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Generalized Elzaki – Tarig Transformation and its Applications to New Fractional Derivative with Non Singular Kernel

Shrinath Manjarekar and Ashok Parasharam Bhadane*

Loknete Vyankatrao Hiray Arts, Science and Commerce College, Nashik,(M.S.), India

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Abstract: In this paper, we have defined the new generalized Elzaki – Tarig transformation and find out its relations with other transformations. Furthermore we have derived the inversion formula, convolution theorem for it. Also as an application we have solve fractional differential equation with non – singular kernel.

Keywords: Elzaki – Tarig transform, fractional derivatives.

1 Introduction

Fractional differential equations play an important role in modelling the dynamics of complex systems (see for example Refs. [1,2,3,4,5] and the references therein) The idea of transformations and hyper geometric functions [6] is generally started with the need of converting problems from one form into another form which is rather simpler to solve and then by using inversion formula again coming back to the original form with the solution.

In recent years many linear boundary value problems, initial value problems are effectively solved by these transformations [7,8,9,10] like Laplace, Fourier, Mellin, wavelet and other transformations with applications increasing rapidly in daily life and branches of science like bioengineering, computational fluid dynamics, Abel's integral equations, biomathematics, capacitor theory, conductance of biological systems [11,12]. The Elzaki–Tarig transform [13] which is still not widely known in the area of fractional calculus

In this paper, we have introduced the generalized Elzaki – Tarig transformations with its relation to other transformations in general way. Moreover, as an application we have solved fractional differential equation with non-local and non-singular kernel [7] by using Tarig transformation [10] as a part of generalized definition.

The paper mainly divided into three parts, in the first part we define the generalized Elzaki–Tarig transform and some of its properties, in the second part we have derive the relation of it with other transformations. In the third part, we have provided an application of it to solve fractional differential equations with non-local and non-singular kernel along with the discussion of the obtained result conclusion part ends our manuscript.
In the following we present some basic definitions needed in proving the main results.
Definition 1: Atangana – Baleanu Riemann fractional derivative. Consider a function $f \in H^1(a,b), b > a, \alpha \in [0,1]$ which is of exponential order then the new ABR fractional derivative [7] of $f(t)$ is defined as ,
$_a^{ABR}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}[\int_a^t f(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)]b > a, \alpha \in [0,1]$ and $B(\alpha)$ is normalization function obeying $B(0) = B(1) = 1.$ (1)
Definition 2: Atangana – Baleanu Caputo fractional derivative. Consider a function $f \in H^1(a,b), b > a, \alpha \in [0,1]$ which is of exponential order then the new ABC fractional derivative [7] of $f(t)$ is defined as,
$_a^{ABC}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\left[\int_a^t f'(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)dx\right] b > a, \alpha \in [0,1]$ and $B(\alpha)$ is normalization function obeying $B(0) = B(1) = 1.$ (2)

* Corresponding author e-mail: shrimathematics@gmail.com

Definition 3: Laplace type Integral Transform
Consider a function f(x) which is piecewise continuous and of exponential order then the Laplace – type integral [8] transform is defined as follows

$$\mathscr{L}_\varepsilon\{f(x);p\} = \int_0^\infty \varepsilon'(x)\, e^{-\Phi(p)\varepsilon(x)} f(x)\,dx. \quad (3)$$

where $\Phi(p)$ is invertible function with $\varepsilon(x) = \int e^{-a(x)}dx$ an exponential function and $a(x)$ as invertible function.
Definition 4: Elzaki – Tarig Transform

Let S = $\{f(t) : \exists k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, t\in(-1)^j X[0,\infty) \wedge M > 0\}$ then for the given function which satisfies the condition of the set S, the Elzaki – Tarig transformation of $f(t)$ is defined as [10]
$T(f(t);p) = p\int_0^\infty f(t)e^{-t/p}dt, p\neq 0.$ (4)
Definition 5: Tarig Transform

Let S = $\{f(t) : \exists k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, t\in(-1)^j X[0,\infty) \wedge M > 0\}$ then for the given function which satisfies the condition of the set S, the Tarig transformation of $f(t)$ is defined as [13]

$$T(f(t);p) = \int_0^\infty \frac{1}{p} f(t)e^{\frac{-t}{p^2}}dt, p\neq 0. \quad (5)$$

Definition 6: Mellin Transform
The Mellin transform of the function $f(t)$ is the integral transform defined as [8]

$$M(f(t);p) = \int_0^\infty t^{p-1} f(t)\,dt. \quad (6)$$

2 Main Result

Definition: We consider the definition of Generalized Elzaki – Tarig Transformation by using the definition (3), (5) and (6) as ,

$$\Im_\varepsilon\{f(x);p\} = \int_0^\infty \Phi\left(\frac{1}{p}\right)\Phi_1(p)\,\varepsilon'(x)\,e^{-\Phi(p)\varepsilon(x)} f(x)\,dx \qquad p\neq 0. \quad (2.1)$$

Here $f(x)\in$ S = $\{f(x) : \exists k_1, k_2 > 0, |f(x)| < Me^{\frac{|x|}{k_j}}, x\in(-1)^j X[0,\infty) \wedge M > 0\}$ and $\Phi\left(\frac{1}{p}\right), \Phi_1(p)$ are invertible functions of p with $\varepsilon(x) = \int e^{-a(x)}dx$ an exponential function and $a(x)$ as invertible function, thus from the definitions above it can be seen that it is the generalization of Elzaki – Tarig transformation.

2.1 Relation with Other Transformations

2.1.1 Elzaki – Tarig Transform

The Elzaki – Tarig Transformations [10,13] of a function $f(x)$ can be obtained by taking $\Phi(p) = \frac{1}{p} \wedge \Phi_1(p) = 1 with \varepsilon(x) = x$ in equation (2.1)

$\Im_x\{f(x);p\} = \int_0^\infty pe^{\frac{-x}{p}} f(x)dx,\ p\neq 0$ i.e.

2.1.2 Tarig Transform

The Tarig Transformation [10] of a function $f(x)$ can be obtained by taking $\Phi(p) = \frac{1}{p^2} \wedge \Phi_1(p) = \frac{1}{p^3} with \varepsilon(x) = x$ in equation (2.1)

$\Im_x\{f(x);p\} = \int_0^\infty p^2 \frac{1}{p^3} e^{\frac{-x}{p^2}} f(x)dx,\ p\neq 0,$

$\Rightarrow \Im_x\{f(x);p\} = \int_0^\infty \frac{1}{p} e^{\frac{-x}{p^2}} f(x)dx,\ p\neq 0.$

2.1.3 Laplace Transformation

The Laplace Transformation [8] of a function $f(x)$ can be obtained by taking $\Phi(p) = p \wedge \Phi_1(p) = p with \varepsilon(x) = x$ in equation (2.1) with Re.$(p)>0$
$\Im_x\{f(x);p\} = \int_0^\infty \frac{1}{p} pe^{-xp} f(x)dx,\ Re.(p) > 0,$
$\Rightarrow \Im_x\{f(x);p\} = \int_0^\infty e^{-xp} f(x)dx,\ Re.(p) > 0.$

2.1.4 Mellin Transform

The Mellin Transformation [8] of a function $f(x)$ can be obtained by taking $\Phi(p) = -p \wedge \Phi_1(p) = -p with \varepsilon(x) = \ln(x)$ in equation (2.1) with $p > 0$

$\Im_{\ln(x)}\{f(x);p\} = \int_0^\infty (\frac{-1}{p})(-p)\frac{1}{x}e^{p\ln(x)}f(x)dx,\ p > 0$

$\Rightarrow \Im_{\ln(x)}\{f(x);p\} = \int_0^\infty \frac{1}{x}e^{\ln(x^p)}f(x)dx,\ p > 0,$

$\Rightarrow \Im_{\ln(x)}\{f(x);p\} = \int_0^\infty \frac{1}{x}x^p f(x)dx,\ p > 0,$

$\Rightarrow \Im_{\ln(x)}\{f(x);p\} = \int_0^\infty x^{p-1}f(x)dx,\ p > 0.$

2.1.5 L_2 Transform

As a particular case of $L_\varepsilon-$ transform; the L_2 transform [8,14] of a function $f(x)$ at a point p can be obtained from (2.1) by substitution of $\Phi(p) = p^2 \wedge \Phi_1(p) = p^2 with \varepsilon(x) = x^2$

$\Im_{x^2}\{f(x);p\} = \int_0^\infty \frac{1}{p^2}p^2 e^{-p^2x^2}f(x)dx,\ \text{Re.}(p^2) > 0,$

which is nothing but L_2- transform of $f(x)$ [8].

2.2 Properties of Generalized Elzaki – Tarig Transform

2.2.1 Inversion Formula

The definition of generalized Elzaki – Tarig transformation tells us that

$F(p) = \Im_\varepsilon\{f(x);p\} = \int\limits_0^\infty \Phi\left(\frac{1}{p}\right)\Phi_1(p)\,\varepsilon'(x)\,e^{-\Phi(p)\varepsilon(x)}f(x)\,dx\ p{\neq}0$ which satisfies the given conditions in (2.1) then the inverse transformation to be defined as,

$$\Im_\varepsilon^{-1}(F(p)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\Phi^{-1}(p)\Phi_1^{-1}(p))e^{\Phi(p)\varepsilon(x)}dp.$$

Proof: By definition of generalized Elzaki – Tarig transformation (2.1) with defining

$\Phi\left(\frac{1}{p}\right) = \frac{1}{r}$ and $\Phi_1(p) = r. \Rightarrow F\left(\Phi^{-1}\left(\frac{1}{r}\right)\Phi_1^{-1}(r)\right) = \int_0^\infty \varepsilon'(x)\,e^{-r\varepsilon(x)}f(x)\,dx.$

Put $\varepsilon(x) = t$ in the above equation

$\Rightarrow F\left(\Phi^{-1}\left(\frac{1}{r}\right)\Phi_1^{-1}(r)\right) = \int_0^\infty e^{-rt}f\left(\varepsilon^{-1}(t)\right)dt = \Im_\varepsilon\left\{f\left(\varepsilon^{-1}(t)\right);r\right\},$

whenever, $\Phi\left(\frac{1}{p}\right) \wedge \Phi_1(p)$ are inverses of each other so that by complex inversion formula for the Laplace transform with $\varepsilon^{-1}(t) = x \wedge r = p.$

$$\Rightarrow f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F\left(\Phi^{-1}(p)\,\Phi_1^{-1}(p)\right)e^{p\varepsilon(x)}dp.$$

2.2.2 Generalized Elzaki – Tarig Transform of Derivative

Let $f(x)$ satisfies all the conditions in equation (2.1) with $f'(x)$ has piecewise continuous derivative then $I_\varepsilon\{f'(x);p\} = \Phi(p)[\Im_\varepsilon\{f(x);p\}] - \Phi\left(\frac{1}{p}\right)\Phi_1(p)f(0^+).$

Proof: By definition of generalized Elzaki – Tarig transformation (2.1), we have

$$\Im_\varepsilon\left\{f'(x);p\right\} = \int_0^\infty \Phi\left(\frac{1}{p}\right)\Phi_1(p)\,\varepsilon'(x)e^{-\Phi(p)\varepsilon(x)}f'(x)dx, p{\neq}0 = \Phi\left(\frac{1}{p}\right)\Phi_1(p)\int_0^\infty \varepsilon'(x)e^{-\Phi(p)\varepsilon(x)}f'(x)dx,$$

applying integration by part to the above we get

$$\Rightarrow \Im_{\varepsilon}\left\{f'(x);p\right\} = \Phi\left(\frac{1}{p}\right)\Phi_1(p)\{\left[\varepsilon'(x)e^{-\Phi(p)\varepsilon(x)}f(x)\right]_0^{\infty} + \Phi(p)\int_0^{\infty}\varepsilon'(x)e^{-\Phi(p)\varepsilon(x)}f(x)\,dx\}$$
$$= \Phi(p)\{\Im_{\varepsilon}\{f(x);p\} - \Phi\left(\frac{1}{p}\right)\Phi_1(p)f(0^{+}).$$

As $f(x)$ is of exponential order

2.2.3 Convolution Theorem

If $F(p)$ and $G(p)$ are the generalized Elzaki – Tarig transformations of two functions $f(x)$ and $g(x)$ respectively then the generalized convolution theorem for definition (2.1) is calculated as follows,

$$F(p)G(p) = \Im_{\varepsilon}\{f * g;p\} = (\int_0^{\infty}\Phi\left(\frac{1}{p}\right)\Phi_1(p)\varepsilon'(y)e^{-\Phi(p)\varepsilon(y)}f(y)\,dy)(\int_0^{\infty}\Phi\left(\frac{1}{p}\right)\Phi_1(p)\varepsilon'(t)e^{-\Phi(p)\varepsilon(t)}g(t)\,dt) =$$
$$\iint_0^{\infty}\phi\left(\frac{1}{p}\right)\psi_1(p)\varepsilon'(y)\varepsilon'(t)e^{-\Phi(p)[\varepsilon(y)+\varepsilon(t)]}f(y)g(t)\,dydt.$$

Substitute $\varepsilon(y)+\varepsilon(t) = \varepsilon(x)$ and changing the order of integration [4] in the double integral we get,

$$F(p)G(p) = \int_0^{\infty}\phi\left(\frac{1}{p}\right)\psi_1(p)\varepsilon'(x)e^{-\Phi(p)\varepsilon(x)}dx\int_0^{x}\varepsilon'(t)g(t)f\left(\varepsilon^{-1}(\varepsilon(x)-\varepsilon(t))\right)dt =$$
$$\Im_{\varepsilon}\left\{\left(\int_0^{x}\Phi\left(\frac{1}{p}\right)\Phi_1(p)\varepsilon'(t)g(t)f\left(\varepsilon^{-1}(\varepsilon(x)-\varepsilon(t))\right)dt\right);p\right\}.$$

3 Application

We present some basic relations needed for giving an application of Generalized Elzaki – Tarig transformation,

3.1 Laplace Transform of ABR Fractional Derivative

Given a function $f(x)$ then the Laplace transform of ABR fractional derivative of it is given by [8,14],

$$L\{{}_a^{ABR}D_t^{\alpha}(f(t));p\} = \frac{B(\alpha)}{1-\alpha}\frac{p^{\alpha}L\{f(t);p\}-p^{\alpha-1}F(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}},\ \alpha\in[0,1).$$

3.1.1 Relation Between Laplace and Tarig Transform

Given a function $f(x)$ then the relation between Laplace and Tarig transform is given by using the definition of generalized Elzaki – Tarig transform, $L\{f(x);p\} = \Im_x\{f(x);p\}$ with $\Phi(p) = p \wedge \Phi_1(p) = p$ and $\varepsilon(x) = x$.

Hence from (3.1.1), (2.1.3) and the definition (5) we get

$$L\{{}_a^{ABC}D_t^{\alpha}(f(t));p\} = \frac{B(\alpha)}{1-\alpha}\frac{p^{\alpha}\Im_x\{f(t);p\}-p^{\alpha-1}F(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}} = \frac{B(\alpha)}{1-\alpha}\frac{p^{\alpha}L\{f(t);p\}-p^{\alpha-1}F(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}},\ \alpha\in[0,1).$$

3.1.2 Applications to Fractional Differential Equation

We consider the following fractional differential equation in the ABC sense [7], namely
$y^{\alpha}(t) = y(t), \alpha \in (0,1)$ with initial condition $y(0) = 1$.

To solve the above ABC fractional differential equation we apply the Laplace transform on both sides of the above equation and use the relation (3.1.2), namely

$$L\{y^{\alpha}(t)\} = L\{y(t)\} \tag{1}$$

which implies

$$L\{{}_{a}^{ABC}D_{t}^{\alpha}(y(t));p\} = L\{y(t)\}, \tag{2}$$

or

$$\frac{B(\alpha)}{(1-\alpha)}\frac{p^{\alpha}\Im_{x}\{y(t);p\}}{p^{\alpha}+\frac{\alpha}{(1-\alpha)}} = \Im_{x}\{y(t);p\}. \tag{3}$$

After some calculations we get

$$[B(\alpha)p^{\alpha} - [(1-\alpha)p^{\alpha}+\alpha]]Y(p) = B(\alpha)p^{\alpha-1} \tag{4}$$

and we conclude that

$$Y(p) = \frac{B(\alpha)p^{\alpha-1}}{[B(\alpha)p^{\alpha} - [(1-\alpha)p^{\alpha}+\alpha]]}. \tag{5}$$

Now by applying the convolution property and taking the inverse Tarig transform we get the desired solution.

4 Conclusion

The paper gives some new ideas in the field of integral transformations as well as in fractional calculus as an application. Also with the help of generalized Elzaki – Tarig transformations we have found relation between other transformations.

We can conclude that by making proper choice of $\Phi(p), \Phi_1(p)$ and $\varepsilon(x)$ the ABC and ABR fractional differential equation which is still not very well known in the field of fractional calculus can be solved using Generalized Elzaki – Tarig transformation.

Thus from generalized Elzaki – Tarig transformation various transform can be obtained by putting different condition on it which will be helpful to find the solutions of fractional differential equations and boundary value problems and might be extend to solve partial fractional differential equation by extending the definition to the higher dimensions.

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