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Traveling Wave Solutions of some Nonlinear Evolution Equations by Sine-Cosine Method

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Abstract: In this paper, we established traveling wave solutions of the nonlinear evolution equation. The sine-cosine method was used to construct travelling wave solutions of the Fitzhugh-Nagumo equation and Cahn-Allen equation. Graphical interpretation shows that obtained results include periodic and Soliton wave solutions. It is also shown that for $\alpha = -1$, solutions of Fitzhugh-Nagumo equation coincide with solutions of Cahn-Allen equation.

Keywords: Traveling wave solutions, nonlinear evolution equations, sine-cosine method

1 Introduction

The study of solitary wave solutions of nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena, which appears in various scientific and engineering fields, such as optical fibers, solid state physics, fluid mechanics, plasma physics, biology, chemical kinematics, chemical physics and geochemistry. In recent years, the direct approach for exact solutions of PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform the complicated and tedious algebraic calculations on computer.

A variety of powerful methods, such as Exp-function method [1], adomian decomposition method [2], bilinear transformation[3], Inverse Scattering Transform [4], the tanh-sech method[5,6], the tanh-coth method[7], homogeneous balance method[8], Exp-function method [9], and many others see [10,11,12,13,14]. In recent years, the sine-cosine method and the rational sine-cosine method [15,16,17,18] have been widely used to search for various exact solutions of nonlinear PDEs. The motivation of the present paper is come from the work of Abdul-Majid Wazwaz [2].

The Travelling wave solutions are useful in the theoretical and numerical studies of the nonlinear equation models due to complexity and challenges in their theoretical study. Therefore, finding travelling wave solutions of nonlinear equations is of fundamental interest to complectly understand the model. In this paper the sine-cosine method will determine the Traveling wave solutions of Fitzhugh-Nagumo equation [19,20,21] and Cahn-Allen equation [22].

The article is prepared as follows: In Section 2, sine-cosine method is discussed. In Section 3, we exert this method to the nonlinear evolution equations pointed out above; in Section 4, graphical interpretation and in Section 5, conclusions are given.

2 Analysis of the method

In this section, we will highlight briefly the main steps of Sine-cosine method for nonlinear PDEs.

(I). We first use the wave variable $\xi = x - ct$ to convert the PDE

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{1}$$

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into an ordinary differential equation (ODE)

$$Q(u,u',u'',u''',...)=0. \quad (2)$$

(II). The sine-cosine method allows us the use of the ansatz

$$u(x,t)=\lambda\cos^{\beta}(\mu\xi), \quad |\xi|\leq\frac{\pi}{2\mu}, \quad (3)$$

or the ansatz

$$u(x,t)=\lambda\sin^{\beta}(\mu\xi),|\xi|\leq\frac{\pi}{\mu}, \quad (4)$$

where λ ,μ and β are parameters that will be determined.

(III). Substituting (3) or (4) into the reduced ODE gives a polynomial equation of cosine or sine terms.

(IV). Balance the terms of the cosine functions when eq. (3) is used, or balance the sine functions when eq. (4) is used, to get a system of algebraic equations among the unknowns λ ,μ, c and β .

(V). Determined λ ,μ, c and β by algebraic calculations or by using Maple, the solutions proposed in eq. (3) and eq. (4) follow immediately.

3 Applications

3.1 Fitzhugh-Nagumo equation

Let us consider the Fitzhugh-Nagumo equation

$$u_t-u_{xx}=u(u-\alpha)(1-u) \quad (5)$$

where α is an arbitrary constant. Eq. (5) is an important nonlinear reaction-diffusion equation and applied to model the transmission of nerve impulses [19] and [20], also used in biology and the area of population genetics, in circuit theory [21].
Using the transformation $u(x,t)=u(\xi)$ where $\xi=x-ct$, (5) yields following ODE,

$$u^3-u^2-\alpha u^2+\alpha u-cu'-u''=0. \quad (6)$$

Substituting eq. (3) into eq. (6) yields

$$\frac{-\lambda(\cos(\mu\xi))^{\beta}}{(\cos(\mu\xi))^2}\begin{pmatrix}-\lambda^2\left((\cos(\mu\xi))^{\beta}\right)^2(\cos(\mu\xi))^2\\+\lambda(\cos(\mu\xi))^{\beta}(\cos(\mu\xi))^2\\+\alpha\lambda(\cos(\mu\xi))^{\beta}(\cos(\mu\xi))^2\\-\beta\sin(\mu\xi)\mu c\cos(\mu\xi)\\+\beta^2(\sin(\mu\xi))^2\mu^2\\-\beta\mu^2(\cos(\mu\xi))^2\\-\beta(\sin(\mu\xi))^2\mu^2\\-\alpha(\cos(\mu\xi))^2\end{pmatrix}=0 \quad (7)$$

or

$$\begin{aligned}&\lambda^3(\cos(\mu\xi))^{3\beta}-\lambda^2(\cos(\mu\xi))^{2\beta}\\&-\lambda^2(\cos(\mu\xi))^{2\beta}\alpha+\lambda(\cos(\mu\xi))^{\beta-1}\beta\sin(\mu\xi)\mu c\\&-\lambda(\cos(\mu\xi))^{\beta-2}\beta^2\mu^2+\lambda(\cos(\mu\xi))^{\beta}\beta^2\mu^2\\&+\lambda(\cos(\mu\xi))^{\beta-2}\beta\mu^2+\alpha\lambda(\cos(\mu\xi))^{\beta}=0\end{aligned} \quad (8)$$

Simplifying this equation and introducing a new variable $Y=\cos(\mu z)$, we obtain

$$\begin{aligned}&\lambda^3Y^{3\beta}-\lambda^2Y^{2\beta}-\lambda^2Y^{2\beta}\alpha-\lambda Y^{\beta-1}\beta\sin(\mu z)\mu c\\&-\lambda Y^{\beta-2}\beta^2\mu^2+\lambda Y^{\beta}\beta^2\mu^2+\lambda Y^{\beta-2}\beta\mu^2+\alpha\lambda Y^{\beta}=0\end{aligned} \quad (9)$$

It is obvious that equation eq.(9) is satisfied if the following system of algebraic equations holds:

$$\begin{aligned}&\beta-1\neq0,\beta-2\neq0,\\&3\beta=\beta-2,\\&2\beta=\beta-1,\\&\lambda^3=\lambda\beta^2\mu^2-\lambda\beta\mu^2,\\&-\lambda^2-\lambda^2\alpha=-\lambda\beta\sin(\mu\xi),\\&\lambda\beta^2\mu^2=-\alpha\lambda.\end{aligned} \quad (10)$$

Solving this system leads to

$$\begin{aligned}&\beta=-1,\\&\lambda=\sqrt{-2\alpha},\\&\mu=\sqrt{-\alpha}.\end{aligned} \quad (11)$$

Consequently, we obtain the following Traveling wave solutions

$$u_1=i\sqrt{2\alpha}sech\left(\sqrt{\alpha}(x-ct)\right), \quad \alpha>0 \quad (12)$$

$$u_2=\sqrt{-2\alpha}\sec\left(\sqrt{-\alpha}(x-ct)\right), \quad \alpha<0 \quad (13)$$

Now, if we use the ansatz eq. (4) instead of eq. (3), we will get the same system of eqations as above

and therefore two more solutions are given by

$$u_3=\sqrt{2\alpha}csch\left(\sqrt{\alpha}(x-ct)\right), \quad \alpha>0 \quad (14)$$

$$u_4=\sqrt{-2\alpha}csc\left(\sqrt{-}\alpha(x-ct)\right), \quad \alpha<0 \quad (15)$$

3.2 Cahn-Allen equation

Now we consider the Cahn-Allen equation

$$u_t-u_{xx}-u+u^3=0 \quad (16)$$

which is a reaction-diffusion equation and describes the process of phase separation in iron alloys, including order-disorder transitions[22].

Using the transformation $u(x,t) = u(\xi)$ where $\xi = x - ct$, equation (16) is converted to the following ODE

$$-u'' - cu' + u^3 - u = 0 \tag{17}$$

Substituting eq. (3) into eq. (17) yields

$$\frac{\lambda(\cos(\mu\xi))^{\beta}}{(\cos(\mu\xi))^2}\begin{pmatrix} \lambda^2\left((\cos(\mu\xi))^{\beta}\right)^2(\cos(\mu\xi))^2 \\ +\beta\sin(\mu\xi)\mu c\cos(\mu\xi) \\ -\beta^2(\sin(\mu\xi))^2\mu^2 \\ +\beta\mu^2(\cos(\mu\xi))^2 \\ +\beta(\sin(\mu\xi))^2\mu^2 \\ -(\cos(\mu\xi))^2 \end{pmatrix} = 0 \tag{18}$$

Or

$$\begin{aligned} &\lambda^3(\cos(\mu\xi))^{3\beta} + \lambda(\cos(\mu\xi))^{\beta-1}\beta\sin(\mu\xi)\mu c \\ &-\lambda(\cos(\mu\xi))^{\beta-2}\beta^2\mu^2 + \lambda(\cos(\mu\xi))^{\beta}\beta^2\mu^2 \\ &+\lambda(\cos(\mu\xi))^{\beta-2}\beta\mu^2 - \lambda(\cos(\mu\xi))^{\beta} = 0 \end{aligned} \tag{19}$$

Simplifying this equation and introducing a new variable $Y = \cos(\mu z)$, we obtain

$$\begin{aligned} &\lambda^3 Y^{3\beta} - \lambda Y^{\beta-1}\beta\sin(\mu z)\mu c - \lambda Y^{\beta-2}\beta^2\mu^2 \\ &+\lambda Y^{\beta}\beta^2\mu^2 + \lambda Y^{\beta-2}\beta\mu^2 - \lambda Y^{\beta} = 0 \end{aligned} \tag{20}$$

It is obvious that equation eq. (20) is satisfied if the following system of algebraic equations holds:

$$\begin{aligned} &\beta - 1 \neq 0, \beta - 2 \neq 0, \\ &3\beta = \beta - 2, \\ &\lambda^3 = \lambda\beta^2\mu^2 - \lambda\beta\mu^2, \\ &-\lambda\beta\sin(\mu\xi)\mu c = 0 \\ &\lambda\beta^2\mu^2 = \lambda. \end{aligned} \tag{21}$$

Solving this system leads to

$$\beta = -1, \mu = 1, \lambda = \sqrt{2}. \tag{22}$$

Consequently, we obtain the following Traveling wave solution

$$u_1 = \frac{\sqrt{2}}{\cos(x - ct)} = \sqrt{2}\sec(x - ct) \tag{23}$$

For ansatz $u(x,t) = \lambda\sin^{\beta}(\mu\xi)$ we obtain following solutions

$$u_2 = \frac{\sqrt{2}}{\sin(x - ct)} = \sqrt{2}\csc(x - ct) \tag{24}$$

4 Graphical Interpretation

In this section, we will put forth the graphical representation of determined traveling wave solutions of Fitzhugh-Nagumo equation and Cahn-Allen equation.

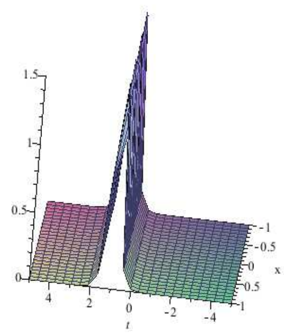

Fig. 1: Soliton profile of (12) with wave speed $c = 1$, $\alpha = 1$ and $x \geq -1$, $t \leq 4$

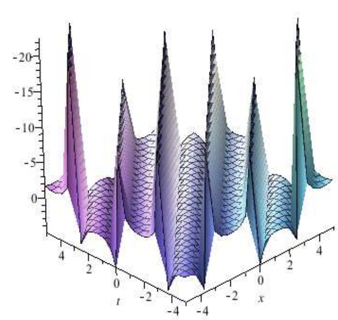

Fig. 2: 3d plot of periodic wave solution, profile of (13) with wave speed $c = 1$, $\alpha = 1$ and $x \geq -4$, $t \leq 4$

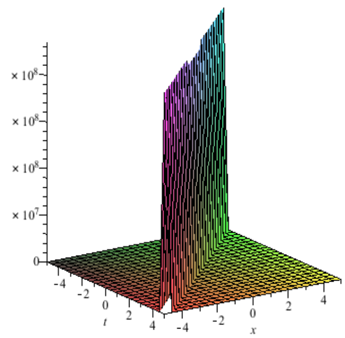

Fig. 3: Soliton profile of (14) with wave speed $c = -1$, $\alpha = 1$ and $x \geq -4$, $t \leq 4$

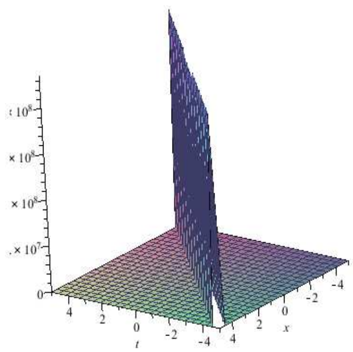

Fig. 4: Soliton profile of (15) with wave speed $c = -1$, $\alpha = 1$ and $x \geq -4$, $t \leq 4$

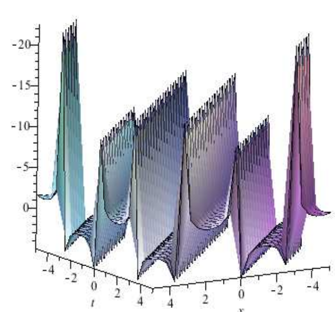

Fig. 5: 3d plot of periodic wave solution, shape of (23) with wave speed $c = 1$,and $x \geq -4$, $t \leq 4$

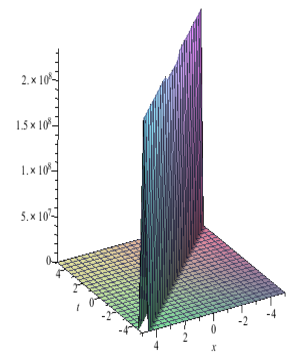

Fig. 6: Soliton profile of (24) with wave speed $c = -1$,and $x \geq -4$, $t \leq 4$

Remark 1 For $\alpha = -1$, the Fitzhugh-Nagumo equation coincide with Cahn-Allen equation, hence their solutions also coincide : (13) coincide with (2.3) and (15) coincide with (24), which can also be seen by their graphical interpretation.

5 Conclusion

We conclude that Sine-cosine method, with the help of symbolic computation provides a powerful mathematical tool for solving nonlinear evolution equations arising in mathematical physics, which may be useful for the explanation of some new nonlinear physical phenomena. This approach can be extended to find traveling wave solutions for the wide class of nonlinear dispersion partial differential equations under certain restrictions, which are arising in the theory of solitons and other areas.

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