0.2,0.2,0.2)$,$\sigma = 1$

Example 2 Consider the time fractional heat conduction problem

$$\begin{aligned} &\frac{\partial^{0.7}U}{\partial t^{0.7}} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + I, \\ U(0,x,y,z) = x^2y^2z^2 - y^4z^4, \quad &U(t,0,y,z) = t^3z^3 \quad \frac{\partial}{\partial x}U(t,0,y,z) = 0, \\ &U(t,x,0,z) = t^3z^3 \quad \frac{\partial}{\partial y}U(t,x,0,z) = 0, \\ &U(t,x,y,0) = (txy)^2, \quad U(t,x,y,1) = t^3 + (xy+txy)^2 + t^4x^4y^4, \end{aligned} \tag{73}$$

Take the source term

$$I(t,x,y,z) = \frac{30108612877763975\,t^{\frac{3}{10}}\left(4000t^3x^4y^4z^4 + 3300t^2z^3 + 2530\,t\,x^2y^2 + 3289x^2y^2z\right)}{44437017523264684032}.$$

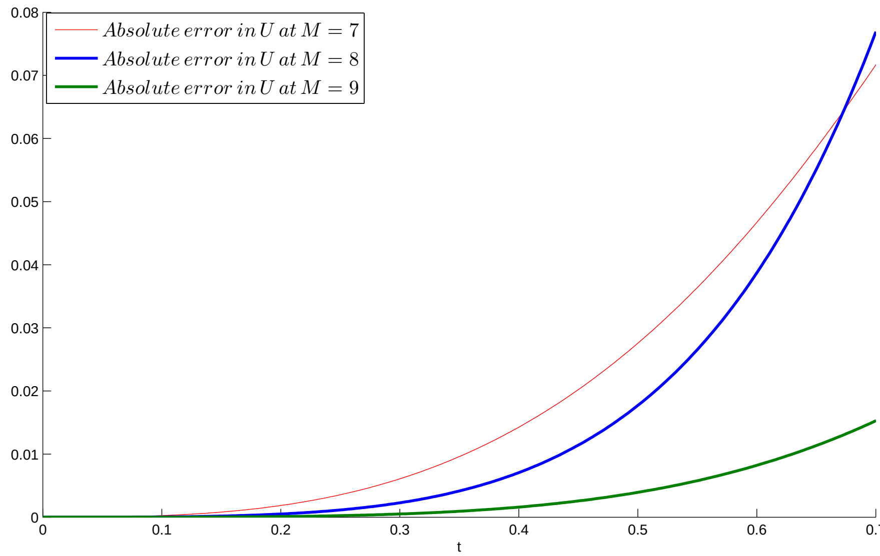

Fig. 7 Absolution difference of exact and approximate solution of example 1 at different value of M.Here we fix $(x,y,z) = (0.8,0.8,0.8)$,$\sigma = 1$

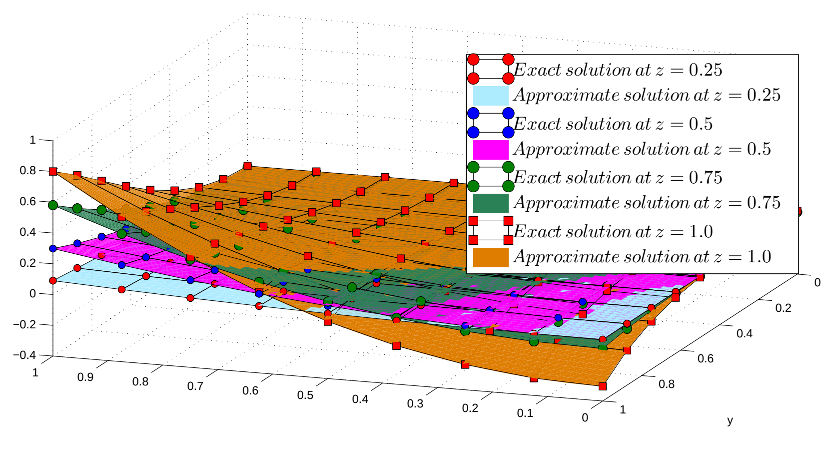

Fig. 8 Comparison of exact(dots) and approximate solution (surface) of example 2 at different value of z $M = 13$, $t = 0.3$.

Then the unique analytic solution of the problem is

$$U(t,x,y,z) = t^3z^3 - y^4z^4 + (txy + xyz)^2 + t^4x^4y^4z^4.$$

We compare the exact with the approximate solution obtained with the proposed method at different value of t and z. The results are displayed in $Fig(8), Fig(9)$ and $Fig(10)$. One can easily see that at $M = 12$ the approximate solution matches very well with the exact solution. The absolute error is approximated at different points of xt-plane and yz-plane. One can see that the absolute error is much more less than 10^{-12}, see $Fig(11)$ and $Fig(12)$.

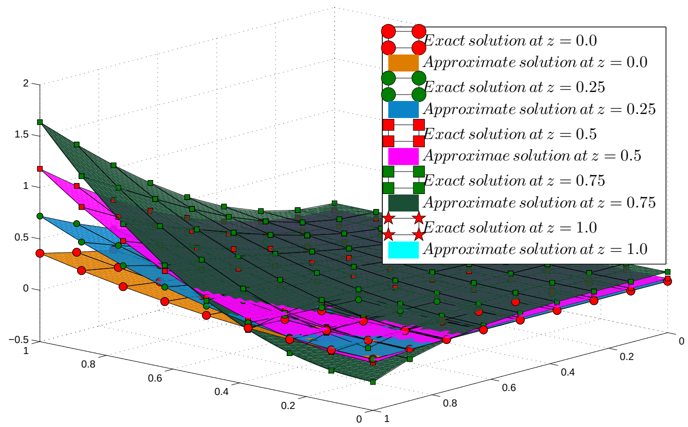

Fig. 9 Comparison of exact(dots) and approximate solution (surface) of example 2 at different value of z $M = 13$, $t = 0.6$.

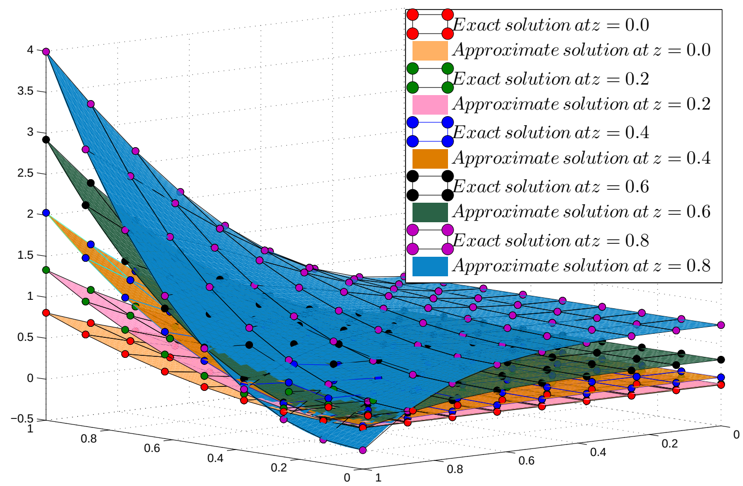

Fig. 10 Comparison of exact(dots) and approximate solution (surface) of example 2 at different value of z $M = 13$, $t = 0.9$.

Example 3 As a third example, we consider the space and time fractional Heat conduction equation as given.

$$\begin{aligned}
&\frac{\partial^{0.7}U}{\partial t^{0.7}} = \frac{\partial^{1.8}U}{\partial x^{1.8}} + \frac{\partial^{1.8}U}{\partial y^{1.8}} + \frac{\partial^{1.8}U}{\partial z^{1.8}} + I,\\
&U(0,x,y,z) = x^2y^2z^2 - yz, \quad U(t,0,y,z) = t^2z^2 - yz \quad \frac{\partial}{\partial x}U(t,0,y,z) = 0,\\
&U(t,x,0,z) = t^2z^2 \quad \frac{\partial}{\partial y}U(t,x,0,z) = -z,\\
&U(t,x,y,0) = (txy)^2, \quad U(t,x,y,1) = t^2 - y + (xy + txy)^2 + t^3x^3y^3,
\end{aligned} \tag{74}$$

where

$$\begin{aligned}
I(t,x,y,z) =& 0.0330t^{1/5}\{75(xyz)^3t^2 + 55(xy)^2t + 55tz^2 + 66(xy)^2z\}\\
&- 1.0891z^{1/5}\{5z(xyt)^3 + 2t^2 + 2(xy)^2\} - 1.0891\phi(t,x,y,z)\{x^2y^{1/5} + x^1/5y^2\}.
\end{aligned}$$

Where

$$\phi(t,x,y,z) = 5xyt^3z^3 + 2t^2 + 4tz + 2z^2.$$

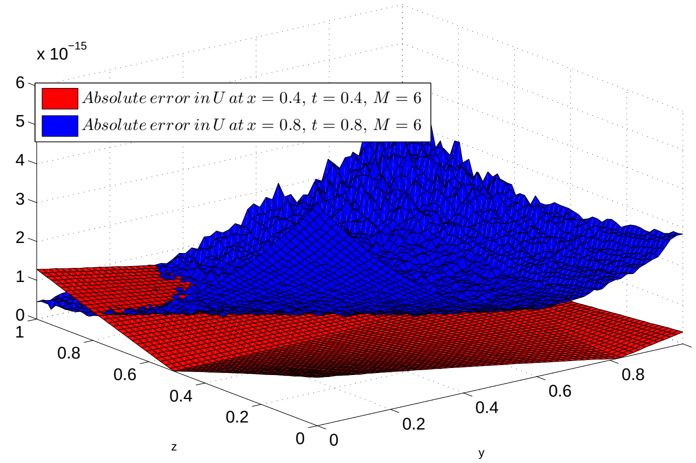

Fig. 11 The absolute error of example 2 at $M = 13$ on two different points of the $xt-plane$.

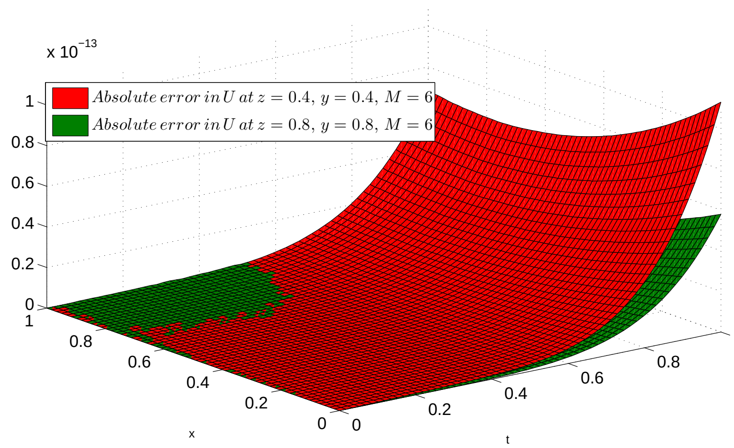

Fig. 12 The absolute error of example 2 at $M = 12$ on two different points of the $yz-plane$.

This problem has exact analytic solution as

$$U = t^2z^2 + (txy + xyz)^2 - yz + t^3x^3y^3z^3.$$

One can easily check it by direct substitution. We approximate the solution of this problem at different scale level. We compare exact and approximate solution of this problem Fig (13)and Fig (14) and observe the high accuracy of the approximate solution. We approximate the absolute error at different scale level. We observe the convergence of approximate solution to the exact solution with the increase of the scale level. We approximate the absolute error of this example at three fix value of t while z is fix to be 0.5, see Fig (15).

7 conclusion

The algorithm presented in this paper is complicated but provides a very high accurate estimate of the approximate solution.The method is spectral method and its accuracy depends on the smoothness of solution. We observe that the method can easily solve fractional order partial differential equations in four variables. It is also expected that the method

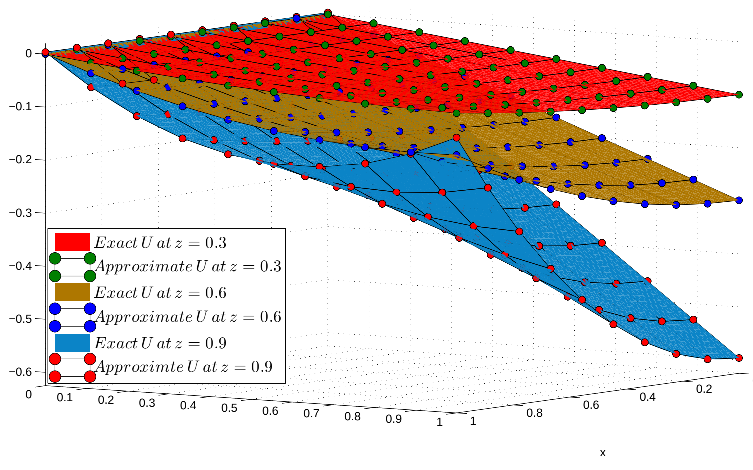

Fig. 13 The comparison of exact solution (surfaces) with approximate solution of example 3 at $M=20$, $t=0.3$.

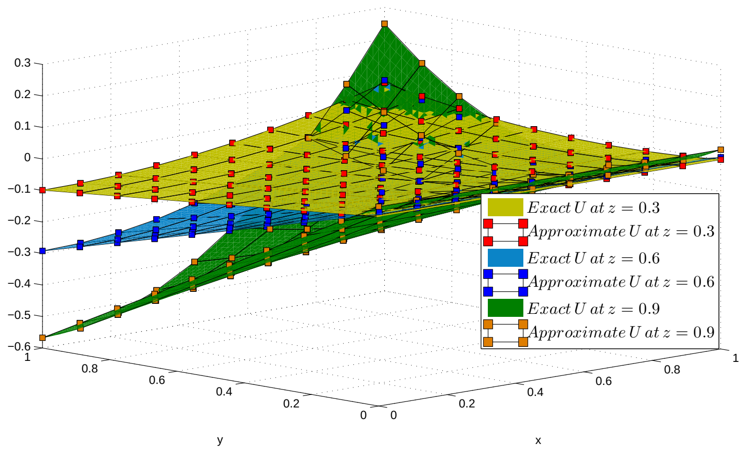

Fig. 14 The comparison of exact solution (surfaces) with approximate solution of example 3 at $M=20$, $t=0.7$.

may provide a more accurate estimate by using some other families of orthogonal polynomials like Brenstein and Laguerre.

Acknowledgement

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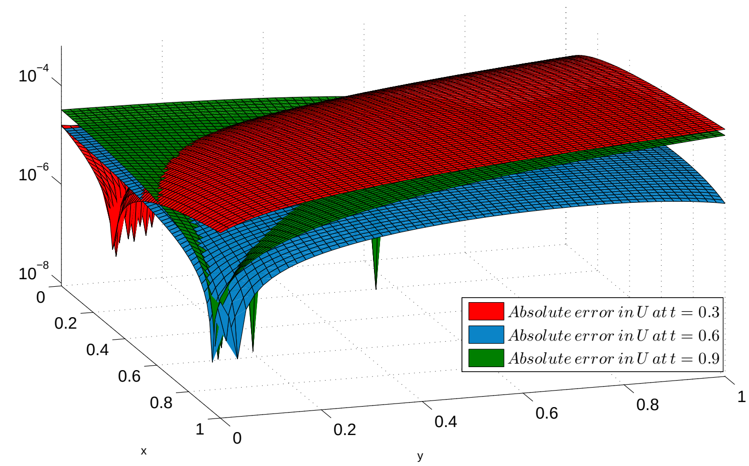

Fig. 15 Absolute difference of exact solution and approximate of example 3 at $z = 0.5$ and different value of t, M=20.

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