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Computational Non-Polynomial Spline Function for Solving Fractional Bagely-Torvik Equation

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Abstract: In this paper, the Bagley-Torvik equation is constructed. A model approach based on non-polynomial numerical methods spline interpolation is developed to solve some problems. We show that the approximate solutions of such problems obtained by the numerical algorithm developed using non-polynomial spline interpolation functions are better than those produced by other numerical methods. The aim of this paper is to compare the performance of the non-polynomial spline method with polynomial spline method. For this purpose, the algorithm is tested on two examples to illustrate the practical usefulness of the approach.

Keywords: Bagley-Torvik Equation, Caputo Derivative, Non-polynomial Spline, Convergence Analysis.

1 Introduction

Fractional differential equations have been of great interest recently. This is due to the intensive development of the theory of fractional derivatives itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, physics and engineer in self similar and porous structures, electrical networks, fluid mechanics, chemical physics, and many other branches of science (see [1,2,3,4,5,8,9]).

Knowing the importance of differential equations of fractional order, lots of authors are working to find the analytic or approximate solutions of the equations. For examples, the Adomian decomposition method [14,24], Pade approximation method [13] and generalized differential transform method [16,17] have been used to find approximately for fractional order differential equations. In [6,**?**], a new fractional spline function of polynomial form with the concept of the lacunary interpolation is considered to find approximate solution for fractional differential equations. Recently, in [10], cubic polynomial spline function is considered to find approximate solution for Bagley-Torvik equation. More recently, in [7], fractional spline of non-polynomial form has studied to solve the generalized Bagley-Torvik equation.

In this article, a new construction has been developed to find the numerical solution of the following Bagley-Torvik equation [20,21]

$$y''(x)+(\eta D^{\alpha}+\mu)=f(x),\ m-1\leq\alpha<m,\ x\in[a,b] \quad (1)$$

Subject to boundary conditions:

$$y(a)-A_1=y(b)-A_2=0 \quad (2)$$

where A_1, A_2, η, μ are all real constants and $m=1$ or 2. The function $f(x)$ is continuous on the interval $[a,b]$ and the operator D^{α} represents the Caputo fractional derivative. When $\alpha=0$, equation (1) is reduced to the classical second order boundary value problem. For more details on non-polynomial spline you can see [11,12,13].

2 Basic Definitions

In this section, we define some definitions and properties of the fractional calculus theory, which are used in this paper. There are several definitions of a fractional derivative of order $\alpha>0$ [1,2,3,4]. The two most commonly used definitions are the Riemann-Liouville and Caputo.

Definition 1. *A real function $f(x), x>0$, is said to be in the space $C_{\mu}, \mu\in\mathbb{R}$ if there exists a real number $p(>\mu)$, such that $f(x)=x^{p}f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_{μ}^{n} iff $f^{n}\in C_{\mu}n\in\mathbb{N}$.*

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Definition 2.[3,4,5] *The Riemann-Liouville integral operator of order $\alpha > 0$ of a function, $f \in C_\mu, \mu \geq -1$, is defined as:*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \; n-1 < \alpha < n \in \mathbb{N}, \quad x > a,$$
$$I_a^0 f(x) = f(x).$$

Definition 3.[3,4,5] *The fractional derivative of $f(x)$ in the Riemann-Liouville sense is defined as:*

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha-1} f(\xi) d\xi,$$

where $n-1 < \alpha < n, n \in \mathbb{N}$ and $f \in C_{-1}^n$.

Definition 4.[1,2,3] *The Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi,$$

where $n-1 < \alpha < n, , \in \mathbb{N}, x > 0$ and $f \in C_{-1}^n$.

Definition 5.[1,2,3] *The Grünwald definition for fractional derivative is:*

$$^G D^\alpha y(x) = \lim_{n \to \infty} \frac{1}{h^\alpha} \sum_{k=0}^n g_{\alpha,k} y(x-kh) \tag{3}$$

where the Grünwald weights are:

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \tag{4}$$

3 Consistency Relations

In the construction of the spline models for the FDEs be develop (1-2) we introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into equal n- parts.

$$x_i = a + ih, \; x_0 = a, \; x_n = b, \; h = \frac{b-a}{n}, \; i = 0(1)n \tag{5}$$

Let $y(x)$ be the analytic solution of (1) and Si be the spline approximation to $y_i = y(x_i)$, and has the spline function $Q_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}) then in each sub-interval the non-polynomial spline interpolation segment $Q_i(x)$ has the form

$$Q_i(x) = a_i + b_i(x-x_i) + c_i \sin k(x-x_i) + d_i \cos k(x-x_i), \; i = 0(1)n \tag{6}$$

where a_i, b_i, c_i and d_i are constants to be determined, and k is free parameter. Following [2] we have

$$S_{i+1} - 2S_i + S_{i-1} = h^2(\lambda M_{i+1} + 2\beta M_i + \lambda M_{i-1}) \tag{7}$$

where

$$\lambda = \frac{1}{\theta^2}(\theta \csc\theta - 1), \beta = \frac{1}{\theta^2}(1 - \theta\cot\theta), \theta = kh,$$

and

$$M_i = f_i - \mu S_i - \eta D^\alpha S(x) \mid_{x=x_i}, i = 0(1)n \tag{8}$$

whenever $k \to 0$, then $\lambda \to \frac{1}{6}$ and $\beta = \frac{1}{3}$ then the method reduces to the method of [8].

As in [10] has mentioned, approximation of the fractional term $D^\alpha \mid_{x=x_i}, i = 0(1)n$ may be determined by the following:

$$D^\alpha \mid_{x=x_i} \approx h^{-\alpha} \sum_{k=0}^{i} g_{\alpha,k} S(x_i - kh), \; i = 0(1)n \tag{9}$$

4 Construction the Spline model

The non-polynomial spline model of Equation (1) with boundary conditions (2) is based on the system of linear equations given by Equation (7).

Let $Y = (y_i), S = (s_i), C = (c_i), T = (t_i)$ and $E = (e_i) = Y - S$ be $n-1$ as see[22] . Then, we obtained the system given by (7) as follows:

$$PS = h^2 BM + C; \tag{10}$$

where the matrices P, B and the vector C are given below

$$P_{i,j} = \begin{cases} -2, & \text{for } i = j = 1(1)n-1 \\ 1, & \text{for } \mid i-j \mid = 1 \\ 0, & \text{Other wise} \end{cases}$$

$$B = \begin{cases} 2\beta, & \text{for } i = j = 1(1)n-1 \\ \lambda, & \text{for } \mid i-j \mid = 1 \\ 0, & \text{Other wise} \end{cases}$$

and

$$C = \begin{pmatrix} -A_1 + h^2 M_0 \\ \vdots \\ -A_2 + h^2 M_n \end{pmatrix}.$$

The vector M can be written as:

$$M = F - \mu S - \eta h^{-\alpha}(GS + G_0), \tag{11}$$

where the vectors F, G_0 and the matrix G are given below respectively:

$$F = (f_1 \; f_2 \; \dots \; f_{n-2} \; f_{n-1})^t, \tag{12}$$

$$G_0 = A_1(g_{\alpha,1} \;\; g_{\alpha,2} \;\; \dots \; g_{\alpha,n-2} \;\; f_{\alpha,n-1})^t, \tag{13}$$

and

$$G = \begin{pmatrix} g_{\alpha,0} & & & & \\ g_{\alpha,1} & g_{\alpha,0} & & & \\ \vdots & \vdots & \ddots & & \\ g_{\alpha,n-3} & g_{\alpha,n-4} & \cdots & g_{\alpha,1} & g_{\alpha,0} \\ g_{\alpha,n-2} & g_{\alpha,n-3} & \cdots & g_{\alpha,2} & g_{\alpha,1} & g_{\alpha,0} \end{pmatrix}.$$

5 Convergence Analysis of the Method

In this section, we discussed the convergence analysis of the method (7) along with (9) based on non-polynomial spline model. Our main purpose is to derive bounds errors and estimate the rate of convergence on E. For this, the following lemma is needed [17, **?**].

Lemma 1. *If N is a square matrix of order n and $\|N\| < 1$, then $(I+N)^{-1}$ exists and $\|(I+N)^{-1}\| < \frac{1}{1-\|N\|}$.*

Now, by substituting from Eq.(11) into Eq.(10), we get:

$$(P+\mu h^2B+\eta h^{2-\alpha}BG)S = h^2B(F-\eta h^{-\alpha}G_0)+C,$$

and

$$(P+\mu h^2B+\eta h^{2-\alpha}BG)Y = h^2B(F-\eta h^{-\alpha}G_0)+C+T.$$

Hence

$$T = (P+\mu h^2B+\eta h^{2-\alpha}BG)E. \tag{14}$$

From which, we can write the error term as

$$E = (I+\mu h^2P^{-1}B+\eta h^{2-\alpha}P^{-1}BG)^{-1}P^{-1}T,$$

that gives

$$\|E\| \le \frac{\|P^{-1}\|\|T\|}{1-\mu h^2\|P^{-1}\|\|B\|-\eta h^{2-\alpha}\|P^{-1}\|\|B\|\|G\|}, \tag{15}$$

provided that

$$\mu h^2\|P^{-1}\|\|B\|-\eta h^{2-\alpha}\|P^{-1}\|\|B\|\|G\| < 1. \tag{16}$$

Now,

$$\|G\| \le 2m, \quad \forall (m-1) < \alpha < m. \tag{17}$$

It was shown, in [18], that

$$\|B\| = 1 \text{ for } \lambda+\beta = \frac{1}{2}, \tag{18}$$

$$\|P^{-1}\| \le \frac{h^{-2}}{8}((b-a)^2+h^2) = wh^{-2}, \tag{19}$$

$$\|T\| = \eta_1 h^4 M_4, \tag{20}$$

where

$$M_4 = \max_{a\le x\le b} \left|y^{(4)}(x)\right|.$$

From Equations (15)(20), it follows that

$$\|E\| \le \frac{\|P^{-1}\|\|T\|}{1-\mu h^2\|P^{-1}\|\|B\|-\eta h^{2-\alpha}\|P^{-1}\|\|B\|\|G\|} \cong O(h^2) \tag{21}$$

which shows that spline interpolation method developed for the solution of boundary value problem (1) with boundary conditions (2), is second-order convergent.

We summarize the above results in the following theorem:

Theorem 1. *Let $y(x)$ be the analytic solution of the continuous boundary value problem (1)-(2) and let $y(x_i), i = 1(1)n-1$, satisfy the discrete boundary value problem (10). Moreover, if we set $e_i = y_i - s_i$, then $E \cong O(h^2)$ as given by equation (21), neglecting all errors due to round off.*

6 Illustrations

To illustrate our method and to demonstrate its estimate of convergence analysis and applicability of our presented methods computationally, we have solved two second order Bagley-Torvik boundary value problems. For the interest of comparison with method [10], we consider the same examples as in [10]. In the process of computation, all the numerical computations are performed by using MATLAB R12b and MAPLE 15.

*Example 1.*Consider the fractional differential equation

$$y''(x)+0.5D^{\alpha}y(x)+y(x) = 3+x^2+\frac{x^{2-\alpha}}{\Gamma(3-\alpha)} \tag{22}$$

subject to

$$y(0) = 1, y(1) = 2.$$

the exact solution of this problem is

$$y(x) = x^2+1.$$

The numerical solutions using cubic polynomial spline [10] and our non-polynomial spline are presented in Table 1 and 2 in case of $n = 8, \alpha = 0.5$ and different values of the parameters λ, β. Also, the exact solution and approximate solutions are compared and plotted for $n = 10, \lambda = \frac{1}{14}, \beta = \frac{3}{7}$ and different values of α in Figure 1.

Table 1: Comparison results for Example 1

For our method $\lambda = \beta = 0:25$			
x	Exact solution	Our method	Method in [10]
0.125	1.015625	1.011600	1.020078
0.250	1.062500	1.068042	1.065554
0.375	1.140630	1.135673	1.141389
0.500	1.250000	1.246099	1.247476
0.625	1.390630	1.380457	1.383746
0.750	1.562500	1.559747	1.550150
0.875	1.765630	1.754927	1.750225
1	2	2	2

Table 2: Comparison results for Example 1

For our method $\lambda = \frac{1}{12}, \beta = \frac{5}{12}$			
x	Exact solution	Our method	Method in [10]
0	1	1	1
0.125	1.015625	1.027789	1.020078
0.250	1.062500	1.053351	1.065554
0.375	1.140630	1.140481	1.141389
0.500	1.250000	1.240644	1.247476
0.625	1.390630	1.404943	1.383746
0.750	1.562500	1.554359	1.550150
0.875	1.765630	1.779840	1.750225
1	2	2	2

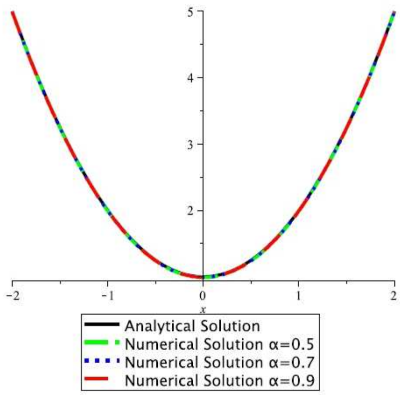

Fig. 1: Comparison results for Example 1 for different values of α.

Example 2. Consider the boundary value problem

$$y''(x) + 0.5D^{\alpha}y(x) + y(x) = f(x), \tag{23}$$

where

$$f(x) = 4x^2(5x-3) + \mu x^4(x-1) + \eta x^{4-\alpha}\left(\frac{120x}{\Gamma(6-\alpha)} - \frac{24}{\Gamma(5-\alpha)}\right)$$

subject to

$$y(0) = y(1) = 0.$$

The exact solution of this problem is

$$y(x) = x^4(x-1).$$

The numerical solution for $\eta = 0.5, \mu = 1, n = 8$ and $\alpha = 0.6$ is represented in Table 3 and 4 for different values of the parameters λ, β. Also, the exact solution and approximate solutions are compared and plotted for $\alpha = 0.75, \lambda = \frac{1}{18}, \beta = 4$ and different values of n in Figure 2.

Table 3: Comparison results for Example 2.

$\lambda = \beta = 0.25$			
x	Exact solution	Our method	Method in [10]
0	0	0	0
0.125	-0.0002140	-0.0004611	-0.0022100
0.250	-0.0029297	-0.0019601	-0.0070100
0.375	-0.0123596	-0.0108561	-0.0181900
0.500	-0.0312500	-0.0283160	-0:0381000
0.625	-0.0572200	-0.0512610	-0.0640300
0.750	-0.0791000	-0.0678123	-0.0846700
0.875	-0.0732730	-0.0537219	-0.0765400
0	0	0	0

Table 4: Comparison results for Example 2.

$\lambda = \frac{1}{12}, \beta = \frac{5}{12}$			
x	Exact Solution	Our method	Method in [10]
0	0	0	0
0.125	-0.0002140	-0.0008307	-0.0022100
0.250	-0.0029297	-0.0008708	-0.0070100
0.375	-0.0123596	-0.0093272	-0.0181900
0.500	-0.0312500	-0.0272481	-0.0381000
0.625	-0.0572200	-0.0521432	-0.0640300
0.750	-0.0791000	-0.0726714	-0.0846700
0.875	-0.0732730	-0.0650553	-0.0765400
0	0	0	0

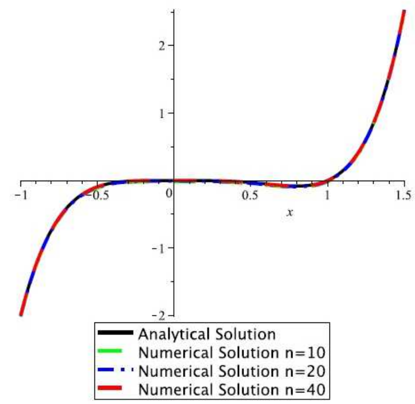

Fig. 2: Comparison results for Example 2 for different values of n.

7 Discussion and Conclusions

In this paper we used a non-polynomial spline model to develop numerical algorithms of Bagley-Torvik equation. In addition, we have compared the performance of the non-polynomial spline method with polynomial spline method. Numerical examples are presented. The results obtained by our algorithm in comparison with results in [10] have shown in Tables 1-4, for the same value of x.

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