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Recurrence Relations for Single and Product Moments of Generalized Order Statistics from Marshall-Olkin Extended General Class of Distributions

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Abstract: Marshall and Olkin [7] introduced a new method of adding parameter to expand a family of distributions. Using this concept, in this paper the Marshall-Olkin extended general class of distributions is introduced. Further, some recurrence relations for single and product moments of generalized order statistics (*gos*) are studied. Also the results are deduced for record values and order statistics.

Keywords: Marshall-Olkin extended general class of distributions; generalized order statistics; order statistics; record values and recurrence relations.

1 Introduction

Kamps [5] introduced the unifying concept of generalized order statistics (*gos*), the use of such concept has been steadily growing along the years. This is due to the fact that such concept describes random variables arranged in ascending order of magnitude and includes important well known concept that have been separately treated in statistical literature. Examples of such concepts are the order statistics, sequential order statistics, progressive type II censored order statistics, record values and Pfeifer's records. Application is multifarious in a variety of disciplines and particularly in reliability.

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \text{ for } 1 \leq i \leq n-1.$$

The random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$ are said to be generalized order statistics from an absolutely continuous distribution function $F()$ with the probability density funtion (pdf) $f()$, if their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

If $m_i = 0,\ i = 1,2,\ldots,n-1$ and $k = 1$, we obtain the joint pdf of the order statistics and for $m_i = -1,\ k \in N$, we get the joint $pdf\ k-th$ record values.

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Let the Marshall-Olkin extended general form of distributions be

$$\bar{F}(x) = \frac{\lambda [ah(x)+b]^c}{\{1-(1-\lambda)[ah(x)+b]^c\}}, \quad \alpha \le x \le \beta, \ \lambda > 0, \tag{1.2}$$

where a, b and c are such that $F(\alpha) = 0, F(\beta) = 1$ and $h(x)$ is a monotonic and differentiable function of x in the interval (α, β).

Also we have,

$$\bar{F}(x) = -\frac{\{[ah(x)+b]-(1-\lambda)[ah(x)+b]^{c+1}\}}{ach'(x)} f(x) \tag{1.3}$$

where, $\bar{F}(x) = 1 - F(x)$

The relation (1.3) will be utilized to establish recurrence relations for moments of *gos*.

2 Single Moments

Case I: $\gamma_i \ne \gamma_j;\ i \ne j = 1, 2, \ldots, n-1.$

In view of (1.1) the pdf of $r-th$ generalized order statistic $X(r,n,\tilde{m},k)$ is

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i - 1} \tag{2.1}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k+n-i+\sum_{j=i}^{n-1} m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \le i \le r \le n.$$

Theorem 2.1. *For the Marshall-Olkin extended general class of distributions as given in (1.2) and* $n \in N,\ \tilde{m} \in \mathbb{R},\ k > 0,\ 1 \le r \le n, \lambda > 0$

$$E\left[\xi\{X(r,n,\tilde{m},k)\}\right] = E\left[\xi\{X(r-1,n,\tilde{m},k)\}\right] - \frac{1}{ca\gamma_r} E\left[\psi\{X(r,n,\tilde{m},k)\}\right]$$
$$+\frac{(1-\lambda)}{ca\gamma_r} E\left[\phi\{X(r,n,\tilde{m},k)\}\right], \tag{2.2}$$

where $\psi(x) = [ah(x)+b]\omega(x),\ \phi(x) = [ah(x)+b]^{c+1}\omega(x),\ \omega(x) = \frac{\xi'(x)}{h'(x)}$

and $\gamma_r = k+n-r+\sum_{j=r}^{n-1} m_j > 0.$

Proof: We have by Athar and Islam [2],

$$E\left[\xi\{X(r,n,\tilde{m},k)\}\right] - E\left[\xi\{X(r-1,n,\tilde{m},k)\}\right]$$
$$= C_{r-2} \int_{\alpha}^{\beta} \xi'(x) \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i} dx. \tag{2.3}$$

Now on using (1.3) in (2.3), we get

$$E\left[\xi\{X(r,n,\tilde{m},k)\}\right]-E\left[\xi\{X(r-1,n,\tilde{m},k)\}\right]$$

$$=-\frac{C_{r-1}}{ca\gamma_r}\int_{\alpha}^{\beta}\frac{\xi'(x)}{h'(x)}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}$$

$$\times\{[ah(x)+b]-(1-\lambda)[ah(x)+b]^{c+1}\}f(x)\,dx,$$

which after simplification yields (2.2).

Case II: $m_i=m,\ i=1,2,\ldots,n-1.$

The pdf of $X(r,n,m,k)$ is given as:

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)), \tag{2.4}$$

where,

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\,,\ \gamma_i=k+(n-i)(m+1),$$

$$h_m(x)=\begin{cases}-\dfrac{1}{m+1}(1-x)^{m+1} & ,\ m\neq-1\\ -\log(1-x) & ,\ m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(0),\ \ x\in(0,1).$$

Theorem 2.2. *For distribution as given in (1.2) and* $n\in N,\ \tilde{m}\in\mathbb{R},\ k>0,\ 1\leq r\leq n,\lambda>0,\ \gamma_r=k+(n-r)(m+1)>0,$

$$E\left[\xi\{X(r,n,m,k)\}\right]=E\left[\xi\{X(r-1,n,m,k)\}\right]-\frac{1}{ca\gamma_r}\ E\left[\psi\{X(r,n,m,k)\}\right]$$

$$+\ \frac{(1-\lambda)}{ca\gamma_r}\ E\left[\phi\{X(r,n,m,k)\}\right] \tag{2.5}$$

Proof: It may be noted that for $\gamma_i\neq\gamma_j$ but at $m_i=m,\ i=1,2,\ldots,n-1,$

$$a_i(r)=\frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}$$

Therefore the pdf of $X(r,n,\tilde{m},k)$ given in (2.1) reduces to (2.4) cf [6].

Hence it can be seen that (2.5) is the partial case of (2.2) and is obtained by replacing $\tilde{m}$ with m in (2.2).

Remark 2.1: Recurrence relation for single moments of order statistics (at $m=0,k=1$) is

$$E\left[\xi(X_{r:n})\right]=E\left[\xi(X_{r-1:n})\right]-\frac{1}{ca(n-r+1)}\ \left\{E\left[\psi(X_{r:n})\right]+(1-\lambda)\ E\left[\phi(X_{r:n})\right]\right\}$$

At $\lambda=1$, we get

$$E\left[\xi(X_{r:n})\right]=E\left[\xi(X_{r-1:n})\right]-\frac{1}{ca(n-r+1)}\ E\left[\psi(X_{r:n})\right]$$

as obtained by Ali and Khan [1].

Remark 2.2: Recurrence relation for single moments of $k-th$ upper record (at $m=-1$) will be

$$E\left[\xi\{X(r,n,-1,k)\}\right] = E\left[\xi\{X(r-1,n,-1,k)\}\right] - \frac{1}{cak}\left\{E\left[\psi\{X(r,n,-1,k)\}\right] + (1-\lambda)\ E\left[\phi\{X(r,n,-1,k)\}\right]\right\}$$

Remark 2.3: Set $\lambda=1$ in (2.5), we get

$$E\left[\xi\{X(r,n,m,k)\}\right] = E\left[\xi\{X(r-1,n,m,k)\}\right] - \frac{1}{ca\gamma_r}\ E\left[\psi\{X(r,n,m,k)\}\right]$$

as obtained by Athar and Islam [2].

Examples

1. Marshall-Olkin-Extended Uniform Distribution

$$\bar{F}(x) = \frac{\lambda\ (\theta-x)}{[\lambda\ \theta+(1-\lambda)x]},\quad 0<x<\theta,\ \ \lambda\ >\ 0.$$

We have $a=-\frac{1}{\theta},\ \ b=1, c=1$ and $h(x)=x.$

Let $\xi(x)=x^{j+1}$, then

$\psi(x)=(j+1)(x^j-\frac{x^{j+1}}{\theta})$ and $\phi(x)=(j+1)[x^j+\frac{x^{j+2}}{\theta^2}-\frac{2}{\theta}x^{j+1}]$

Thus from relation (2.5), we have

$$E\left[X^{j+2}(r,n,m,k)\right] = \frac{\lambda\ \theta^2}{(1-\lambda)}\ E\left[X^j(r,n,m,k)\right] + \frac{\theta\ \gamma_r}{(1-\lambda)(j+1)}\ E\left[X^{j+1}(r-1,n,m,k)\right] - \frac{\theta[\gamma_r-(j+1)(1-2\lambda)]}{(1-\lambda)(j+1)} E\left[X^{j+1}(r,n,m,k)\right]$$

as obtained by Athar and Nayabuddin [3].

2. Marshall-Olkin-Extended Weibull Distribution

$$\bar{F}(x) = \frac{\lambda\ e^{-x^{\theta}}}{[1-(1-\lambda)e^{-x^{\theta}}]},\quad x\geq 0,\ \ \lambda\ >\ 0, \theta >\ 0$$

Here we have $a=1,\ \ b=0,\ \ c=1$ and $h(x)=e^{-x^{\theta}}$.

Assuming $\xi(x)=x^j$, we get

$\psi(x)=-\frac{j}{\theta}x^{j-\theta}$ and $\phi(x)=\frac{j}{\theta}\ \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}x^{j-\theta(1-l)}$

Thus from relation (2.5),

$$E\left[X^j(r,n,m,k)\right] = E\left[X^j(r-1,n,m,k)\right] + \frac{j}{\theta\gamma_r}\Big\{E\left[X^{j-\theta}(r,n,m,k)\right] + (1-\lambda)\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}E\left[X^{j-\theta(1-l)}(r,n,m,k)\right]\Big\}$$

as obtained by Athar *et al.* [4].

3. Marshall-Olkin-Extended Lomax Distribution

$$\bar{F}(x) = \frac{\lambda\ (1+\frac{x}{\theta})^{-p}}{[1-(1-\lambda)(1+\frac{x}{\theta})^{-p}]},\quad 0<x<\infty,\ \ \lambda,\theta,p\ >\ 0.$$

We have, $a=\frac{1}{\theta}$, $b=1$, $c=-p$ and $h(x)=x$.

Let $\xi(x)=x^{j+1}$, then

$\psi(x)=(j+1)(x^j+\frac{x^{j+1}}{\theta})$ and $\phi(x)=(j+1)\Sigma_{t=0}^{1-p}\binom{1-p}{t}\frac{1}{\theta^t}\ x^{j+t}$

Thus from relation (2.5), we have

$$E\left[X^{j+1}(r,n,m,k)\right]=\left(\frac{p\gamma_r}{p\gamma_r-(j+1)}\right)E\left[X^{j+1}(r-1,n,m,k)\right]+\left(\frac{(j+1)\theta}{p\gamma_r-(j+1)}\right)$$
$$\times\left\{E\left[X^{j}(r,n,m,k)\right]-(1-\lambda)\sum_{t=0}^{1-p}\binom{1-p}{t}\frac{1}{\theta^t}E\left[X^{j+t}(r,n,m,k)\right]\right\}$$

4. Marshall-Olkin-Extended Log- Logistic Distribution

$$\bar{F}(x)=\frac{\lambda\ (1+\theta x^p)^{-1}}{[1-(1-\lambda)(1+\theta x^p)^{-1}]},\quad 0<x<\infty,\ \ \lambda,\theta,p>0$$

Here we have, $a=\theta$, $b=1$, $c=-1$ and $h(x)=x^p$.

Let $\xi(x)=x^{j+1}$, then

$\psi(x)=\frac{(j+1)}{p}(x^{j-p+1}+\theta\ x^{j+1})$ and $\phi(x)=\frac{(j+1)}{p}\ x^{j-p+1}$

Thus from relation (2.5), we get

$$E\left[X^{j+1}(r,n,m,k)\right]=\left(\frac{p\gamma_r}{p\gamma_r-(j+1)}\right)E\left[X^{j+1}(r-1,n,m,k)\right]$$
$$+\left(\frac{\lambda(j+1)}{\theta[p\gamma_r-(j+1)]}\right)E\left[X^{j-p+1}(r,n,m,k)\right]$$

5. Marshall-Olkin-Extended Beta of II Kind Distribution

$$\bar{F}(x)=\frac{\lambda\ (1+x)^{-1}}{[1-(1-\lambda)(1+x)^{-1}]},\ \ 0<x<\infty,\ \lambda>0$$

Here we have $a=1$, $b=1$, $c=-1$ and $h(x)=x$.

Suppose that $\xi(x)=x^{j+1}$, then

$\psi(x)=(j+1)(x^j+x^{j+1})$ and $\phi(x)=(j+1)\ x^j$.

Thus from relation (2.5), we get

$$E\left[X^{j+1}(r,n,m,k)\right] = \left(\frac{\gamma_r}{\gamma_r-(j+1)}\right)E\left[X^{j+1}(r-1,n,m,k)\right]$$
$$-\left(\frac{\lambda\,(j+1)}{\gamma_r-(j+1)}\right)E\left[X^{j}(r,n,m,k)\right].$$

Similarly several recurrence relations based on Theorem 2.2 can be established with proper choice of $a,b,c,$ and $h(x)$.

3 Product Moments

Case I: $\gamma_i \neq \gamma_j;\ i \neq j = 1,2,\ldots,n-1$

The joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is given as

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1}\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\right)\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)$$
$$\times\ \frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)},\qquad \alpha \le x < y \le \beta, \tag{3.1}$$

where,

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i},\quad r+1 \le i \le s \le n.$$

Theorem 3.1. *For the Marshall-Olkin extended generall class of distributions as given in (1.2). Fix a positive integer k and for $n \in N$, $\tilde{m} \in \mathbb{R}$, $1 \le r < s \le n$, $\lambda > 0$,*

$$E\left[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}\right] = E\left[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}\right]$$
$$+\ \frac{(1-\lambda)}{ca\gamma_s}E\left[\phi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}\right] - \frac{1}{ca\gamma_s}E\left[\psi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}\right] \tag{3.2}$$

where,

$$\psi(x,y) = [ah(y)+b]\frac{\frac{\partial}{\partial y}\,\xi(x,y)}{h'(y)},\quad \phi(x,y) = [ah(y)+b]^{c+1}\frac{\frac{\partial}{\partial y}\,\xi(x,y)}{h'(y)}$$

and $\gamma_s = k+n-s+\sum_{j=s}^{n-1}m_j > 0$.

Proof: We have by Athar and Islam [2],

$$E\left[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}\right] - E\left[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}\right]$$
$$=\ C_{s-2}\int\int_{\alpha\le x<y\le\beta}\frac{\partial}{\partial y}\,\xi(x,y)\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}$$

$$\times \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i} \frac{f(x)}{F(x)}\, dy\, dx. \tag{3.3}$$

Now in view of (1.3) and (3.3),we have

$$E\left[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}\right] - E\left[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}\right]$$

$$= -\frac{C_{s-1}}{c a \gamma_s} \int\int_{\alpha \le x < y \le \beta} \frac{\frac{\partial}{\partial y}\,\xi(x,y)}{h'(y)} \Big(\sum_{i=r+1}^{s} a_i^{(r)}(s) \Big[\frac{\bar{F}(y)}{\bar{F}(x)}\Big]^{\gamma_i} \Big) \Big(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i} \Big)$$

$$\times \{[ah(y)+b] - (1-\lambda)[ah(y)+b]^{c+1}\} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)}\, dy\, dx. \tag{3.4}$$

which leads to (3.2).

Case II: $m_i = m;\ i = 1,2,\ldots,n-1.$

The joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k),\ 1 \le r < s \le n$ is given as

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x)\, g_m^{r-1}(F(x))$$

$$\times\ [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y),$$

$$\alpha \le x < y \le \beta \tag{3.5}$$

Theorem 3.2. *For distribution as given in (1.2) and condition as stated in Theorem 3.1*

$$E\left[\xi\{X(r,n,m,k),X(s,n,m,k)\}\right] = E\left[\xi\{X(r,n,m,k),X(s-1,n,m,k)\}\right]$$

$$+\frac{(1-\lambda)}{c a \gamma_s} E\left[\phi\{X(r,n,m,k),X(s,n,m,k)\}\right] - \frac{1}{c a \gamma_s} E\left[\psi\{X(r,n,m,k),X(s,n,m,k)\}\right] \tag{3.6}$$

Proof: We have when $\gamma_i \ne \gamma_j$ but at $m_i = m,\ i = 1,2,\ldots,n-1$

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}} (-1)^{s-i} \frac{1}{(i-r-1)!(s-i)!}$$

Hence, joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ given in (3.1) reduces to (3.5). cf [6].

Therefore, Theorem 3.2 can be established by replacing $\tilde{m}$ with m in Theorem 3.1.

Remark 3.1: Recurrence relation for product moments of order statistics (at $m = 0,\ k = 1$) is

$$E\left[\xi(X_{r,s:n})\right] = E\left[\xi(X_{r,s-1:n})\right] - \frac{1}{ca(n-s+1)} \left\{E\left[\psi(X_{r,s:n})\right] + (1-\lambda)\ E\left[\phi(X_{r,s:n})\right]\right\}$$

Remark 3.2: Recurrence relation for product moments of $k-th$ record values will be

$$E\left[\xi\{X(r,n,-1,k),X(s,n,-1,k)\}\right]=E\left[\xi\{X(r,n,-1,k),X(s-1,n,-1,k)\}\right]$$
$$-\frac{1}{cak}\Big\{E\left[\psi\{X(r,n,-1,k),X(s,n,-1,k)\}\right]+(1-\lambda)E\left[\phi\{X(r,n,-1,k),X(s,n,-1,k)\}\right]\Big\}$$

Remark 3.3: Set $\lambda=1$ in (3.6),we get

$$E\left[\xi\{X(r,n,m,k),X(s,n,m,k)\}\right]=E\left[\xi\{X(r,n,m,k),X(s-1,n,m,k)\}\right]$$
$$-\frac{1}{ca\gamma_s}E\left[\psi\{X(r,n,m,k),X(s,n,m,k)\}\right]$$

as obtained by Athar and Islam [2].

Examples

1. Marshall-Olkin-Extended Uniform Distribution

$$\bar{F}(x)=\frac{\lambda\ (\theta-x)}{[\lambda\ \theta+(1-\lambda)x]},\quad 0<x<\theta,\ \ \lambda\ >\ 0.$$

We have, $a=-\frac{1}{\theta},\ \ b=1,\ \ c=1$ and $h(x)=x$.

Let $\xi(x,y)=x^iy^{j+1}$, then

$\psi(x,y)=[ah(y)+b]\frac{\frac{\partial}{\partial y}\ \xi(x,y)}{h'(y)}=(j+1)(x^iy^j-\frac{x^iy^{j+1}}{\theta})$

and $\phi(x,y)=(j+1)[x^iy^j+\frac{x^iy^{j+2}}{\theta^2}-\frac{2}{\theta}x^iy^{j+1}]$.

Thus from relation (3.6), we have

$$E\left[X^i(r,n,m,k).X^{j+2}(s,n,m,k)\right]=\frac{\lambda\ \theta^2}{(1-\lambda)}\ E\left[X^i(r,n,m,k).X^j(s,n,m,k)\right]$$
$$+\frac{\theta\ \gamma_s}{(1-\lambda)(j+1)}\ E\left[X^i(r,n,m,k).X^{j+1}(s-1,n,m,k)\right]$$
$$-\frac{\theta[\gamma_r-(j+1)(1-2\lambda)]}{(1-\lambda)(j+1)}E\left[X^i(r,n,m,k).X^{j+1}(s,n,m,k)\right],$$

as obtained by Athar and Nayabuddin [3].

2. Marshall-Olkin-Extended Weibull Distribution

$$\bar{F}(x)=\frac{\lambda\ e^{-x^{\theta}}}{[1-(1-\lambda)e^{-x^{\theta}}]},\quad 0<x<\infty,\ \ \lambda\ >\ 0,\theta,>\ 0.$$

Here , $a=1,\ \ b=0,\ \ c=1$ and $h(x)=e^{-x^{\theta}}$.

Let $\xi(x,y)=x^iy^j$, then

$$\psi(x,y)=[ah(y)+b]\frac{\frac{\partial}{\partial y}\ \xi(x,y)}{h'(y)}=-\frac{j}{\theta}x^iy^{j-\theta}$$

and $\phi(x,y)=[ah(y)+b]^{c+1}\frac{\frac{\partial}{\partial y}\ \xi(x,y)}{h'(y)}=\frac{j}{\theta}\ \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}x^iy^{j-\theta(1-l)}$

Thus from relation (3.6), we get

$$E\left[X^i(r,n,m,k).X^j(s,n,m,k)\right]=E\left[X^i(r,n,m,k).X^j(s-1,n,m,k)\right]$$
$$+\frac{j}{\theta\gamma_s}\Big\{E\left[X^i(r,n,m,k).X^{j-\theta}(s,n,m,k)\right]$$
$$+(1-\lambda)\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}E\left[X^i(r,n,m,k).X^{j-\theta(1-l)}(s,n,m,k)\right]\Big\},$$

as obtained by Athar *et al.* [4].

3. Marshall-Olkin-Extended Lomax Distribution

$$\bar{F}(x)=\frac{\lambda\ (1+\frac{x}{\theta})^{-p}}{[1-(1-\lambda)(1+\frac{x}{\theta})^{-p}]},\quad 0<x<\infty,\ \lambda,\theta,p>0.$$

Here we have , $a=\frac{1}{\theta},\ \ b=1,\ \ c=-p,\ \ h(x)=x.$

Suppose $\xi(x,y)=x^iy^{j+1}$, then

$\psi(x,y)=[ah(y)+b]\frac{\frac{\partial}{\partial y}\ \xi(x,y)}{h'(y)}=(j+1)(x^iy^j+\frac{x^iy^{j+1}}{\theta})$

and $\phi(x,y)=[ah(y)+b]^{c+1}\frac{\frac{\partial}{\partial y}\ \xi(x,y)}{h'(y)}=(j+1)\sum_{t=0}^{1-p}\binom{1-p}{t}\frac{1}{\theta^t}\ \ y^{j+t}x^i$

Thus from relation (3.6), we have

$$E\left[X^i(r,n,m,k)X^{j+1}(s,n,m,k)\right]=\left(\frac{p\gamma_s}{p\gamma_s-(j+1)}\right)E\left[X^i(r,n,m,k)X^{j+1}(s-1,n,m,k)\right]$$
$$+\left(\frac{(j+1)\theta}{p\gamma_s-(j+1)}\right)\Big\{E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]$$
$$-(1-\lambda)\sum_{t=0}^{1-p}\binom{1-p}{t}\frac{1}{\theta^t}E\left[X^i(r,n,m,k)X^{j+t}(s,n,m,k)\right]\Big\}.$$

4. Marshall-Olkin-Extended Log-Logistic Distribution

$$\bar{F}(x)=\frac{\lambda\ (1+\theta x^p)^{-1}}{[1-(1-\lambda)(1+\theta x^p)^{-1}]},\quad 0<x<\infty,\ \lambda,\theta,p>0.$$

We have, $a=\theta,\ \ b=1,\ \ c=-1$ and $h(x)=x^p$.

Let $\xi(x,y)=x^iy^{j+1}$, then

$\psi(x,y) = [ah(y)+b]\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)} = \frac{(j+1)}{p}(x^i y^{j-p+1} + \theta x^i y^{j+1})$

and $\phi(x,y) = [ah(y)+b]^{c+1}\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)} = \frac{(j+1)}{p}\ x^i\ y^{j-p+1}$.

Thus from relation (3.6), we have

$$E\left[X^i(r,n,m,k).X^{j+1}(s,n,m,k)\right] = \left(\frac{p\gamma_s}{p\gamma_s-(j+1)}\right)E\left[X^i(r,n,m,k).X^{j+1}(s-1,n,m,k)\right]$$
$$+\left(\frac{\lambda(j+1)}{\theta(p\gamma_s-(j+1))}\right)E\left[X^i(r,n,m,k).X^{j-p+1}(s,n,m,k)\right].$$

5. Marshall-Olkin-Extended Beta Of II Kind Distribution

$$\bar{F}(x) = \frac{\lambda\ (1+x)^{-1}}{[1-(1-\lambda)(1+x)^{-1}]},\quad 0<x<\infty,\ \lambda>0.$$

Here, $a=1,\ b=1,\ c=-1$ and $h(x)=x$.

Suppose $\xi(x,y) = x^i y^{j+1}$, then

$\psi(x,y) = [ah(y)+b]\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)} = (j+1)(x^i y^j + x^i y^{j+1})$

and $\phi(x,y) = [ah(y)+b]^{c+1}\frac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)} = (j+1)\ x^i\ y^j$

Thus from relation (3.6), we have

$$E\left[X^i(r,n,m,k).X^{j+1}(s,n,m,k)\right] = \left(\frac{\gamma_s}{\gamma_s-(j+1)}\right)E\left[X^i(r,n,m,k).X^{j+1}(s-1,n,m,k)\right]$$
$$+\left(\frac{\lambda(j+1)}{(\gamma_s-(j+1))}\right)E\left[X^i(r,n,m,k)X^j.(s,n,m,k)\right].$$

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