On a Fractional Oscillator Equation with Natural Boundary Conditions

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Abstract: We prove existence of solutions for a nonlinear fractional oscillator equation with both left Riemann–Liouville and right Caputo fractional derivatives subject to natural boundary conditions. The proof is based on a transformation of the problem into an equivalent lower order fractional boundary value problem followed by the use of an upper and lower solutions method. To succeed with such approach, we first prove a result on the monotonicity of the right Caputo derivative.

Keywords: Boundary value problems, fractional derivatives, upper and lower solutions method, existence of solutions, integral equations.

1 Introduction

Fractional calculus is an interesting field of research due to its ability to describe memory properties of materials and, therefore, providing a better representation of physical models. Because of this, the study of nonlinear fractional differential equations has attracted a lot of attention and many papers and monographs are devoted to the subject [1,2,3]. Here, we are concerned with the solvability of a nonlinear fractional oscillator equation involving both Riemann–Liouville and Caputo fractional derivatives with natural boundary conditions:

$$\omega^2 u(t) - {}^C D_{1^-}^p D_{0^+}^q u(t) = f(t, u(t)), \quad 0 \le t \le 1, \quad \omega \in \mathbb{R}, \quad \omega \neq 0, \tag{1}$$

with the initial condition

$$u(0) = 0 \tag{2}$$

and the natural condition (see [4,5])

$$D_{0^+}^q u(1) = 0, \tag{3}$$

where $0 < p, q < 1$, ${}^C D_{1^-}^p$ is the right side Caputo derivative, $D_{0^+}^q$ denotes the left side Riemann–Liouville derivative, u is the unknown function, and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. We denote problem (1)–(3) by (P_1). Note that if $p = q \to 1$, then problem (P_1) is a classical oscillator boundary value problem [6].

Oscillator equations appear in different fields of science, such as classical mechanics, electronics, engineering, and fractional calculus, being a subject of strong current research: see, e.g., [7,8,9] and references therein. Different methods are used to solve such equations, for example, by the Laplace transform method or by using numerical methods [10]. Since some phenomena obey an equation of motion with fractional derivatives, oscillator equations with fractional derivatives are a particularly interesting subject to study [4,5,10,7,11,12].

Blaszczyk studied numerically the associated linear problem of (P_1) with $f(t, u(t)) = Ag(t)$, see [5]. In [4], Agrawal discussed the relationship between transversality and natural boundary conditions in order to solve fractional differential

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equations. Moreover, he gave some interesting examples [4]. To the best of our knowledge, most works in the literature have studied problem (P_1) only numerically and with a term f in the right-hand side of equation (1) that does not depend on u. Differently, here we study problem (P_1) by the lower and upper solutions method, considering a more general situation where the nonlinear term f is a function of u. This is important since the physical phenomena described by the differential equations are mainly of nonlinear nature.

The method of upper and lower solutions is an efficient tool in the study of differential equations [13]. Indeed, when we apply this method, we prove not only existence of solution, but we also get its location between the lower and upper solutions. The method was first introduced by Picard in 1893, later developed by Dragoni, and then becoming a useful tool to prove existence of a solution for ordinary as well as fractional differential equations [14,15,16,17,18].

The paper is organized as follows. Section 2 is devoted to some definitions on fractional calculus and properties that will be used later. We also define the upper and lower solutions for problem (P_1). Our results are given in Sections 3 and 4. The main result is Theorem 2, which establishes existence of solution for problem (P_1). To prove it, we make use of several auxiliary results. The first of them is given in Section 3, where we provide a monotonicity result for the right Caputo derivative. In Section 4, we convert problem (P_1) into an equivalent Caputo boundary value problem of order p that, under some conditions on the nonlinear term f, is used to prove existence of solutions for problem (P_1) between the reversed ordered lower and upper solutions. Moreover, we construct explicitly the upper and lower solutions. The new results of the paper are then illustrated through an example in Section 5.

2 Preliminaries

This section is devoted to recall some essential definitions on fractional calculus [1,2,3]. We also define some concepts related to upper and lower solutions.

Definition 1. *Let g be a real function defined on $[0,1]$ and $\mu > 0$. Then the left and right Riemann–Liouville fractional integrals of order μ of g are defined respectively by*

$$I_{0+}^{\mu} g(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{g(s)}{(t-s)^{1-\mu}} ds$$

and

$$I_{1-}^{\mu} g(t) = \frac{1}{\Gamma(\mu)} \int_t^1 \frac{g(s)}{(s-t)^{1-\mu}} ds.$$

The left Riemann–Liouville and the right Caputo fractional derivatives of order $0 < \mu < 1$ of function g are

$$D_{0+}^{\mu} g(t) = \frac{d}{dt} \left(I_{0+}^{1-\mu} g \right)(t)$$

and

$$^{C}D_{1-}^{\mu} g(t) = -I_{1-}^{1-\mu} g'(t),$$

respectively.

With respect to the properties of Riemann–Liouville and Caputo fractional derivatives, we recall here two of them. Let $0 < \mu < 1$ and $f \in L_1[0,1]$. Then,

1. $I_{0+}^{\mu} D_{0+}^{\mu} f(t) = f(t) + ct^{\mu-1}$ almost everywhere on $[0,1]$;
2. $I_{1-}^{\mu} {}^{C}D_{1-}^{\mu} f(t) = f(t) - f(1)$.

Now, we give the definition of lower and upper solutions for problem (P_1). By $AC^2[0,1]$ we denote the following space of functions:

$$AC^2[0,1] := \left\{ u \in C^1[0,1] \mid u' \text{ is an absolutely continuous function on } [0,1] \right\}.$$

Definition 2. *Functions $\alpha, \beta \in AC^2[0,1]$ are called, respectively, lower and upper solutions of problem (P_1) if*

1. *$\omega^2 \alpha(t) - {}^{C}D_{1-}^r D_{0+}^q \alpha(t) - f(t, \alpha(t)) \leq 0$ for all $t \in [0,1]$ and all $r \in [p,1)$ and, moreover, $\alpha(0) \geq 0$, $D_{0+}^q \alpha(1) \geq 0$;*
2. *$\omega^2 \beta(t) - {}^{C}D_{1-}^r D_{0+}^q \beta(t) - f(t, \beta(t)) \geq 0$ for all $t \in [0,1]$ and all $r \in [p,1)$ and, moreover, $\beta(0) \leq 0$, $D_{0+}^q \beta(1) \leq 0$.*

Functions α and β are lower and upper solutions in reverse order if $\alpha(t) \geq \beta(t)$, $0 \leq t \leq 1$.

Remark. If α and β are, respectively, lower and upper solutions of problem (P_1), then they are still lower and upper solutions for the sequence of problems generated by the boundary conditions (2)–(3) and the fractional differential equations obtained from (1) by replacing p by r for all $r \in [p,1)$.

3 Monotonicity for the Right Caputo Derivative

We begin by proving a useful monotonicity result for the right Caputo derivative. Theorem 1 provides the right counterpart of the main result of [19], which was recently obtained for the left Caputo fractional derivative $^{C}D_{0+}^{r}f(t)$. It will be needed in the proof of our Lemma 4.

Theorem 1. *Assume that $f \in C^{1}[0,1]$ is such that $^{C}D_{1-}^{r}f(t) \geq 0$ for all $t \in [0,1]$ and all $r \in (p,1)$ with some $p \in (0,1)$. Then f is monotone decreasing. Similarly, if $^{C}D_{1-}^{r}f(t) \leq 0$ for all t and r mentioned above, then f is monotone increasing.*

Proof. The proof is based on the following well-known propriety:

$$0 \leq \lim_{r \to 1^{-}} {}^{C}D_{0+}^{r}f(t) = \lim_{r \to 1^{-}} I_{0+}^{1-r}f'(t) = f'(t)$$

(see Theorem 2.10 of [20]). For the right Caputo fractional derivative $^{C}D_{1-}^{r}f(t)$, one can prove the following analogue property:

$$0 \leq \lim_{r \to 1^{-}} {}^{C}D_{1-}^{r}f(t) = \lim_{r \to 1^{-}} -I_{1-}^{1-r}f'(t) = -f'(t). \tag{4}$$

Using (4), the proof follows in the same way as in [19].

Remark. Property (4) and Theorem 1 can be obtained straightforwardly from the results of [20,19] by using the duality theory of Caputo–Torres between left and right fractional operators [21].

4 Existence of Solutions

First we solve a Riemann–Liouville fractional problem of order q:

$$\begin{cases} D_{0+}^{q}u(t) = v(t), & 0 \leq t \leq 1, \\ u(0) = 0. \end{cases} \tag{P_2}$$

Lemma 1. *For $0 < q < 1$, the solution of problem (P_2) is given by*

$$u(t) = \frac{1}{\Gamma(q-1)} \int_{0}^{t} (t-s)^{q-1} v(s) \, ds. \tag{5}$$

Proof. We get (5) by applying the properties of the Riemann–Liouville derivative and the initial condition $u(0) = 0$.

Let $E := C([0,1], \mathbb{R})$ be equipped with the uniform norm $||u|| = \max_{t \in [0,1]} |u(t)|$. Define the operator T on E by

$$Tv(t) = \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} v(s) \, ds = I_{0+}^{q} v(t), \quad t \in [0,1].$$

Thus, $u(t) = Tv(t)$. Since $D_{0+}^{q}u(1) = 0$, problem (P_1) is equivalent to the following Caputo boundary value problem:

$$\begin{cases} \omega^2 Tv(t) - {}^{C}D_{1-}^{p}v(t) = f(t, Tv(t)), & 0 \leq t \leq 1, \\ v(1) = 0. \end{cases} \tag{P_3}$$

Let us make the following hypotheses:

(H_1) there exists a nonnegative constant A such that

$$\omega^2 x - f(t,x) \leq A(1-t)^{1-r}$$

for $0 \leq t \leq 1$, $0 \leq x \leq \frac{A}{\Gamma(q+1)}$, and for all $r \in [p,1)$;

(H_2) there exists a constant $B \leq 0$ such that $A \geq |B|$ and

$$\omega^2 x - f(t,x) \geq B(1-t)^{1-r}$$

for $0 \leq t \leq 1$, $\frac{B}{\Gamma(q+1)} \leq x \leq 0$ and for $r \in [p,1)$.

Lemma 2. *If hypotheses (H_1) and (H_2) hold, then problem (P_1) has a lower and an upper solution.*

Proof. Setting $\varphi(t) = A(1-t)$, it follows that

$$0 \le T\varphi(t) = I^q_{0+}\varphi(t) = \frac{At^q(q+1-t)}{\Gamma(q+2)} \le \frac{A}{\Gamma(q+1)}.$$

Now we prove that $\alpha(t) = T\varphi(t)$ is an upper solution of problem (P_1). We have for all $r \in [p,1)$ that

$$\begin{aligned}
\omega^2 T\varphi(t) &- {}^C D^r_{1-}\varphi(t) - f(t, T\varphi(t)) \\
&= \frac{-A}{\Gamma(2-r)}(1-t)^{1-r} + \omega^2 T\varphi(t) - f(t, T\varphi(t)) \\
&\le -A(1-t)^{1-r} + \omega^2 T\varphi(t) - f(t, T\varphi(t)) \\
&\le 0.
\end{aligned}$$

In addition, $\alpha(0) = T\varphi(0) = 0$ and $D^q_{0+}\alpha(1) = \varphi(1) = 0$. Thus, $\alpha(t) = T\varphi(t)$ is a lower solution of problem (P_1). Similarly, if we set $\psi(t) = B(1-t)$, then $\beta(t) = T\psi(t)$ is an upper solution of problem (P_1).

Lemma 3. *Under hypotheses (H_1) and (H_2), the upper and lower solutions of problem (P_1) satisfy $\beta(t) \le \alpha(t)$ and $D^q_{0+}\beta(t) \le D^q_{0+}\alpha(t)$ for all $0 \le t \le 1$.*

Proof. Since $\alpha(t) = T\varphi(t)$ and $\beta(t) = T\psi(t)$ are, respectively, lower and upper solutions of problem (P_1), then from

$$\alpha(t) = \frac{A(q+1-t)t^q}{\Gamma(q+2)} \ge 0, \quad \beta(t) = \frac{B(q+1-t)t^q}{\Gamma(q+2)} \le 0,$$

we get that

$$D^q_{0+}\alpha(t) = \varphi(t) = A(1-t) \ge B(1-t) = \psi(t) = D^q_{0+}\beta(t).$$

This completes the proof.

We consider a sequence of modified problems

$$\begin{cases} -{}^C D^r_{1-}v(t) = Fv(t), & 0 \le t \le 1, \\ v(1) = 0 \end{cases} \qquad ((P_4)_r)$$

for $r \in [p,1)$, where the operator $F : E \to E$ is defined by

$$Fv(t) = -\omega^2 T\min[\varphi, \max(v,\psi)] + f(t, T\min[\varphi, \max(v,\psi)]), \quad 0 \le t \le 1.$$

Next lemma gives the relation between the solution of a modified problem $((P_4)_r)$ and the solution of problem (P_1).

Lemma 4. *If v is a solution of problem $((P4)_p)$, then $u = Tv$ is solution of problem (P_1) satisfying*

$$\beta(t) \le u(t) \le \alpha(t) \quad and \quad D^q_{0+}\beta(t) \le D^q_{0+}u(t) \le D^q_{0+}\alpha(t)$$

for all $0 \le t \le 1$.

Proof. Firstly, for $r \in [p,1)$, we prove that if v_r is a solution of problem $((P_4)_r)$, then $\psi(t) \le v_r(t) \le \varphi(t)$. Putting $\varepsilon(t) = v_r(t) - \varphi(t)$, and using the initial conditions $v_r(1) = \varphi(1) = 0$, it yields $\varepsilon(1) = 0$. Suppose the contrary, i.e., that there exists $t_1 \in [0,1[$ such that $v_r(t_1) > \varphi(t_1)$. From the continuity of ε, we conclude that there exist $b \in [t_1,1)$ and $a \in [0,t_1]$ such that $\varepsilon(b) = 0$ and $\varepsilon(t) \ge 0, t \in [a,b]$. Applying the right Caputo fractional derivative and taking into account the definition of lower solution, we get

$$\begin{aligned}
{}^C D^r_{1-}\varepsilon(t) &= {}^C D^r_{1-}v_r(t) - {}^C D^r_{1-}\varphi(t) \\
&= \omega^2 T\min[\varphi, \max(v_r,\psi)] - f(t, T\min[\varphi, \max(v_r,\psi)]) - {}^C D^r_{1-}D^q_{0+}\alpha(t) \\
&\le 0
\end{aligned}$$

for $t \in [a,b]$. Thanks to Theorem 1, we know that ε is increasing on $[a,b]$. Since $\varepsilon(b) = 0$, we conclude that $v_r(t) \leq \varphi(t)$, $t \in [a,b]$, which leads to a contradiction. Similarly, we prove that $\psi(t) \leq v_r(t)$, $t \in [0,1]$. From the above discussion, if v is a solution of problem $((P4)_p)$, then

$$-{}^C D_{1-}^p v(t) = (Fv)(t) = -\omega^2 T v(t) + f(t, Tv(t)).$$

Thus, v is a solution of (P_3) and, therefore, $u = Tv$ is a solution of (P_1). Finally, the monotonicity of operator T implies

$$T\psi(t) \leq Tv(t) \leq T\varphi(t), \quad t \in [0,1].$$

This achieves the proof.

Now we are ready to formulate and prove our main result of existence of solution for problem (P_1).

Theorem 2. *Assume that hypotheses (H_1) and (H_2) hold. Then, problem (P_1) has at least one solution u such that*

$$\beta(t) \leq u(t) \leq \alpha(t)$$

and

$$D_{0+}^q \beta(t) \leq D_{0+}^q u(t) \leq D_{0+}^q \alpha(t)$$

for all $0 \leq t \leq 1$.

Proof. Define the operator R on E by $Rv(t) = I_{1-}^p Fv(t)$, $0 \leq t \leq 1$. Set

$$\Omega := \{v \in C[0,1], \psi(t) \leq v(t) \leq \varphi(t), 0 \leq t \leq 1\},$$

where

$$M := \max\left\{\left|\omega^2 x - f(t,x)\right|, \beta(t) \leq x \leq \alpha(t), 0 \leq t \leq 1\right\}.$$

Let $v \in \Omega$. Taking into account that $\beta(t) \leq T(\min[\varphi, \max(v, \psi)]) \leq \alpha(t)$, then

$$|Rv(t)| \leq I_{1-}^p \left|-\omega^2 T(\min[\varphi, \max(v, \psi)]) + f(t, T\min[\varphi, (\max(v, \psi))])\right|$$
$$\leq \frac{M}{\Gamma(p+1)}.$$

Thus, $R(\Omega)$ is uniformly bounded and $R(\Omega) \subset \Omega$. For simplicity, denote

$$g(t) = -\omega^2 T(\min[\varphi, \max(v, \psi)]) + f(t, T\min[\varphi, \max(v, \psi)]).$$

For $0 \leq t_1 < t_2 \leq 1$, we have

$$|Rv(t_1) - Rv(t_2)| \leq \left|I_{1-}^p g(t_1) - I_{1-}^p g(t_2)\right|$$
$$\leq \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} (s-t_1)^{p-1} |g(s)| ds$$
$$+ \frac{1}{\Gamma(p)} \int_{t_2}^{1} \left((s-t_1)^{p-1} - (s-t_2)^{p-1}\right) |g(s)| ds$$
$$\leq \frac{M}{\Gamma(p+1)} \left((1-t_1)^p - (1-t_2)^p\right) \to 0 \text{ as } t_1 \to t_2.$$

Therefore, $R(\Omega)$ is equicontinuous. We conclude, by the Arzela–Ascoli theorem, that R is completely continuous. Then, by Schauder's fixed point theorem, R has a fixed point $v \in \Omega$. We conclude that $u = Tv$ is a solution of (P_1) satisfying, by Lemma 4, $\beta(t) \leq u(t) \leq \alpha(t)$ and $D_{0+}^q \beta(t) \leq D_{0+}^q u(t) \leq D_{0+}^q \alpha(t)$, $0 \leq t \leq 1$. The proof is complete.

5 An Illustrative Example

We present a simple example to illustrate our results. Consider problem (P_1) with $\omega = 1$, $p = q = \frac{1}{2}$, and

$$f(t,x) = x - \frac{1}{100}(1-t)^{\frac{1}{2}}, \quad 0 \le t \le 1.$$

If we choose $A = \frac{1}{100}$ and $B = -\frac{1}{100}$, then we get

$$\omega^2 x - f(t,x) = \frac{1}{100}(1-t)^{\frac{1}{2}} \le A(1-t)^{1-r}$$

and

$$\omega^2 x - f(t,x) = \frac{1}{100}(1-t)^{\frac{1}{2}} \ge 0 \ge B(1-t)^{1-q}$$

for $0 \le t \le 1$ and for all $r \in [p,1)$. Then all assumptions of Theorem 2 hold. Consequently, problem

$$u(t) - {}^C D_{1-}^{1/2} D_{0+}^{1/2} u(t) = u(t) - \frac{1}{100}(1-t)^{\frac{1}{2}}, \quad 0 \le t \le 1,$$
$$u(0) = 0, \quad D_{0+}^{1/2} u(1) = 0,$$

has a solution u such that $\beta(t) \le u(t) \le \alpha(t)$. By direct computations we get

$$\alpha(t) = \frac{A(q+1-t)t^q}{\Gamma(q+2)} = \frac{t^{\frac{1}{2}}(\frac{3}{2}-t)}{100\Gamma(\frac{5}{2})} \ge 0, \quad \beta(t) = -\frac{t^{\frac{1}{2}}(\frac{3}{2}-t)}{100\Gamma(\frac{5}{2})} \le 0,$$

and

$$u(t) = \frac{t^{\frac{1}{2}}}{100}\left(1 - \frac{2t}{3}\right).$$

6 Conclusion

In this paper we have proved a useful monotonicity result for the right Caputo derivative. We solved the Riemann–Liouville fractional problem $D_{0+}^q u(t) = v(t)$ of order q, $0 \le t \le 1$, subject to the initial condition $u(0) = 0$. Then, under hypotheses (H_1) and (H_2), we proved that the nonlinear fractional oscillator problem (1)–(3) with both left Riemann–Liouville and right Caputo fractional derivatives, denoted by (P_1), has a lower and an upper solution, respectively $\alpha(t)$ and $\beta(t)$, such that $\beta(t) \le \alpha(t)$ and $D_{0+}^q \beta(t) \le D_{0+}^q \alpha(t)$ for all $0 \le t \le 1$. A relation between the solution of problem (P_1) and the solution of a sequence of modified problems was established, which allowed us to prove existence of solution for problem (P_1). We finished by considering a simple illustrative example. Our results show the usefulness and effectiveness of the upper and lower solutions method in the study of a general class of fractional oscillator equations with physical relevance.

Acknowledgments

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