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Biconvex Functions and Mixed Bivariational Inequalities

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Abstract: Some concepts of the biconvex sets and biconvex functions are considered in this paper. Properties of the strongly biconvex convex functions are investigated under suitable conditions. The minimum of the sum of differentiable biconvex functions and nondifferentiable biconvex functions is characterized by variational inequality, which is called mixed bivariational inequality. The auxiliary principle technique is used to propose and investigate some iterative methods along with convergence criteria. Some important special cases as applications are discussed. Results obtained in this paper can be viewed as significant refinement and improvement of previously known results.

Keywords: Convex functions, biconvex functions, mixed bivariational inequalities, iterative methods, convergence criteria.
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1 Introduction

Convexity theory is a branch of mathematical sciences with a wide range of applications in industry, physics, social, regional and engineering sciences. It is worth mentioning that variational inequalities represent the optimality conditions for the differentiable convex functions on the convex sets. The convex sets and convex functions have been extended and generalized in several directions using innovative ideas to consider completed problems. See an excellent book by Cristescu and Lupsa [1]. Inspired by the research work going on in this field, Noor and Noor [2–4] introduced and and considered a new class of nonconvex sets and nonconvex functions with respect to an arbitrary bifunction. This class of nonconvex set is called the biconvex set and the noncovex function is called biconvex function. functions is called the biconvex functions. Noor et al [2–4] have studied some basic properties of the biconvex functions. It have been shown that the biconvex functions have characterizations as the convex functions enjoy.
Mixed variational inequalities involving term can be viewed as novel extension of variational inequalities. which were introduced and studied by Stampacchia [5] and Lions et al [6]. Mixed variational inequalities have witnessed an explosive growth in theoretical advances, algorithmic developments and applications across almost all disciplines of engineering, pure and applied sciences.and There are several methods for solving mixed variational inequalities. Due to the nature of the mixed variational inequalities, projection and Wiener-Hopf methods cannot be applied for solving mixed variational inequalities. In recent years, the auxiliary principle technique is being used to suggest and analyze some iterative methods for solving variational inequalities and equilibrium problems. Glowinski et al [7] used this technique to study the existence problem for mixed variational inequalities, whereas Noor [8–11] and Zhu et al. [12] have used this approach to suggest and analyze some iterative methods for solving various classes of variational inequalities and equilibrium problems. For more details, see [5–16] and the references therein.
In this paper, we consider the mixed bivariational inequalities, which arise as a sum of differentiable biconvex function and nondiferentiable biconvex function. The auxiliary principle technique is used to suggest several new iterative schemes for mixed bivariational inequalities. We also prove that the convergence of these methods require either pseudomonotonicity or partially relaxed strongly monotonicity. These are weaker conditions than monotonicity. As a special case, we obtain new iterative schemes for solving mixed bivariational inequalities,

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variational inequalities and optimization problem. The comparison of these methods with other methods is a subject of future research.

2 Preliminaries and basic results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. Let $F : K_\beta \to R$ be a continuous function and let $\beta(.-.) : K_\beta \times K_\beta \to R$ be an arbitrary continuous bifunction.
We now recall the known concepts and basic results, which are mainly due to Noor and Noor [2–4].

Definition 1. *The set K_β in H is said to be biconvex set with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$u+\lambda\beta(v-u) \in K_\beta, \qquad \forall u,v \in K_\beta, \lambda \in [0,1].$$

The biconvex set K_β is also called β-connected set. Note that the biconvex set with $\beta(v,u) = v-u$ is a convex set K, but the converse is not true.
For example, the set $K_\beta = R - (-\frac{1}{2},\frac{1}{2})$ is an biconvex set with respect to η, where

$$\beta(v-u) = \begin{cases} v-u, \text{ for } & v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, \text{ for } & v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K_β is not a convex set.

From now onward K_β is a nonempty closed biconvex set in H with respect to the bifunction $\beta(\cdot - \cdot)$, unless otherwise specified.

We now introduce some new concepts of biconvex functions and their variants forms.

Definition 2. *The function F on the biconvex set K_β is said to be a strongly biconvex with respect to the bifunction $\beta(\cdot-\cdot)$, if there exists a constant $\mu > 0$ such that*

$$F(u+\lambda\beta(v-u)) \le (1-\lambda)F(u) + \lambda F(v) \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \\ \forall u,v \in K_\beta, \lambda \in [0,1].$$

The function F is said to be strongly biconcave, if and only if, $-F$ is strongly biconvex function. Consequently, we have a new concept.

Definition 3. *A function F is said to be strongly affine involving an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$F(u+\lambda\beta(v-u)) = (1-\lambda)F(u) + \lambda F(v) \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \\ \forall u,v \in K_\beta, \lambda \in [0,1].$$

Note that every strongly biconvex function is a strongly affine biconvex, but the converse is not true.

If $\beta(v-u) = v-u$, then the strongly biconvex function becomes a strongly convex function, that is,

$$F(u+\lambda(v-u)) \le (1-\lambda)F(u) + \lambda F(v) \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \\ \forall u,v \in K, \lambda \in [0,1].$$

For the properties of the convex functions in variational inequalities and equilibrium problems, see Noor [7–10, 14], Zhu et al. [12] and Patriksson [16].

Definition 4. *The function F on the biconvex set K_β is said to be strongly quasi biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u+\lambda\beta(v-u)) \le \max\{F(u),F(v)\} \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \\ \forall u,v \in K_\beta, \lambda \in [0,1].$$

Definition 5. *The function F on the biconvex set K_β is said to be strongly* log-*biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u+\lambda\beta(v-u)) \le (F(u))^{1-\lambda}(F(v))^{\lambda} \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \\ \forall u,v \in K_\beta, \lambda \in [0,1].$$

where $F(\cdot) > 0$.

We can rewrite the Definition 5 in the following form

Definition 6. *The function F on the biconvex set K_β is said to be strongly* log-*biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$\log F(u+\lambda\beta(v-u)) \le (1-\lambda)\log F(u) + \lambda \log F(v) \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \\ \forall u,v \in K_\beta, \lambda \in [0,1].$$

where $F(\cdot) > 0$.

This definition can be used to discus the properties of the differentiable strongly log-biconvex functions.

From the above definitions, we have

$$F(u+\lambda\beta(v-u)) \le (F(u))^{1-\lambda}(F(v))^{\lambda} \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2 \\ \le (1-\lambda)F(u) + \lambda F(v) \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2 \\ \le \max\{F(u),F(v)\} \\ -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2.$$

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This shows that every strongly log-biconvex function is strongly biconvex function and every strongly biconvex function is a strongly quasi-biconvex function. However, the converse is not true.

For $\lambda = 1$, Definition 2 and 5 reduce to the following condition.

Assumption 1

$$F(u+\beta(v-u)) \leq F(v), \quad \forall v,u \in K_\beta,$$

which is called the Condition A.

To derive the main results, we need the following assumptions regarding the bifunction $\beta(\cdot - \cdot)$.

Assumption 2. *The bifunction $\beta(,-,)$ said to satisfy the assumptions, if*

(i). $\beta(\gamma\beta(v-u)) = \gamma\beta(v-u), \forall u,v \in K_\beta, \gamma \in R^n$.
(ii). $\beta(v-u-\gamma\beta(v-u)) = (1-\gamma)\beta(v-u), \forall u,v \in K_\beta$,

which is called the Condition M.

Remark. Let $\beta(\cdot - \cdot) : K_\beta \times K_\beta \to H$ satisfy the assumption

$$\beta(v-u) = \beta(v-z) + \beta(z-u), \forall u,v,z \in K_\beta.$$

One can easily show that $\beta(v-u) = 0 \quad \forall u,v \in K_\beta$. Consequently

$$\beta(v-u) = 0 \quad \Leftrightarrow \quad v = u, \forall u,v \in K_\beta.$$

Also

$$\beta(v-u) + \beta(u-v) = 0, \quad \forall u,v \in K_\beta.$$

This implies that the bifunction $\beta(.-.)$ is skew symmetric.

Theorem 1. *Let K_β be a biconvex function in H and the condition M hold. If the function F is a differentiable strongly biconvex function with constant $\mu > 0$, then the following are equivalent.*

(i). *The function F is a strongly biconvex function.*
(ii). $F(v) - F(u) \geq \langle F'(u), \beta(v-u)\rangle + \mu\|\beta(v-u)\|^2, \forall u,v \in K_\beta$.
(iii). $\langle F'(u), \beta(v-u)\rangle + \langle F'(v), \beta(u-v)\rangle \leq -\mu\{\|\beta(v-u)\|^2 + \|\beta(u-v)\|^2\|\}, \forall u,v \in K_\beta$.

3 Main Results

In this section, we consider and study the mixed bivariational inequalities. Some iterative methods are suggested for finding the approximate solution of the mixed bivariational inequalities. First of all, we discuss the optimality conditions for the differentiable biconvex functions. To be more precise, we consider the energy functional $I[v]$ defined as:

$$I[v] = F(v) + \phi(v), \quad \forall v \in H, \tag{3.1}$$

where F and ϕ are two suitable biconvex functions.

Theorem 2. *Let F be a differentiable biconvex function and ϕ be a nondifferentiable biconvex function. If $u \in K_\beta$ is the minimum of the energy functional $I[v]$, if and only if, $u \in K_\beta$ satisfies the*

$$\langle F'(u), \beta(v-u)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in K_\beta. \tag{3.2}$$

Proof. Let $u \in K_\beta$ be a minimum of the functional $I[v]$. Then

$$I(u) \leq I(v), \forall v \in K_\beta. \tag{3.3}$$

Since K_β is a biconvex set, so, $\forall u,v \in K_\beta, \quad \lambda \in [0,1]$,

$$v_\lambda = u + \lambda\beta(v-u) \in K_\beta.$$

Taking $v = v_\lambda$ in (3.3), we have

$$\begin{aligned} F(u) + \phi(u) &\leq F(u+\lambda\beta(v-u)) + \phi(u+\lambda\beta(v-u)) \\ &\leq F(u+\lambda\beta(v-u)) + \phi(u) + \lambda(\phi(v) - \phi(u)), \end{aligned}$$

from which, we have

$$\begin{aligned} 0 &\leq \lim_{\lambda\to 0}\{\frac{F(u+\lambda\beta(v-u)) - F(u)}{\lambda}\} + \phi(v) - \phi(u) \\ &\leq \langle F'(u), \beta(v-u)\rangle + \phi(v) - \phi(u), \end{aligned} \tag{3.4}$$

which is the inequality (3).
Conversely, let $u \in K_\beta$ satisfy (3). We have to show that $u \in K_\beta$ is the minimum of the functional $I[v]$ defined by (3.1).
Since F is differentiable biconvex function, so

$$F(u+\lambda\beta(v-u)) \leq F(u) + \lambda(F(v) - F(u)), \forall u,v \in K_\beta,$$

which implies that

$$F(v) - F(u) \geq \lim_{\lambda\to 0}\{\frac{F(u+\lambda\beta(v-u)) - F(u)}{\lambda}\} \tag{3.5}$$

From (3.1), (3) and (3.5), we obtain

$$\begin{aligned} I[u] - I[v] &= -\{F(v) - G(u) + \phi(v) - \phi(u)\} \\ &\leq -\{\langle F'(u), \beta(v-u)\rangle + \phi(v) - \phi(u)\} \\ &\leq 0. \end{aligned}$$

This implies that

$$I[u] \leq I[v], \quad \forall v \in K_\beta,$$

This shows that $u \in K_\beta$ is the minimum of the functional $I[v]$ defined by (3).

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The inequality of the type (3) is called the mixed bivariational inequality and appears to new one.

It is worth mentioning that inequalities of the type (3) may not arise as a minimization of the biconvex functions. This motivated us to consider a more general mixed bivariational inequality of which (3) is a special case.

For a given operator T, bifunction $\beta(.-.)$ and continous function ϕ, consider the problem of finding $u\in H$, such that

$$\langle Tu,\beta(v-u)\rangle+\phi(v)-\phi(u)\geq 0,\forall v\in H, \quad (3.6)$$

which is called mixed bivariational inequality.

It is worth mentioning that for suitable and appropriate choice of the operators, biconvex sets and spaces, one can obtain a wide class of variational inequalities and optimization problems. This shows that the mixed bivariational inequalities are quite flexible and unified ones.

Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these bivariational inequalities. To overcome these drawback, one uses the auxiliary principle technique of Glowinski et al. [7] to suggest and analyze some iterative methods for solving the mixed bivariational-like inequalities(3.6). This technique does not involve the concept of the projection, which is the main advantage of this technique. We again use the auxiliary principle technique coupled with Bergman functions. These applications are based on the type of convex functions associated with the Bregman distance. We now suggest and analyze some iterative methods for mixed bivariational inequalities (3.6) using the auxiliary principle technique coupled with Bregman distance functions.

For a given $u\in K_\beta$ satisfying the bivariational inequality (3.6), we consider the auxiliary problem of finding a $w\in K$ such that

$$\langle\rho Tw,\beta(v-w)+\langle E'(w)-E'(u),\beta(v-w)\rangle +\phi(v)-\phi(u)\geq 0,\quad \forall v\in H, \quad (3.7)$$

where $\rho>0$ is a constant and $E'(u)$ is the differential of a strongly biconvex function $E(u)$ at $u\in K_\beta$. Since $E(u)$ is a strongly biconvex function, this implies that its differential E' is strongly β-monotone. Consequently it follows that the problem (3.6) has an unique solution.

Remark. The function

$$B(w,u)=E(w)-E(u)-\langle E'(u),\beta(w,u)\rangle$$

associated with the biconvex function $E(u)$ is called the generalized Bregman function. By the strongly boiconvexity of the function $E(u)$, the Bregman function $B(.,.)$ is nonnegative and $B(w,u)=0$, if and only if $u=w,\forall u,w\in K_\beta$.

We note that, if $w=u$, then clearly w is solution of the mixed bivariational inequality (3.7). This observation enables us to suggest and analyze the following iterative method for solving (3.7).

Algorithm 1. *For a given $u_0\in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle\rho Tu_{n+1},\beta(v-u_{n+1})\rangle+\langle E'(u_{n+1})-E'(u_n),\beta(v-u_{n+1})\rangle + \phi(v)-\phi(u_{n+1})\geq 0,\quad \forall v\in H, \quad (3.8)$$

where $\rho>0$ is a constant. Algorithm 1 is called the proximal method for solving mixed bivariational inequalities (3.6). In passing we remark that the proximal point method was suggested in the context of convex programming problems as a regularization technique.

If $\beta(v-u)=v-u$, then Algorithm 1 collapses to:

Algorithm 2. *For a given $u_0\in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle\rho T(u_{n+1}),v-u_{n+1}\rangle+\langle E'(u_{n+1})-E'(u),v-u_{n+1} +\phi(v)-\phi(u)\geq 0,\quad \forall v\in H,$$

for solving the mixed variational inequality.

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational inequalities and related problems.

Theorem 3. *Let the operator T be pseudomonotone. Let E be differentiable higher order strongly biconvex function with module $\nu>0$ and Condition M hold. If $\rho\mu\leq\nu$, then the approximate solution u_{n+1} obtained from Algorithm 1 converges to a solution $u\in K$ satisfying the mixed bivariational inequality(3.6).*

*Proof.*Let $u\in H$ be a solution of mixed bivariational inequality(3.6). Then

$$\langle Tu,\beta(v-u)\rangle+\phi(v)-\phi(u)\geq 0,\quad \forall v\in H,$$

implies that

$$-\langle Tv,\beta(u-v))+\phi(v)-\phi(u)\rangle\geq 0,\quad \forall v\in H, \quad (3.9)$$

since T is β-pseudomonotone.

Taking $v=u$ in (3.8) and $v=u_{n+1}$ in (3.9), we have

$$\langle\rho T(u_{n+1}),\beta(u,u-n+1)\rangle +\langle E'(u_{n+1})-E'_k(u_n,\beta(u-u_{n+1})\rangle +\phi(u)-\phi(u_{n+1})\geq 0. \quad (3.10)$$

and

$$-\langle Tu_{n+1},\beta(u-u_{n+1})\rangle+\phi(v)-\phi(u_{n+1})\geq 0. \quad (3.11)$$

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We now consider the Bregman distance function

$$B(u,w) = E(u) - E(w) - \langle E'(w, \beta(u-w)\rangle \geq \nu\|\beta(u-w)\|^2, \quad (3.12)$$

using higher order strongly biconvexity of E.
Now combining (3.12),(3.10) and (3.11), we have

$$\begin{aligned} &B(u,u_n) - B(u,u_{n+1}) \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \beta(u-u_n)\rangle \\ &\quad + \langle E'(u_{n+1}), \beta(u-u_{n+1})\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}, \beta(u-u_{n+1})\rangle \\ &\quad - \langle E'(u_n, u_{n+1}-u_n\rangle \\ &\geq \nu\|\beta(u_{n+1}-u_n)\|^2 + \langle E'(u_{n+1}) - E'(u_n), \beta(u-u_{n+1})\rangle \\ &\geq \nu\|\beta(u_{n+1}-u_n)\|^2 - \rho\langle T(u_{n+1}), \beta(u-u_{n+1})\rangle \\ &\quad - \rho\mu\|\beta(u-u_{n+1})\|^2 \\ &\geq (\nu - \rho\mu)\|\beta(u_{n+1}-u_n)\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the problem(3.6). Otherwise, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(u_{n+1}-u_n)\| = 0.$$

from which, we have

$$\lim_{n\to\infty} \|u_{n+1}-u_n\| = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [12], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the mixed bivariational inequality(3.6).

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the mixed bivariational inequality(3.6) using the auxiliary principle technique.

For a given $u \in H$, find $w \in K_\beta$ such that

$$\langle \rho T(u, \beta(v-w)\rangle + \langle E'(w) - E', \beta(v-w)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \quad (3.13)$$

where $E'(u)$ is the differential of a biconvex function $E(u)$ at $u \in H$. Problem (3.13) has a unique solution, since E is strongly biconvex function. Note that problems (3.13) and (3.7) are quite different problems.
It is clear that for $w = u$, w is a solution of (3.6). This fact allows us to suggest and analyze another iterative method for solving the mixed bivariational inequality (3.6).

Algorithm 3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho T u_n, \beta(v-u_{n+1})\rangle \\ &+ \langle E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle \\ &+ \phi(v) - \phi(u_{n+1}) \geq 0, \forall v \in H, \end{aligned} \quad (3.14)$$

for solving the mixed bivariational inequality (3.6).

If $\beta(v,u)) = v-u$, Algorithm 3 collapses to:

Algorithm 4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle T u_n, v-u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle + \phi(v) - \phi(u_{n+1}) \geq 0, \forall v \in H,$$

for solving the variational inequalities and appears to be a new one.

We now again use the auxiliary principle to suggest some more iterative methods for solving bivariational inequalities.
For a given $u \in H$ satisfying (3.6), find $w \in H$ such that

$$\begin{aligned} &\langle \rho T(w, \beta(v-w)\rangle + \langle w-u+\alpha(u-u), v-w\rangle \\ &+ \phi(v) - \phi(u) \geq 0, \forall v \in H, \end{aligned} \quad (3.15)$$

which is the auxiliary mixed bivariational inequality. We note that, if $w = u$, w is a solution of (3.6). This fact allows us to suggest and analyze another iterative method for solving the mixed bivariational inequality (3.6).

Algorithm 5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} &\rho\langle T u_{n+1}, \beta(v-u_{n+1})\rangle \\ &+ \langle u_{n+1}-u_n) + \alpha(u_n - u_{n-1}), v-u_{n+1}\rangle \\ &+ \phi(v) - \phi(u_{n+1}) \geq 0, \forall v \in H, \end{aligned} \quad (3.16)$$

where α is a constant. Algorithm 5 is called the inertial proximal method for solving the mixed bivariational inequalities (3.6).
For $\alpha = 0$, Algorithm 5 becomes:

Algorithm 6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle T u_{n+1}, \beta(v-u_{n+1})\rangle + \langle u_{n+1}-u_n), v-u_{n+1}\rangle + \phi(v) - \phi(u_{n+1}) \geq 0, \forall v \in H,$$

which is called the proximal method for solving the mixed bivariational inequalities (3.6).

If $\beta. - .) = v-u$, then the boconvex set K_β becomes the convex set K. Consequently Algorithm 3.6 reduces to:

Algorithm 7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle T u_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n) + \alpha(u_n-u_{n-1}), v-u_{n+1}\rangle \geq 0, \quad \forall v \in H.$$

Algorithm 7 is known as the inertial proximal method for solving variational inequalities.
We now consider the convergence analysis of Algorithm 5.

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Theorem 4. *Let $\bar{u} \in H$ be a solution of (3.6) and let u_{n+1} be the approximate solution obtained from Algorithm 5. If the $T : H \longrightarrow R$ is pseudo β-monotone, then*

$$\begin{aligned}&\|u_{n+1}-\bar{u}\|^2\\ \leq\ &\|u_n-\bar{u}\|^2-\|u_{n+1}-u_n-\alpha_n(u_n-u_{n-1})\|^2\\ &+\alpha_n\{\|u_n-\bar{u}\|^2-\|u_{n-1}-\bar{u}\|^2\\ &+2\|u_n-u_{n-1}\|^2\}.\end{aligned} \tag{3.17}$$

*Proof.*Let $\bar{u} \in H$ be a solution of (3.6). Then

$$\langle Tu, \beta(v-u)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

implies that

$$-\langle Tv, \beta(\bar{u}-v)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{3.18}$$

since T is pseudo β-monotone.
Taking $v = u_{n+1}$ in (3.18), we have

$$\langle Tu_{n+1}, \beta(\bar{u}-u_{n+1})\rangle \geq 0. \tag{3.19}$$

Now taking $v = \bar{u}$ in (3.16), we obtain

$$\begin{aligned}&\langle \rho Tu_{n+1}, \beta(\bar{u}-u_{n+1})\rangle\\ +\ &\langle u_{n+1}-u_n-\alpha_n(u_n-u_{n-1}), \bar{u}-u_{n+1}\rangle \geq 0.\end{aligned} \tag{3.20}$$

From (3.19) and (3.20), we have

$$\begin{aligned}&\langle u_{n+1}-u_n-\alpha_n(u_n-u_{n-1}), \bar{u}-u_{n+1}\rangle\\ \geq\ &-\langle \rho Tu_{n+1}, \beta(\bar{u}-u_{n+1})\rangle \geq 0,\end{aligned} \tag{3.21}$$

One can write (3.21) in the form

$$\begin{aligned}&\langle u_{n+1}-u_n, \bar{u}-u_{n+1}\rangle\\ \geq\ &\alpha_n\langle u_n-u_{n-1}, \bar{u}-u_n+u_n-u_{n+1}\rangle.\end{aligned} \tag{3.22}$$

Using the inequality $2\langle u,v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \forall u,v \in H$ and rearranging the terms in (3.21), one can easily obtain (3.17), the required result.

Theorem 5. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 5 and $\bar{u} \in H$ be a solution of (3.6). If there exists $\alpha \in (0,1)$ such that $0 \leq \alpha_n \leq \alpha$, $\quad \forall n \in N$ and*

$$\sum_{n=1}^{\infty} \alpha_n \|u_n-u_{n-1}\|^2 \leq \infty,$$

then $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.

*Proof.*Let $\bar{u} \in K_\beta$ be a solution of (3.6). First we consider the case $\alpha_n = 0$. In this case, we see from (3.17) that the sequence $\{\|\bar{u}-u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (3.17), we have

$$\sum_{n=0}^{\infty} \|u_{n+1}-u_n\|^2 \leq \|u_0-\bar{u}\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1}-u_n\| = 0. \tag{3.23}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (3.17) and taking the limit $n_j \longrightarrow \infty$ and using (3.23), we have

$$\langle T\hat{u}, \beta(v-\hat{u})\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the mixed bihemivariational inequality problem (3.6) and

$$\|u_{n+1}-u_n\|^2 \leq \|u_n-\bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty} u_n = \hat{u}.$$

Now we consider the case $\alpha_n > 0$. From (3.17), we have

$$\sum_{n+1}^{\infty} \|u_{n+1}-u_n - \alpha_n(u_n-u_{n-1})\|^2 \leq \|u_0-\bar{u}\|^2$$
$$+\sum_{n=1}^{\infty}\{\alpha\|u_n-\bar{u}\|^2+2\|u_n-u_{n-1}\|^2\} \leq \infty,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1}-u_n-\alpha_n(u_n-u_{n-1})\|^2 = 0.$$

Repeating the above arguments as in the case $\alpha_n = 0$, one can easily show that $\lim_{n\to\infty} u_n = \hat{u}$, the required result.

For a given $u \in H$ satisfying the mixed bivariational inequality (3.6), consider the auxiliary problem of finding $w \in H$ such that

$$\begin{aligned}&\langle \rho Tu, \beta(v-u)\rangle + \langle w-u, v-w\rangle\\ &\qquad +\phi(v)-\phi(u) \geq 0, \quad \forall v \in H,\end{aligned} \tag{3.24}$$

where $\rho > 0$ is a constant. Problem (3.24) is known as the auxiliary bivariational inequality. We note that, if $w = u$, then clearly w is a solution of the problem (3.6). This observation enables us to suggest and analyze the following iterative method for solving the problem(3.6).

Algorithm 8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho Tw_n, \beta(v-w_n)\rangle\\ &+\langle u_{n+1}-w_n, v-u_{n+1}\rangle + \phi(v) - \phi(u_{n+1}) \geq 0, \forall v \in H,\\ &\langle \nu T(u_n, \beta(v-u_n)\rangle\\ &+\langle w_n-u_n, v-w_n\rangle + \phi(v) - \phi(w_n) \geq 0, \quad \forall v \in H,\end{aligned}$$

where $\rho > 0$ and $\nu > 0$ are constants. Algorithm 8 is two-step predictor-corrector method for solving the mixed bivariational inequalities (3.6).

Remark. For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving mixed bivariational inequality (3.6) and related optimization problems. Convergence analysis of these new algorithms can be considered and investigated using the above techniques and ideas.

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4 Conclusion

In this paper, we have shown that the optimality conditions of a sum of differentiable biconvex functions and nondifferentiable biconvex can be characterised by a class of bivariational inequalities. This result is used to introduce a more general class of mixed bivariational inequalities. Auxiliary principle techniques is used to suggest and analyze some iterative methods for solving the mixed bivariational inequalities. Convergence analysis of the proposed methods is condition using the pseudo monotonicity which is a weaker condition than monotonicity. Our method of proofs is very simple as compared with other techniques. Despite the current activities in these fields, much clearly remains to be done in these fields. It is expected that the ideas and techniques of this paper may be starting point for future research activities.

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