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# On the $l_2$ Stability of Crank-Nicolson Difference Scheme for Time Fractional Heat Equations

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**Abstract:** In this work, the matrix stability of the finite difference scheme based on Crank Nicolson method, for solving time-fractional heat equations, is investigated. An iterative formula is presented for the coefficient matrices of the error equations. Upper bounds for $l_2-$ norms of the coefficient matrices are obtained by a new method based on matrix diagonalization. A detailed numerical analysis, including tables figures and error comparisons, are given to demonstrate the theoretical results.

**Keywords:** Time fractional heat equations, Riemann–Liouville fractional derivative, Crank-Nicolson method, matrix stability, matrix diagonalization.

## 1 Introduction

The fractional partial differential equations (FPDEs) are extension of classical partial differential equations which involved in non-integer order derivative or integral. During the last decades of twentieth century, fractional calculus theory has achieved significant attention due to its widespread ability to model processes in the fields of science and engineering. The most important and fundamental theoretical achievements about fractional calculus and FDEs can be found in [1,2,3,4,5,6]. Many phenomenon has been modeled more efficiently by using fractional calculus than classical calculus. Hence, the FPDEs have been used to describe many processes in finance, physics, chemistry, biology and viscoelasticity [7,8,9,10,11]. As a result of this, scientists started to investigate new analytical and numerical methods for solving the FPDEs. In recent years, several numerical methods, including convergence and stability analysis, have been developed for different type of FPDEs [12,13,14,18,19,20,21,22,23,24]. As in classical numerical methods, stability analysis is the central and the most critical point in fractional numerical methods and so far several stability methods ([25,26,27,28,29,30,31] and references therein) have been used for analysis of fractional methods.

In this work, we firstly, constructed a high order direct difference scheme which is based on Crank-Nicolson method for time-fractional heat equations with the accuracy of order $O(\tau^2+h^2)$ at the point $(t_N,x_j)$. Then, we proved the stability of the proposed difference scheme via matrix stability which is developed by using matrix diagonalization.
We consider the following time fractional heat equation;

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \quad x\in(0,1)\,, t\in(0,1) \\ u(0,x)=r(x), \quad x\in[0,1]\,, \\ u(t,0)=p(t),\ u(t,1)=q(t), \quad t\in[0,1]\,. \end{cases} \tag{1}$$

where $f(t,x),\ p(t)$ and $q(t)$ are all given and sufficiently smooth functions and the term $\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}$ denotes $\alpha$-order Riemann-Liouville fractional derivative given with the formula:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int\limits_0^t \frac{u(s,x)}{(t-s)^\alpha}ds, \text{if } 0<\alpha<1, \\ \frac{\partial}{\partial t}u(t,x), \text{ if } \alpha=1, \end{cases}$$

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where $\Gamma(.)$ is the Gamma function. We assume here that the problem (1) has an exact solution $u(t,x) \in C^{2,4}_{t,x}([0,1]\times[0,1])$ which is smooth enough to satisfy the requirements of discretization.

The framework of the paper is as follows. In Section 2, we propose an extension of the Crank-Nicolson different scheme, to be used in the numerical simulations of the fractional heat equation. In Section 3, we prove the stability of the proposed method. This is followed by the simulations of the model on Section 4. The work is concluded in Section 5.

## 2 Discretization of the Problem

In this section, we introduce the basic ideas for the numerical solutions of the time fractional heat equation (1) by an extension of the Crank-Nicolson type difference scheme.

For any two positive integers $M$ and $N$, we denote $x_j = jh$, $t_k = k\tau$, $\Omega_h = \left\{x_j | 0 \le j \le M\right\}$, $\Omega_\tau = \{t_k | 0 \le k \le N\}$. The computational domain $[0,1]\times[0,1]$ is covered by $\Omega_h^\tau = \Omega_h \times \Omega_\tau$, where $h = 1/M$, $\tau = 1/N$ are the uniform spatial and temporal mesh sizes, respectively. Suppose $V_h = \{u | u = (u_0, u_1, ..., u_M)\}$ is a grid function space defined on $\Omega_h$. For any grid function $u \in V_h$, introduce the following notations:

$$\delta_x u_{i+\frac{1}{2}} = \frac{1}{2}(u_{i+1} - u_i), \quad \delta_x u_i = \frac{1}{2h}(u_{i+1} - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h}\left(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}\right),$$

where $u_{i+\frac{1}{2}} = \frac{1}{2}(u_{i+1} + u_i)$.

Let $V^\tau = \left\{u | \ u = \left(u^0, u^1, ..., u^N\right)\right\}$ be grid function space defined on $\Omega_\tau$. For any grid function $u \in V^\tau$, introduce the following notations

$$\delta_t^\alpha u^{k+\frac{1}{2}} = \omega_0 u_j^{k+1} + (\omega_1 - \omega_0) u_j^k + \sum_{r=1}^{k-1} (\omega_{r+1} - \omega_r) u_j^{k-r} - (\omega_k - b_k) u_j^0$$

as the discrete fractional derivative operator, where $\omega_0 = b_0 - a_0$; $\omega_r = a_{r-1} - a_r - (r-1)b_{r-1} + (r+1)b_r$ for $1 \le r \le N$ and $a_r = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\left[\frac{(r+1)^{2-\alpha} - r^{2-\alpha}}{(2-\alpha)}\right]$, $b_r = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\left[\frac{(r+1)^{1-\alpha} - r^{1-\alpha}}{(1-\alpha)}\right]$ for $0 \le r \le N$.

**Lemma 1.** *Suppose* $0 < \alpha < 1$, $y \in C^2[0, t_{k+\frac{1}{2}}]$, *it holds that*

$$\left| \left.\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}\right|_{t=t_{k+\frac{1}{2}}} - \delta_t^\alpha u^{k+\frac{1}{2}} \right| \le \frac{1}{\Gamma(2-\alpha)} O(\tau^{2-\alpha+\alpha\frac{\ln k}{\ln N}}) + O(\tau^2).$$

**Proof**.
Let $H(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y(s)}{(t-s)^\alpha} ds$. Then, we have the following approximation for $H(t_k)$;

$$\begin{aligned} H(t_k) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{y(s)}{(t_k - s)^\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{r=1}^{k} \int_{(r-1)\tau}^{r\tau} \frac{y(s)}{(t_k - s)^\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{r=1}^{k} \int_{(r-1)\tau}^{r\tau} \left[\frac{(s - t_r)}{-\tau} y^{r-1} + \frac{(s - t_{r-1})}{\tau} y^r + \frac{(s - t_r)(s - t_{r-1})}{2!} y''(\xi)\right] \frac{1}{(t_k - s)^\alpha} ds \\ &= \tau \sum_{r=0}^{k-1} (a_r - r b_r) y^{k-r-1} - \tau \sum_{r=0}^{k-1} (a_r - (r+1) b_r) y^{k-r} + R_k. \end{aligned}$$

Now, using the idea of Crank-Nicolson method, we construct a discrete approximation to the fractional derivative $\frac{d^{\alpha}y(t)}{dt^{\alpha}}$ at the point $t = t_{k+\frac{1}{2}}$;

$$\left.\frac{d^{\alpha}y(t)}{dt^{\alpha}}\right|_{t=t_{k+\frac{1}{2}}} = \left.\frac{d}{dt}H(t)\right|_{t=t_{k+\frac{1}{2}}} = \frac{H(t_{k+1})-H(t_k)}{\tau} + O(\tau^2)$$
$$= \omega_0 y^{k+1} + (\omega_1 - \omega_0)y^k + \sum_{r=1}^{k-1}(\omega_{r+1} - \omega_r)y^{k-r} - (\omega_k - b_k)y^0 + R_{k+\frac{1}{2}}$$

and

$$\left|R_{k+\frac{1}{2}}\right| = \frac{1}{\Gamma(2-\alpha)}\left[\frac{(k+1)^{1-\alpha}-k^{1-\alpha}}{\tau}\right]O(\tau^{3-\alpha}) + O(\tau^2)$$
$$= \frac{1}{\Gamma(2-\alpha)}\left[\frac{(\tau(k+1))^{1-\alpha}-(\tau k)^{1-\alpha}}{\tau}\right]O(\tau^2) + O(\tau^2).$$

On the other hand, putting $f(x) = x^{1-\alpha}$ and using the mean-value theorem, we get $\left[\frac{\tau^{1-\alpha}\left((k+1)^{1-\alpha}-k^{1-\alpha}\right)}{\tau}\right] = \frac{1}{\tau^a}f'(c)$ where $k \le c \le k+1$. Then

$$\left[\frac{((k+1)^{1-\alpha} - k^{1-\alpha})}{\tau^{\alpha}}\right] = \frac{(1-\alpha)c^{-\alpha}}{\tau^{\alpha}}, \quad k \le c \le (k+1)$$
$$\le \frac{(1-\alpha)}{k^{\alpha}\tau^{\alpha}}$$
$$= (1-\alpha)\tau^{-\alpha+\alpha\frac{\ln k}{\ln N}}, (\text{since} \frac{1}{k^{\alpha}} = \tau^{\alpha\frac{\ln k}{\ln N}}).$$

$$\left|R_{k+\frac{1}{2}}\right| \le \frac{1}{\Gamma(2-\alpha)}\left[(1-\alpha)\tau^{-\alpha+\alpha\frac{\ln k}{\ln N}}\right]O(\tau^2) + O(\tau^2)$$
$$= \frac{1}{\Gamma(2-\alpha)}O(\tau^{2-\alpha+\alpha\frac{\ln k}{\ln N}}) + O(\tau^2).$$

It is obvious that the convergence order of difference derivative is 2 as $k \to N$.
Neglecting $R_{k+\frac{1}{2}}$ gives, the following approximation for fractional derivative

$$\frac{\partial^{\alpha}u(t_{k+\frac{1}{2}}, x_j)}{\partial t^{\alpha}} = \omega_0 u_j^{k+1} + (\omega_1 - \omega_0)u_j^k + \sum_{r=1}^{k-1}(\omega_{r+1} - \omega_r)u_j^{k-r} - (\omega_k - b_k)u_j^0. \quad (2)$$

Here, it can be easily proved that the sequences which are the coefficients of the approximation formula (2) $a_k$, $b_k$ and $\omega_k$ satisfy the following Lemma.

**Lemma 2.**

- $a_k$ *is increasing,* $0 < a_0 < a_1 < a_2 < \cdots < a_N$.
- $b_k$ *is decreasing,* $b_0 > b_1 > b_2 > \cdots > b_N > 0$.
- $\omega_k$ *is decreasing,* $\omega_1 > \omega_2 > \omega_3 > \cdots > \omega_N > 0$ *for* $k \ge 1$ *and* $\omega_0 > \omega_1$ *when* $\alpha \ge 2 - \frac{ln3}{ln2} \approx 0.42$
- $a_k - kb_k$ *is decreasing,* $a_0 - 0b_0 > a_1 - 1b_1 > a_2 - 2b_2 > \cdots > a_N - Nb_N > 0$.
- $\omega_k - b_k > 0$ *for all* $1 \le k \le N$.

Using the notations given above, we obtain the following difference scheme for the problem (1)

$$\begin{cases} \delta_t^{\alpha}u_j^{k+\frac{1}{2}} = \delta_x^2 u_j^{k+\frac{1}{2}} + f_j^{k+\frac{1}{2}}, 0 \le k \le N-1,\ 1 \le j \le M-1, \\ u_j^0 = r(x_j), \qquad 0 \le j \le M, \\ u_0^k = p(t_k),\ u_M^k = q(t_k), \quad 0 \le k \le N. \end{cases} \quad (3)$$

and substituting formulas, we get

$$\begin{cases} \omega_0 u_j^{k+1} + (\omega_1 - \omega_0)u_j^k + \sum_{r=1}^{k-1}(\omega_{r+1} - \omega_r)u_j^{k-r} - (\omega_k - b_k)u_j^0 \\ = \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{2h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{2h^2}\right] + f\left(t_{k+1/2}, x_j\right), \quad 0 \le k \le N-1,\ 1 \le j \le M-1, \\ u_j^0 = 0, \quad 0 \le j \le M, \\ u_0^k = p(t_k),\ u_M^k = q(t_k), \quad 0 \le k \le N. \end{cases} \quad (4)$$

## 3 Stability of the Method

### 3.1 The Matrix Stability of the Method

To analyze the matrix stability of the method we rewrite the difference scheme (4) in the following matrix form:

$$\begin{cases} AU^1 = BU^0 + F^0, \\ AU^{k+1} = RU^k + \sum_{r=1}^{k-1}(\omega_r - \omega_{r+1})U^{k-r} + (\omega_k - b_k)U^0 + F^k, 1 \le k \le N-1, \end{cases} \tag{5}$$

where $A_{(M-1)\times(M-1)}, B_{(M-1)\times(M-1)}$ and $R_{(M-1)\times(M-1)}$ are the matrices of the form

$$A = \begin{bmatrix} \omega_0 + \frac{1}{h^2} & -\frac{1}{2h^2} & 0 & \cdots & 0 \\ -\frac{1}{2h^2} & \omega_0 + \frac{1}{h^2} & -\frac{1}{2h^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\frac{1}{2h^2} & \omega_0 + \frac{1}{h^2} & -\frac{1}{2h^2} \\ 0 & \cdots & 0 & -\frac{1}{2h^2} & \omega_0 + \frac{1}{h^2} \end{bmatrix},$$

$$B = \begin{bmatrix} 2\omega_0 - \omega_1 - b_0 - \frac{1}{h^2} & \frac{1}{2h^2} & 0 & 0 \\ \frac{1}{2h^2} & \cdots & \cdots & 0 \\ \cdots & \frac{1}{2h^2} & 2\omega_0 - \omega_1 - b_0 - \frac{1}{h^2} & \frac{1}{2h^2} \\ 0 & \cdots & \frac{1}{2h^2} & 2\omega_0 - \omega_1 - b_0 - \frac{1}{h^2} \end{bmatrix}$$

and

$$R = \begin{bmatrix} \omega_0 - \omega_1 - \frac{1}{h^2} & \frac{1}{2h^2} & 0 & \cdots & 0 \\ \frac{1}{2h^2} & \omega_0 - \omega_1 - \frac{1}{h^2} & \frac{1}{2h^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{1}{2h^2} & \omega_0 - \omega_1 - \frac{1}{h^2} & \frac{1}{2h^2} \\ 0 & \cdots & 0 & \frac{1}{2h^2} & \omega_0 - \omega_1 - \frac{1}{h^2} \end{bmatrix}.$$

Besides, $F^k$ and $U^k$ are the vectors as follows

$$F^k = \begin{bmatrix} f(t_{k+\frac{1}{2}}, x_1) + \frac{1}{2h^2}\left(p(t_k) + p(t_{k+1})\right) \\ f(t_{k+\frac{1}{2}}, x_2) \\ \cdots \\ f(t_{k+\frac{1}{2}}, x_{M-2}) \\ f(t_{k+\frac{1}{2}}, x_{M-1}) + \frac{1}{2h^2}\left(q(t_k) + q(t_{k+1})\right) \end{bmatrix} \text{ and } U^k = \begin{bmatrix} u_1^k \\ u_2^k \\ \cdots \\ u_{M-2}^k \\ u_{M-1}^k \end{bmatrix}.$$

Setting

$$P = \begin{bmatrix} \frac{1}{h^2} & -\frac{1}{2h^2} & 0 & \cdots & 0 \\ -\frac{1}{2h^2} & \frac{1}{h^2} & -\frac{1}{2h^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\frac{1}{2h^2} & \frac{1}{h^2} & -\frac{1}{2h^2} \\ 0 & \cdots & 0 & -\frac{1}{2h^2} & \frac{1}{h^2} \end{bmatrix}$$

we can rewrite the matrices $A, B$ and $R$;

$A = \omega_0 I + P$

$B = (2\omega_0 - \omega_1 - b_0)I - P$

$R = (\omega_0 - \omega_1)I - P.$

Actually, since the coefficient matrices are symmetric, we can easily analyze $l_2$ norm of the matrices. Because we know that $l_2$ norm of the symmetric matrix equals to maximum of the absolute value of its eigenvalues. So, the norms of the coefficient matrices are

$\|A\|_2 = \max_{1\le s\le M-1} |\omega_0 + \lambda_{P_s}| = \rho(A)$

$\|B\|_2 = \max_{1\le s\le M-1} |2\omega_0 - \omega_1 - b_0 - \lambda_{P_s}| = \rho(B)$

$$\|R\|_2 = \max_{1\le s\le M-1} |\omega_0 - \omega_1 - \lambda_{P_s}| = \rho(R)$$

where $\lambda_{P_s}$ denotes $s-th$ eigenvalue of matrix $P$, and if we formulate the eigenvalues of the matrix $P$,

$$\begin{aligned}\lambda_{P_s} &= \frac{1}{h^2} + 2\frac{1}{2h^2}\cos\left(\frac{s\pi}{M}\right)\\ &= M^2\left(1+\cos\left(\frac{s\pi}{M}\right)\right),\quad s=1,2,...,M-1,\end{aligned}$$

and it is obvious that all eigenvalues of the matrix $P$ are positive.

Let $\tilde{u}_j^k$ be approximate solution of (5) and define $e_j^k = U_j^k - \tilde{u}_j^k$, $k=0,1,...,N;\ j=0,1,2,...,M$. Then we obtain the following error equations for the difference scheme (5)

$$\begin{cases} Ae^1 = B\,e^0, \\ Ae^{k+1} = R\,e^k \sum_{r=1}^{k-1}(\omega_r - \omega_{r+1})e^{k-r} + (\omega_k - b_k)\,e^0, 1\le k\le N-1. \end{cases}$$

To simplify analyzing the coefficient matrices, we rearrange the system above and rewrite $e^k$ according to $e^0$ for all $1\le k\le N-1$, then we get following formulas for the coefficient matrices $Z_k$ for $1\le k\le N-1$;

$$\begin{cases} Z_1 = A^{-1}B, \\ Z_{k+1} = A^{-1}RZ_k + A^{-1}(\omega_1-\omega_2)Z_{k-1} + \cdots + A^{-1}(\omega_{k-1}-\omega_k)Z_1 + A^{-1}(\omega_k - b_k), 1\le k\le N-1. \end{cases} \tag{6}$$

which are obtained by following calculations;

$$e^1 = A^{-1}Be^0 = Z_1e^0,$$

$$e^2 = A^{-1}R\,e^1 + A^{-1}(\omega_1-b_1)\,e^0 = A^{-1}RZ_1e^0 + A^{-1}(\omega_1-b_1)\,e^0 = \left(A^{-1}RZ_1 + A^{-1}(\omega_1-b_1)\right)e^0 = Z_2e^0,$$

$$\vdots$$

and generally

$$\begin{aligned} e^{k+1} &= A^{-1}R\,e^k + A^{-1}\sum_{r=1}^{k-1}(\omega_r-\omega_{r+1})e^{k-r} + A^{-1}(\omega_k-b_k)\,e^0 \\ &= A^{-1}RZ_ke^0 + A^{-1}(\omega_1-\omega_2)Z_{k-1}e^0 + \cdots + A^{-1}(\omega_{k-1}-\omega_k)Z_1e^0 + A^{-1}(\omega_k-b_k)\,e^0 \\ &= \left(A^{-1}RZ_k + A^{-1}(\omega_1-\omega_2)Z_{k-1} + \cdots + A^{-1}(\omega_{k-1}-\omega_k)Z_1 + A^{-1}(\omega_k-b_k)\right)e^0 \\ &= Z_{k+1}e^0,\quad 1\le k\le N-1. \end{aligned}$$

We note that the iteration matrix of the Crank-Nicolson method for solving classical heat equation, is of the form $Z_1 = A^{-1}R$. But, in fractional problems there are $N$ different types of iteration matrices (i.e., $Z_k$). So, the matrix stability of the fractional case has some difficulties according to classical case. In this work, we deal with these difficulties.
First, we will denote some lemmas to ease stability analysis.

**Lemma 3.**[32] *If $\lambda$ is an eigenvalue of a matrix $M$ and $r(M)$ is any rational function of $M$, then $r(\lambda)$ is an eigenvalue of $r(M)$ corresponding to the same eigenvectors.*

**Lemma 4.**[32] *Similar matrices have the same eigenvalues.*

Let $P$ be the matrix satisfying $V^{-1}PV = D_P$, where the diagonal entries of $D_p$ are eigenvalues of $P$ and the columns of $V$ are the corresponding eigenvectors. Since, the matrices $Z_k$ -s can be expressed as a rational function of the matrix $P$ ( i.e.; $Z_k = r(P)$), from the Lemma 3 and Lemma 4, we have that $(D_{Z_k})_{s,s} = r((D_P)_{s,s})$.

**Theorem 1.** *Let $Z_n$ be a coefficient matrix of the difference scheme (4) which is formulated by (6). Then,*

$$\|Z_n\|_2 \le 1,\quad 1\le n\le N.$$

**Proof**. To show the matrix stability of the difference scheme (4), we will prove that the $l_2$ norm of the all iteration matrices are less then unity via mathematical induction.

Firstly, $Z_1 = A^{-1}B = (\omega_0 I + P)^{-1}((2\omega_0 - \omega_1 - b_0)I - P)$ and thus

$$\begin{aligned}
\|Z_1\|_2 &= \left\|A^{-1}B\right\|_2 \\
&= \left\|\left(VD_AV^{-1}\right)^{-1}\left(VD_BV^{-1}\right)\right\|_2 \\
&= \left\|\left(V(D_A)^{-1}V^{-1}\right)\left(VD_BV^{-1}\right)\right\|_2 \\
&= \left\|V(D_A)^{-1}D_BV^{-1}\right\|_2 \\
&= \left\|V\begin{bmatrix}\frac{1}{(\lambda_A)_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \frac{1}{(\lambda_A)_{M-1}}\end{bmatrix}\begin{bmatrix}(\lambda_B)_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & (\lambda_B)_{M-1}\end{bmatrix}V^{-1}\right\|_2 \\
&= \left\|V\begin{bmatrix}\frac{2\omega_0-\omega_1-b_0-(\lambda_P)_1}{\omega_0+(\lambda_P)_1} & & & \\ & \frac{2\omega_0-\omega_1-b_0-(\lambda_P)_2}{\omega_0+(\lambda_P)_2} & & \\ & & \ddots & \\ & & & \frac{2\omega_0-\omega_1-b_0-(\lambda_P)_{M-1}}{\omega_0+(\lambda_P)_{M-1}}\end{bmatrix}V^{-1}\right\|_2 \\
&= \left\|VD_{Z_1}V^{-1}\right\|_2.
\end{aligned}$$

As we mention above, $\|Z_1\|_2 = \rho(Z_1)$. Since similar matrices have the same eigenvalues and $\omega_0 > 0$, $(\lambda_P)_s > 0$, we have that

$$\begin{aligned}
\|Z_1\|_2 &= \rho(Z_1) = \rho(D_{Z_1}) \\
&= \max_{1\le s\le M-1}\left|\frac{2\omega_0-\omega_1-b_0-(\lambda_P)_s}{\omega_0+(\lambda_P)_s}\right| \\
&= \max_{1\le s\le M-1}\frac{|2\omega_0-\omega_1-b_0-(\lambda_P)_s|}{\omega_0+(\lambda_P)_s}.
\end{aligned}$$

◇ If $2\omega_0 - \omega_1 - b_0 - (\lambda_P)_s \ge 0$, then

$$\begin{aligned}
\|Z_1\|_2 &= \frac{2\omega_0-\omega_1-b_0-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} = \frac{\omega_0+(\omega_0-\omega_1-b_0-(\lambda_P)_s)}{\omega_0+(\lambda_P)_s} = \frac{\omega_0+(-a_0-\omega_1-(\lambda_P)_s)}{\omega_0+(\lambda_P)_s} \\
&< \frac{\omega_0}{\omega_0+(\lambda_P)_s} < 1,
\end{aligned}$$

◇ if $2\omega_0 - \omega_1 - b_0 - (\lambda_P)_s < 0$, then

$$\begin{aligned}
\|Z_1\|_2 &= \frac{-2\omega_0+\omega_1+b_0+(\lambda_P)_s}{\omega_0+(\lambda_P)_s} \\
&= \frac{-2\omega_0+\omega_1+b_0+(\lambda_P)_s}{\omega_0+(\lambda_P)_s} \\
&= \frac{(\lambda_P)_s+(-2\omega_0+\omega_1+b_0)}{(\lambda_P)_s+\omega_0}.
\end{aligned}$$

Here, we have two subcases;

□ if $-2\omega_0 + \omega_1 + b_0 < 0$, then it is obvious that $\|Z_1\|_2 < 1$,

- if $-2\omega_0+\omega_1+b_0 \geq 0$, then

$$\begin{aligned}
\|Z_1\|_2 &= \frac{(\lambda_P)_s + (-2\omega_0+\omega_1+b_0)}{(\lambda_P)_s+\omega_0} < 1 \\
&\Leftrightarrow -2\omega_0+\omega_1+b_0 < \omega_0 \\
&\Leftrightarrow \omega_1+b_0 < 3\omega_0 \\
&\Leftrightarrow \omega_0+b_0 < 3\omega_0 \\
&\Leftrightarrow b_0 < 2\omega_0 \\
&\Leftrightarrow 2a_0 < b_0
\end{aligned}$$

and it is always true, since $b_0 - 2a_0 = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\left(\frac{1}{1-\alpha} - \frac{2}{2-\alpha}\right) = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\left(\frac{\alpha}{(1-\alpha)(2-\alpha)}\right) > 0.$
So, we have obtained that

$$\|Z_1\|_2 < 1.$$

Assume that $\|Z_n\|_2 < 1$ for all $n = 1,2,...,k$ and a rational function of $P$. Then, $Z_{k+1} = A^{-1}RZ_k + A^{-1}(\omega_1-\omega_2)Z_{k-1} + A^{-1}(\omega_2-\omega_3)Z_{k-2} + \cdots + A^{-1}(\omega_{k-1}-\omega_k)Z_1 + A^{-1}(\omega_k - b_k)$ is also a rational function of $P$ and

$$\begin{aligned}
\|Z_{k+1}\|_2 &= \left\|A^{-1}RZ_k + A^{-1}(\omega_1-\omega_2)Z_{k-1} + \cdots + A^{-1}(\omega_{k-1}-\omega_k)Z_1 + A^{-1}(\omega_k-b_k)\right\|_2 \\
&= \left\| \left(V(D_A)V^{-1}\right)^{-1}\left(VD_RV^{-1}\right)\left(VD_{Z_k}V^{-1}\right) + \left(V(D_A)^{-1}V^{-1}\right)(\omega_1-\omega_2)\left(V^{-1}D_{Z_{k-1}}V\right) + \cdots \right. \\
&\quad \left. + \left(V(D_A)^{-1}V^{-1}\right)(\omega_{k-1}-\omega_k)\left(V^{-1}D_{Z_1}V\right) + \left(V^{-1}D_{A^{-1}}V\right)(\omega_k-b_k) \right\|_2 \\
&= \left\| V\left[(D_A)^{-1}D_RD_{Z_k} + (D_A)^{-1}(\omega_1-\omega_2)D_{Z_{k-1}} + \cdots + (D_A)^{-1}(\omega_{k-1}-\omega_k)D_{Z_1} + (D_A)^{-1}(\omega_k-b_k)\right]V^{-1}\right\|_2 \\
&= \max_{1\leq s\leq M-1}\left| \frac{(\lambda_R)_s}{(\lambda_A)_s}\left(\lambda_{Z_k}\right)_s + \frac{(\omega_1-\omega_2)}{(\lambda_A)_s}\left(\lambda_{Z_{k-1}}\right)_s + \cdots + \frac{(\omega_{k-1}-\omega_k)}{(\lambda_A)_s}\left(\lambda_{Z_1}\right)_s + \frac{(\omega_k-b_k)}{(\lambda_A)_s} \right| \\
&= \max_{1\leq s\leq M-1}\left| \frac{\omega_0-\omega_1-(\lambda_P)_s}{\omega_0+(\lambda_P)_s}\left(\lambda_{Z_k}\right)_s + \frac{\omega_1-\omega_2}{\omega_0+(\lambda_P)_s}\left(\lambda_{Z_{k-1}}\right)_s + \cdots \right. \\
&\quad \left. + \frac{\omega_{k-2}-\omega_{k-1}}{\omega_0+(\lambda_P)_s}\left(\lambda_{Z_2}\right)_s + \frac{\omega_{k-1}-\omega_k}{\omega_0+(\lambda_P)_s}\left(\lambda_{Z_1}\right)_s + \frac{\omega_k-b_k}{\omega_0+(\lambda_P)_s} \right|,
\end{aligned}$$

where $\omega_{k-1}-\omega_k > 0$ for $1 \leq k \leq N$, $\max_{1\leq s\leq M-1}(\lambda_P)_s > 0$, and $\left|\max_{1\leq s\leq M-1}\left(\lambda_{Z_k}\right)_s\right| < 1$.

Here, we obtain the following linear maximization problem of the form;

$$\max_{1\leq s\leq M-1}\left|c_1.\left(\lambda_{Z_k}\right)_s + c_2.\left(\lambda_{Z_{k-1}}\right)_s + \cdots + c_k.\left(\lambda_{Z_1}\right)_s + (\omega_k-b_k)\right| \tag{7}$$

with the constraint $\left|\left(\lambda_{Z_k}\right)_s\right| < 1$, and $c_2, c_3, ...c_k$ are positive constants. At the expression (7), the first term; $c_1.\lambda_{Z_{k_s}}$ and the remaining terms; $c_2.\left(\lambda_{Z_{k-1}}\right)_s + \cdots + c_k.\left(\lambda_{Z_1}\right)_s + (\omega_k-b_k)$, can be considered, separately.

Some estimates for the expression (7) are obtained, when we substitute $\lambda_{Z_{k_s}} = \pm 1$ and all $\lambda_{Z_{n_s}} = \pm 1$, simultaneously for $1 \leq n \leq k-1$ and then, $\|Z_{k+1}\|_2$ is bounded by the maximum of $\{T_1, T_2, T_3, T_4\}$ where

1. $T_1 = \|Z_{k+1}\|_2$ ,when $\left(\lambda_{Z_k}\right)_s = +1$ and $\left(\lambda_{Z_n}\right)_s = +1$,for all $1 \leq n \leq k-1$,
2. $T_2 = \|Z_{k+1}\|_2$ ,when $\left(\lambda_{Z_k}\right)_s = +1$ and $\left(\lambda_{Z_n}\right)_s = -1$,for all $1 \leq n \leq k-1$,
3. $T_3 = \|Z_{k+1}\|_2$ ,when $\left(\lambda_{Z_k}\right)_s = -1$ and $\left(\lambda_{Z_n}\right)_s = +1$,for all $1 \leq n \leq k-1$,
4. $T_4 = \|Z_{k+1}\|_2$ ,when $\left(\lambda_{Z_k}\right)_s = -1$ and $\left(\lambda_{Z_n}\right)_s = -1$,for all $1 \leq n \leq k-1$.

Now, we will investigate each cases in detail to show $T_k < 1$ for all $1 \leq k \leq 4$.

**1. CASE:**

$$\begin{aligned}
T_1 &= \left| \frac{\omega_0-\omega_1-(\lambda_P)_s}{\omega_0+(\lambda_P)_s}(1) + \frac{\omega_1-\omega_2}{\omega_0+(\lambda_P)_s}(1) + \cdots + \frac{\omega_{k-2}-\omega_{k-1}}{\omega_0+(\lambda_P)_s}(1) + \frac{\omega_{k-1}-\omega_k}{\omega_0+(\lambda_P)_s}(1) + \frac{\omega_k-b_k}{\omega_0+(\lambda_P)_s} \right| \\
&= \left| \frac{\omega_0-\omega_1-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} + \frac{\omega_1-b_k}{\omega_0+(\lambda_P)_s} \right| \\
&= \left| \frac{\omega_0-b_k-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} \right|.
\end{aligned}$$

◇ If $\omega_0 - b_k - (\lambda_P)_s \geq 0$, then

$$T_1 = \frac{\omega_0 - b_k - (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} < 1, \text{ since } (\lambda_P)_s > 0, b_k > 0 \text{ and } \omega_0 > 0.$$

◇ If $\omega_0 - b_k - (\lambda_P)_s < 0$,

$$\begin{aligned} T_1 &= \frac{-\omega_0 + b_k + (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} = \frac{(\lambda_P)_s + (-\omega_0 + b_k)}{(\lambda_P)_s + \omega_0} < 1 \\ &\Leftrightarrow -\omega_0 + b_k < \omega_0 \\ &\Leftrightarrow b_k < 2\omega_0, \end{aligned}$$

and it is always true, because we have already showed that $b_k < b_0 < 2\omega_0$.
So, $T_1 < 1$.

**2. CASE:**

$$\begin{aligned} T_2 &= \left| \frac{\omega_0 - \omega_1 - (\lambda_P)_s}{\omega_0 + (\lambda_P)_s}(1) + \frac{\omega_1 - \omega_2}{\omega_0 + (\lambda_P)_s}(-1) + \cdots + \frac{\omega_{k-2} - \omega_{k-1}}{\omega_0 + (\lambda_P)_s}(-1) + \frac{\omega_{k-1} - \omega_k}{\omega_0 + (\lambda_P)_s}(-1) + \frac{\omega_k - b_k}{\omega_0 + (\lambda_P)_s} \right| \\ &= \left| \frac{\omega_0 - \omega_1 - (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} + \frac{-\omega_1 + 2\omega_k - b_k}{\omega_0 + (\lambda_P)_s} \right| \\ &= \left| \frac{\omega_0 - 2\omega_1 + 2\omega_k - b_k - (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} \right|. \end{aligned}$$

◇ If $\omega_0 - 2(\omega_1 - \omega_k) - b_k - (\lambda_P)_s \geq 0$, then

$$T_2 = \frac{\omega_0 - (2(\omega_1 - \omega_k) + b_k + (\lambda_P)_s)}{\omega_0 + (\lambda_P)_s} < \frac{\omega_0}{\omega_0 + (\lambda_P)_s} < 1,$$

since $2(\omega_1 - \omega_k) > 0, b_k > 0$, $(\lambda_P)_s > 0$ and $\omega_0 > 0$,

◇ if $\omega_0 - 2(\omega_1 - \omega_k) - b_k - (\lambda_P)_s < 0$,

$$\begin{aligned} T_2 &= \frac{-\omega_0 + 2(\omega_1 - \omega_k) + b_k + (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} \\ &= \frac{(\lambda_P)_s + (-\omega_0 + 2(\omega_1 - \omega_k) + b_k)}{(\lambda_P)_s + \omega_0}. \end{aligned}$$

Here, we have two subcases;

- ◻ if $-\omega_0 + 2(\omega_1 - \omega_k) + b_k < 0$, then it is obvious that $T_2 < 1$,
- ◻ if $-\omega_0 + 2(\omega_1 - \omega_k) + b_k > 0$, then

$$\begin{aligned} T_2 &= \frac{(\lambda_P)_s + (-\omega_0 + 2(\omega_1 - \omega_k) + b_k)}{(\lambda_P)_s + \omega_0} < 1 \\ &\Leftrightarrow -\omega_0 + 2(\omega_1 - \omega_k) + b_k < \omega_0 \\ &\Leftrightarrow 2(\omega_1 - \omega_k) + b_k < 2\omega_0 \\ &\Leftrightarrow 2(\omega_0 - \omega_k) + b_k < 2\omega_0 \\ &\Leftrightarrow b_k < 2\omega_k, \end{aligned}$$

it is true, since $b_k < \omega_k < 2\omega_k$ for all $k \geq 1$.
So, in the second case $T_2 < 1$.

**3. CASE:**

$$\begin{aligned} T_3 &= \left| \frac{\omega_0 - \omega_1 - (\lambda_P)_s}{\omega_0 + (\lambda_P)_s}(-1) + \frac{\omega_1 - \omega_2}{\omega_0 + (\lambda_P)_s}(1) + \cdots + \frac{\omega_{k-2} - \omega_{k-1}}{\omega_0 + (\lambda_P)_s}(1) + \frac{\omega_{k-1} - \omega_k}{\omega_0 + (\lambda_P)_s}(1) + \frac{\omega_k - b_k}{\omega_0 + (\lambda_P)_s} \right| \\ &= \left| \frac{-\omega_0 + \omega_1 + (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} + \frac{\omega_1 - b_k}{\omega_0 + (\lambda_P)_s} \right| \\ &= \left| \frac{-\omega_0 + 2\omega_1 - b_k + (\lambda_P)_s}{\omega_0 + (\lambda_P)_s} \right|. \end{aligned}$$

◇ If $-\omega_0+2\omega_1-b_k+(\lambda_P)_s \geq 0$,

$$T_3 = \frac{-\omega_0+2\omega_1-b_k+(\lambda_P)_s}{\omega_0+(\lambda_P)_s} = \frac{(\lambda_P)_s+(-\omega_0+2\omega_1-b_k)}{(\lambda_P)_s+\omega_0},$$

and here we have two subcases;

□ if $-\omega_0+2\omega_1-b_k<0$, then it is obvious that $T_3<1$,
□ if $-\omega_0+2\omega_1-b_k>0$, then

$$\begin{aligned} T_3 &= \frac{(\lambda_P)_s+(-\omega_0+2\omega_1-b_k)}{(\lambda_P)_s+\omega_0} < 1 \\ &\Leftrightarrow -\omega_0+2\omega_1-b_k<\omega_0 \\ &\Leftrightarrow 2\omega_1-b_k<2\omega_0 \end{aligned}$$

and it is always true, since $2\omega_1-b_k<2\omega_1<2\omega_0$.

◇ If $-\omega_0+2\omega_1-b_k+(\lambda_P)_s<0$, then

$$\begin{aligned} T_3 &= \frac{\omega_0-2\omega_1+b_k-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} < \frac{\omega_0-2\omega_1+b_1-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} < \frac{\omega_0-2\omega_1+\omega_1-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} < \frac{\omega_0-(\omega_1+(\lambda_P)_s)}{\omega_0+(\lambda_P)_s} \\ &< \frac{\omega_0}{\omega_0+(\lambda_P)_s} < 1. \end{aligned}$$

So, in the third case $T_3<1$.

**4. CASE:**

$$\begin{aligned} T_4 &= \left| \frac{\omega_0-\omega_1-(\lambda_P)_s}{\omega_0+(\lambda_P)_s}(-1) + \frac{\omega_1-\omega_2}{\omega_0+(\lambda_P)_s}(-1) + \cdots + \frac{\omega_{k-2}-\omega_{k-1}}{\omega_0+(\lambda_P)_s}(-1) + \frac{\omega_{k-1}-\omega_k}{\omega_0+(\lambda_P)_s}(-1) + \frac{\omega_k-b_k}{\omega_0+(\lambda_P)_s} \right| \\ &= \left| \frac{-\omega_0+\omega_1+(\lambda_P)_s}{\omega_0+(\lambda_P)_s} + \frac{-\omega_1+2\omega_k-b_k}{\omega_0+(\lambda_P)_s} \right| \\ &= \left| \frac{-\omega_0+2\omega_k-b_k+(\lambda_P)_s}{\omega_0+(\lambda_P)_s} \right|. \end{aligned}$$

◇ If $-\omega_0+2\omega_k-b_k+(\lambda_P)_s \geq 0$,

$$\begin{aligned} T_4 &= \frac{-\omega_0+2\omega_k-b_k+(\lambda_P)_s}{\omega_0+(\lambda_P)_s} \\ &= \frac{(\lambda_P)_s+(-\omega_0+2\omega_k-b_k)}{(\lambda_P)_s+\omega_0} \end{aligned}$$

and here we have two subcases;

□ if $-\omega_0+2\omega_k-b_k<0$, then it is obvious that $T_4<1$,
□ if $-\omega_0+2\omega_k-b_k>0$, then

$$\begin{aligned} T_4 &= \frac{(\lambda_P)_s+(-\omega_0+2\omega_k-b_k)}{(\lambda_P)_s+\omega_0} < 1 \\ &\Leftrightarrow -\omega_0+2\omega_k-b_k<\omega_0 \\ &\Leftrightarrow 2\omega_k-b_k<2\omega_0 \end{aligned}$$

and it is always true since $2\omega_k-b_k<2\omega_k<2\omega_0$.

◇ If $-\omega_0+2\omega_k-b_k+(\lambda_P)_s<0$, then

$$T_4 = \frac{\omega_0-2\omega_k+b_k-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} < \frac{\omega_0-2\omega_k+\omega_k-(\lambda_P)_s}{\omega_0+(\lambda_P)_s} < \frac{\omega_0-(\omega_k+(\lambda_P)_s)}{\omega_0+(\lambda_P)_s} < \frac{\omega_0}{\omega_0+(\lambda_P)_s} < 1.$$

So, in the fourth case $T_4<1$.
Since, $\|Z_{k+1}\|_2$ is bounded by the maximum of $\{T_1,T_2,T_3,T_4\}$ we obtain;

$$\|Z_{k+1}\|_2 \leq \max\{T_1,T_2,T_3,T_4\} \leq 1.$$

We have proved that, all possible upper bounds for the coefficient matrix of every step of the calculation is less than unity. Since $\|Z_n\|_2 \leq 1$ for all $n$,

$$\|e^n\|_2 \leq \|Z_n\|_2 \left\|e^0\right\|_2 \leq \left\|e^0\right\|_2.$$

Therefore, we can obviously claim that the difference scheme is stable for any values of $M$ and $N$, on the grounds that the round-off errors do not increase.

## 4 Numerical Analysis

*Example 1.*

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + \left(\frac{t^{1-\alpha}}{\Gamma(3/2)} - \frac{2t^{3-\alpha}}{\Gamma(7/2)} + \frac{24t^{5-\alpha}}{\Gamma(11/2)} - \frac{720t^{7-\alpha}}{\Gamma(15/2)}\right)\sin(2\pi x) \\ \qquad + 4\pi^2(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7})\sin(2\pi x); \;\; 0 < x < 1, 0 < t < 1, \\ u(0,x) = 0,\; 0 \leq x \leq 1, \\ u(t,0) = 0,\;\; u(t,1) = 0,\;\; 0 \leq t \leq 1. \end{cases}$$

Exact solution of this problem is $U(t,x) = (t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7})\sin(2\pi x)$. The approximate solutions by the proposed method, exact solutions and errors are given in Figure 1. The errors when solving this problem are listed in the Table 1 for various values of time and space nodes. The errors in the table is calculated by the formula

$$E_k = \max_{\substack{0\leq n\leq M \\ 0\leq k\leq N}} \left|u(t_k,x_n) - U_n^k\right|$$

and the **error rate** formula is $E_k/E_{k+1}$.
On the other hand, the norms of the coefficient matrices and possible upper bounds for these norms are shown in Table 2.

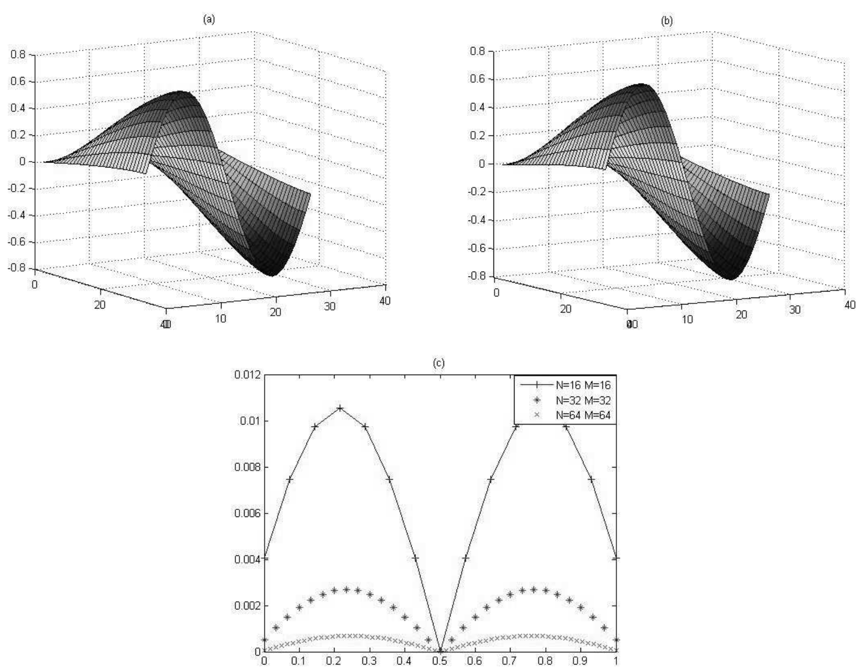


**Fig. 1:** (a) The approximate solutions of Example 1 by the proposed method when $N$=32, $M$=32 and $\alpha = 0.5$. (b) The exact solutions of Example 1 when $N$=32, $M$=32 and $\alpha = 0.5$. (c) The errors for some values of $M$ and $N$ when t=1 and $\alpha = 0.5$.

**Table 1:** The errors for some values of $M$, $N$ and $\alpha$

| | | $\alpha = 0.45$ | | $\alpha = 0.6$ | | $\alpha = 0.9$ | |
|---|---|---|---|---|---|---|---|
| $M$ | $N$ | Error | Rate | Error | Rate | Error | Rate |
| 8 | 8 | 0.04049702 | - | 0.04048122 | - | 0.04031204 | |
| 16 | 16 | 0.01056806 | 3.83 | 0.01051883 | 3.85 | 0.01051606 | 3.83 |
| 32 | 32 | 0.00269188 | 3.92 | 0.00269155 | 3.91 | 0.00269780 | 3.90 |
| 64 | 64 | 0.00068101 | 3.95 | 0.00068226 | 3.95 | 0.00068405 | 3.94 |
| 128 | 128 | 0.00017149 | 3.97 | 0.00020520 | 3.32 | 0.00017220 | 3.97 |

**Table 2:** Norms and some upper bounds for the norms of each iteration matrices.

| $k$ | $\|Z_k\|$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $\max\limits_{1\le i\le 4}\{T_i\}$ |
|---|---|---|---|---|---|---|
| 1 | 0.9960384 | | | | | 0.9960384 |
| 2 | 0.9903764 | 0.9918672 | 0.9918672 | 0.9943106 | 0.9943106 | 0.9943106 |
| 3 | 0.9848617 | 0.9910144 | 0.9944255 | 0.9951634 | 0.9917523 | 0.9951634 |
| 4 | 0.9793399 | 0.9905730 | 0.9952014 | 0.9956048 | 0.9909765 | 0.9956048 |
| 5 | 0.9738669 | 0.9902909 | 0.9956228 | 0.9958869 | 0.9905550 | 0.9958869 |
| 6 | 0.9684144 | 0.9900905 | 0.9958971 | 0.9960873 | 0.9902808 | 0.9960873 |
| 7 | 0.9629987 | 0.9899387 | 0.9960937 | 0.9962392 | 0.9900841 | 0.9962392 |
| 8 | 0.9576091 | 0.9898184 | 0.9962435 | 0.9963594 | 0.9899343 | 0.9963594 |
| 9 | 0.9522527 | 0.9897202 | 0.9963625 | 0.9964576 | 0.9898154 | 0.9964576 |
| 10 | 0.9469240 | 0.9896379 | 0.9964599 | 0.9965399 | 0.9897179 | 0.9965399 |
| 11 | 0.9416268 | 0.9895678 | 0.9965416 | 0.9966101 | 0.9896362 | 0.9966101 |
| 12 | 0.9363579 | 0.9895070 | 0.9966114 | 0.9966708 | 0.9895664 | 0.9966708 |
| 13 | 0.9311196 | 0.9894537 | 0.9966720 | 0.9967242 | 0.9895059 | 0.9967242 |
| 14 | 0.9259097 | 0.9894064 | 0.9967251 | 0.9967715 | 0.9894527 | 0.9967715 |
| 15 | 0.9207296 | 0.9893641 | 0.9967722 | 0.9968138 | 0.9894056 | 0.9968138 |
| 16 | 0.9155780 | 0.9893259 | 0.9968144 | 0.9968519 | 0.9893634 | 0.9968519 |

## 5 Conclusion

The matrix stability analysis of fractional and classical heat equations are discussed. The iteration matrices of the difference scheme to solve a time fractional heat equations are obtained. The problem of finding the norm of the iteration matrices, is reduced to a linear maximization problem by the matrix diagonalization method. After finding some estimates, upper bounds are obtained for the norm of the iteration matrices less than unity which shows the matrix stability of the difference scheme. A numerical example is presented and the results are in good agreement with the theoretical claims.

## References


[1] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon, 1993.

[2] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.

[3] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[4] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York, 1974.

[5] M. D. Ortigueira, *Fractional calculus for scientists and engineers*, Springer, Berlin, Heidelberg and New York, 2011.

[6] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus*: *models and numerical nethods*, Series on Complexity, Nonlinearity and Chaos, Vol. **3**, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2012.

[7] F. Mainardi, *Fractional aalculus and waves in linear viscoelasticity*: *an introduction to mathematical models*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2010.
[8] S. Vitali, G. Castellani and F. Mainardi, Time fractional cable equation and applications in neurophysiology, arXiv:1702.05339 [physics.bio-ph].
[9] F. M. Atici and S. Sengul, Modeling with fractional difference equations, *J. Math. Anal. Appl.* **369**, 1-9 (2010).
[10] R. C. Koeller, Application of fractional calculus to the theory of viscoelasticity, *J. Appl. Mech.* **51**, 229–307 (1984).
[11] F. Mainardi, M. Raberto, R. Gorenflo and E. Scalas, Fractional calculus and continuous-time finance II: the waiting-time distribution, *Physica A* **287**, 468-481 (2000).
[12] D. Baleanu, A. Mousalou and S. Rezapour, A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative, *Adv. Differ. Equ.* **2017**, 51 (2017).
[13] A. Akgul, M. Inc and D. Baleanu, On solutions of variable-order fractional differential equations, *Int. J. Optim. Contr. Theor. Appl.* **7** 1, 112–116 (2017).
[14] A El-Ajou, O. A. Arqub, S. M. Momani, D. Baleanu and A. Alsaedi, A novel expansion iterative method for solving linear partial differential equations of fractional order, *Appl. Math. Comput.* doi: 10.1016/j.amc.2014.12.121, (2015).
[15] X. L. Ding and J. J. Nieto, Numerical Analysis of Fractional Neutral Functional Differential Equations Based on Generalized Volterra-Integral Operators, *J. Comput. Nonlin. Dynam.* **12**, doi: 10.1115/1.4035267 (2017).
[16] M. Ghasemi, Y. Jalilian and J. J. Trujillo, Existence and numerical simulation of solutions for nonlinear fractional pantograph equations, *Int. J. Comput. Math.* http://dx.doi.org/10.1080/00207160.2016.1274745 (2017).
[17] M. R. Hooshmandasl, M. H. Heydari and C. Cattani, Numerical solution of fractional sub-diffusion and time-fractional diffusion-wave equations via fractional-order Legendre functions, *Eur. Phys. J. Plus* **131**, (2016).
[18] M. Chen and W. Deng, Fourth order difference approximations for space Riemann-Liouville derivatives based on weighted and shifted Lubich difference operators, *Commun. Comput. Phys.* **16**, 516–540 (2014).
[19] J. Ren and Z. Z. Sun, Efficient and stable numerical methods for multi-term time fractional sub-diffusion equations, *East Asian J. Appl. Math.* **4**, 242–266 (2014).
[20] C. M. Chen, F. Liu, I. Turner and V. Anh, A Fourier method for the fractional diffusion equation describing sub-diffusion, *J. Comput. Phys.* **227**, 886–897 (2007).
[21] C. Tadjeran, M. M. Meerschaert and H. P. Scheffler, A second-order accurate numerical approximation for the fractional diffusion equation, *J. Comput. Phys.* **213**, 205–213 (2006).
[22] S. B. Yuste and L. Acedo, An explicit finite difference method and a new von Neumann-tape stability analysis for fractional diffusion equations, *SIAM J. Numer. Anal.* **42**, 1862–1874 (2005).
[23] P. Zhuang, F. Liu, V. Anh and I. Turner, New solution and analytical techniques of the implicit numerical methods for the anomalous sub-diffusion equation, *SIAM J. Numer. Anal.* **46**, 1079–1095 (2008).
[24] C. P. Li and F. H. Zeng, Finite difference methods for fractional differential equations, *Int. J. Bifurc. Chaos* **22** (2012).
[25] D. Qian, C. Li, R. P. Agarwal and P. J. Y. Wong, Stability analysis of fractional differential system with Riemann–Liouville derivative, *Math. Comput. Model.* **52**, 862–874 (2010).
[26] E. Kaslik and S. Sivasundaram, Analytical and numerical methods for the stability analysis of linear fractional delay differential equations, *J. Comput. Appl. Math.* **236**, 4027–4041 (2012).
[27] J. C. Trigeassou, N. Maamri, J. Sabatier and A. Oustaloup, A Lyapunov approach to the stability of fractional differential equations, *Signal Proces.* **91**, 437–445 (2011).
[28] J. J. Nieto and B. Samet Solvability of an implicit fractional integral equation via a measure of noncompactness argument, *Acta Math. Sci.* **37** 1, 195–204 (2017).
[29] S. Abbas, W. Albarakati, M. Benchohra and J. J. Trujillo, Ulam Stabilities for Partial Hadamard Fractional Integral Equations, *Arabian J. Math.* **5** 1, 1–7 (2016).
[30] W. Deng, Smoothness and stability of the solutions for nonlinear fractional differential equations, *Nonlinear Anal.* **72**, 1768–1777 (2010).
[31] V. S. Borisov and M. Mond, On stability of difference schemes. Central schemes for hyperbolic conservation laws with source terms, *Applied Numer. Math.* **62**, 895–921 (2012).
[32] K. Hoffman and R. Kunze, *Linear algebra*, Prentice-Hall Inc., New Jersey, 1971.