Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.12785/pfda/010106

Modeling the Under-Actuated Mechanical System with Fractional Order Derivative

Trisha Srivastava[1], *Abhaya Pal Singh*[2,*] *and Himanshu Agarwal*[1]

[1]E&TC Department, Symbiosis Institute of Technology, Symbiosis International University, Pune, India
[2]Research Scholar, Symbiosis Institute of Technology, Symbiosis International University, Pune, India

Received: 17 Dec. 2014, Revised: 27 Dec. 2014, Accepted: 29 Dec. 2014
Published online: 1 Jan. 2015

Abstract: This paper presents the fractional order modeling of an under-actuated mechanical system. The under-actuated system taken in this paper is an overhead crane that has two degrees of freedom and one control input. The modeling equation is derived with the help of Euler-Lagrange equation. The proposed mathematical model is helpful in understanding the under-actuated mechanical system's fractional characteristics.

Keywords: Fractional order calculus (FOC), fractional Hamiltonian, under-actuated mechanical system, IOC, modeling of systems.

1 Introduction

A fractional-order system is defined by a fractional differential equation or a fractional integral equation or a system of such equations. Traditional calculus is based on integer order differentiation and integration. Fractional order calculus allows us to use differentiations and integrations of arbitrary order that may include integer as well as non-integer orders. Control of mechanical systems is currently among one of the most active fields of research due to the diverse applications of mechanical systems in real-life. Fractional controllers have two parameters more than the conventional PID controller; therefore, two more specifications can be met, improving the performance of the system [1, 2, 3]. Using fractional order calculus we can increase the flexibility of controlling any system from a point to a space.

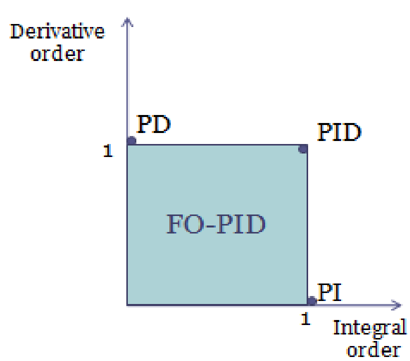

Fig.1. Regions for FOC and IOC

Fractional calculus is not a new topic. It is as old as ordinary calculus. The birth of fractional calculus is dated back to

* *E-mail: abhaya.aps@gmail.com*

the seventeenth century when the concept of fractional differential operator was first mentioned by Leibniz in a letter to L'Hopital in 1965. Subsequently the developments of fractional differential operators were contributed by many famous scientists in that period [4]. Several mathematicians like Leibniz (1695), Euler (1730), Lagrange (1772), Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), and Riemann (1876) made major contributions to the theory of fractional calculus. In spite of its long history, fractional calculus was not considered eligible for any applications. This was due to its high complexity and lack of physical and geometric interpretation [5]. Applications of fractional order calculus to real-world problems are only four decades old. During this period fractional differential equations have gained considerable importance and attention due to their applications in science and engineering, i.e. in control, in porous media, in electrochemistry, in viscoelasticity, and in electromagnetism theory [6, 7].

The fractional or arbitrary order systems can describe dynamical behavior of materials and processes over vast time and frequency scales with very concise and computable models [8]. Nowadays well known concepts are being extended to the development of robust control systems [9] as well as signal filtering methods [10], observer discussed in [11] can also be extend to fractional observer. In [12] it is proposed a generalization of the PID controller, namely the $PI^{\lambda}D^{\mu}$ controller that involves an integrator of order 'l' and a differentiator of arbitrary order 'm'. It also demonstrated the better response of this type of controller as compared to the classical PID controller when used for the control of fractional order systems.

Systems which cannot be commanded to follow arbitrary trajectories are called under-actuated systems [13]. The simplest reason behind this is the number of actuators being less than the degrees of freedom. These systems are said to be trivially under-actuated systems. For example, the act of standing with one foot flat on the ground is not considered as dexterous as a headstand. The contact point between the body and ground in headstands is acting as a pivot without actuation. A system may show the properties of under-actuation due to following reasons: (i) system dynamics, (ii) design of the system for cost reduction or some other practical purposes, (iii) actuator failure and (iv) imposed artificially to create complex low- order nonlinear systems for study purpose.

Overhead crane is one of the best examples of under-actuated systems. It is used for transporting a load from one place to another using trolley and load system. This type of cranes can handle huge loads and especially used in factories, ships, platforms, depots, dockyards, etc. So modeling of such systems is the concern of this paper. In [23] overhead system is considered and the system is modeled to its integer equivalent model, for this model they designed fractional controller and fuzzy base controller then the results are compared. This paper considers the fractional equivalent modeling of the overhead crane system. Design of a controller for an under-actuated mechanical system named Double Inverted Pendulum on a Cart system (DIPOAC) is proposed in [2] that has three degrees-of-freedom and one control input. In [3] state space modeling technique is used for the modeling of fractional order systems.

In last few decades fractional differential equations have gained considerable importance and attention due to their applications in science and engineering, i.e. in control, in porous media, in electrochemistry, in viscoelasticity, and in electromagnetism theory [19]. In [20] it is shown that the order of the fractional operator is the function of the temperature variable $T(t)$.in [10, 21, 22] behavior of chaotic system and chaos control of a fractional order autonomous chaotic system is studied.

1.1 Preliminaries

FOC is a generalization of the IOC to a real or complex order. Formally the real order generalization is introduced as follows:

$$aD_t^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \displaystyle\int_a^t d\tau^{\alpha}, & \alpha < 0, \end{cases}$$

where $\alpha \epsilon R$.

There are many concepts of FOC given by different people. Some of them for fractional derivatives are [5, 14]:

1. Riemann-Liouville (RL): For the case of $0 < \alpha < 1$ the expression of fractional derivative is:-

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (1)$$

where Γ(.) is Gamma function.

2. Grünwald-Letnikov (GL):- If we consider $n = \frac{t-a}{h}$, where a is a real constant, which expresses a limit value we can write:

$$aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\frac{t-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m!\, \Gamma(\alpha-m+1)} f(t-mh). \tag{2}$$

3. Caputo:- It is defined for '$n-1 < \alpha < n$':-

$$aD_t^{\alpha} f(t) = \left[\frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{(\alpha+1-m)}} d\tau \right]. \tag{3}$$

The Caputo and Riemann-Liouville formulation coincide when the initial conditions are zero.

From the above definitions, we can observe that definition of fractional derivative involves integration. Since integration is a non-local operator (as it is defined on an interval), fractional derivative is also a non-local operator [5]. Calculating time-fractional derivative of a function $f\ (t)$ at some $t = t_1$ requires all the past history, i.e. all $f(t)$ from $t = 0$ to$t = t_1$. Fractional derivatives can be used for modeling systems with memory. Calculating space-fractional derivative of a function $f(x)$ at $x = x_1$ requires all non-local$f(x)$ values. Thus fractional derivatives can be used for modeling distributed parameter systems.

2 Modeling of the system

For a basic overhead crane, the defining nonlinear equations of motion can be derived as follows. First we assume that the cable is mass-free and that the trolley mass and the load mass at the end of the cable are denoted as M and m, respectively. Length of the cable is assumed to be l. There is an externally x −directed force on the trolley $u(t)$, and a gravity force $(m * g)$ always acts on the load where g is acceleration due to gravity. $x(t)$ represents the trolley position and $\theta(t)$is the tilt angle referenced to the vertically upward direction.

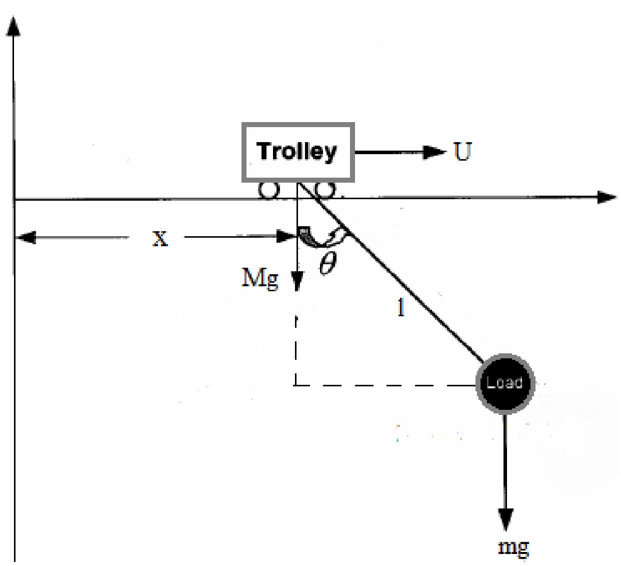

Fig.2. Schematic diagram of an overhead crane

Here we use Euler-Lagrangian model for analysis. In this we have to find the kinetic energy and potential energy of the system and the Lagrangian will be the difference of the kinetic energy and potential energy.

$$Ł = T - V, \tag{4}$$

where, $T = Kinetic\ Energy, V = Potential\ Energy$, Coordinates of the load: $(x + l sin\theta, -l cos\theta)$.
Therefore, the total potential energy and the kinetic energy of the whole system can be obtained as:

$$V_{total} = V_{trolly} + V_{load}, \tag{5}$$

$$V_{total} = Mgl + mg\,(l - l\,cos\theta), \tag{6}$$

$$T_{total} = T_{trolly} + T_{load}, \tag{7}$$

$$T_{total} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\,cos\theta). \tag{8}$$

Therefore, the Lagrangian of the system can be found as:

$$Ł = T_{total} - V_{total}, \tag{9}$$

$$Ł = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\,cos\theta) - Mgl - mg\,(l - l\,cos\theta). \tag{10}$$

Substituting the value of Ł from equation (10) into Euler-Lagrange equations below,

$$\frac{d}{dt}\left(\frac{\partial Ł}{\partial \dot{x}}\right) - \frac{\partial Ł}{\partial x} = u\,, \tag{11}$$

$$\frac{d}{dt}\left(\frac{\partial Ł}{\partial \dot{\theta}}\right) - \frac{\partial Ł}{\partial \theta} = 0. \tag{12}$$

And we get,

$$(M + m)\ddot{x} + ml\,\ddot{\theta}cos\theta - ml\,\dot{\theta}^2 sin\theta = u\,, \tag{13}$$

$$l\ddot{\theta} + \ddot{x}\,cos\theta + g\,sin\theta = 0, \tag{14}$$

After separating the values of $\ddot{x}$ and $\ddot{\theta}$ we get,

$$\ddot{x} = \frac{u + ml\,sin\theta\,\dot{\theta}^2 + gm\,sin\theta\,cos\theta}{M + m - mcos^2\theta}, \tag{15}$$

$$\ddot{\theta} = \frac{u\,cos\theta + ml\,\dot{\theta}^2 sin\theta\,cos\theta + (M + m)g\,sin\theta}{ml\,cos^2\theta - (M + m)l} \tag{16}$$

3 Transfer function

The effect of input force (u) on swing angle (φ) can be observed by deriving transfer function of the system. Since MATLAB can work only with linear functions, the set of equations (13) and (14) should be linearized about a stationary point where angle $\theta = 0$. Assume that $\theta = ø$ (where ø represents a small angle from the vertical downward direction). Therefore, $cos\ ø = 1, sin\ ø = ø$, and $\dot{ø}^2$ is negligible. Substituting these values in equations (13) and (14) we get:

$$(M + m)\ddot{x} + ml\,\ddot{ø} = u, \tag{17}$$

$$l\ddot{ø} + \ddot{x} + g\,ø = 0\,. \tag{18}$$

Laplace transform of the above equations,

$$(M+m)\,X(s)\,s^2 + ml\,ø(s)\,s^2 = u, \tag{19}$$

$$l\,ø(s)\,s^2 + X(s)\,s^2 + g\,ø(s) = 0. \tag{20}$$

From equation (16) we find the value of *X(s)* as:

$$X(s) = -\frac{g + ls^2}{s^2} \text{ø}(s). \tag{21}$$

Substituting the value of *X(s)* from equation (17) into equation (15) we get,

$$\frac{\text{ø}(s)}{u(s)} = \frac{-1}{Mls^2 + (M+m)g}. \tag{22}$$

For the purpose of comparison the fractional order transfer function of the system can be derived by using [4, 15] and written as:

$$\frac{\text{ø}(s)}{u(s)} = \frac{-1}{Mls^{2\alpha} + (M+m)g}. \tag{23}$$

From [18] we get the values of *M*, *m* and l as:

Table 1: List of parameters

Parameter	Description	Value
M	Trolley mass	0.25 kg
M	Load mass	1 kg
L	Cable length	0.6 m
G	Gravitational constant	9.8 m/s^2

The obtained transfer function of the system will be,

$$\frac{\text{ø}(s)}{u(s)} = \frac{-1}{0.15s^2 + 12.25}. \tag{24}$$

And the fractional order transfer function of the system will be,

$$\frac{\text{ø}(s)}{u(s)} = \frac{-1}{0.15s^{2\alpha} + 12.25}. \tag{25}$$

The integer order impulse response and fractional order impulse response of the system is plotted as,

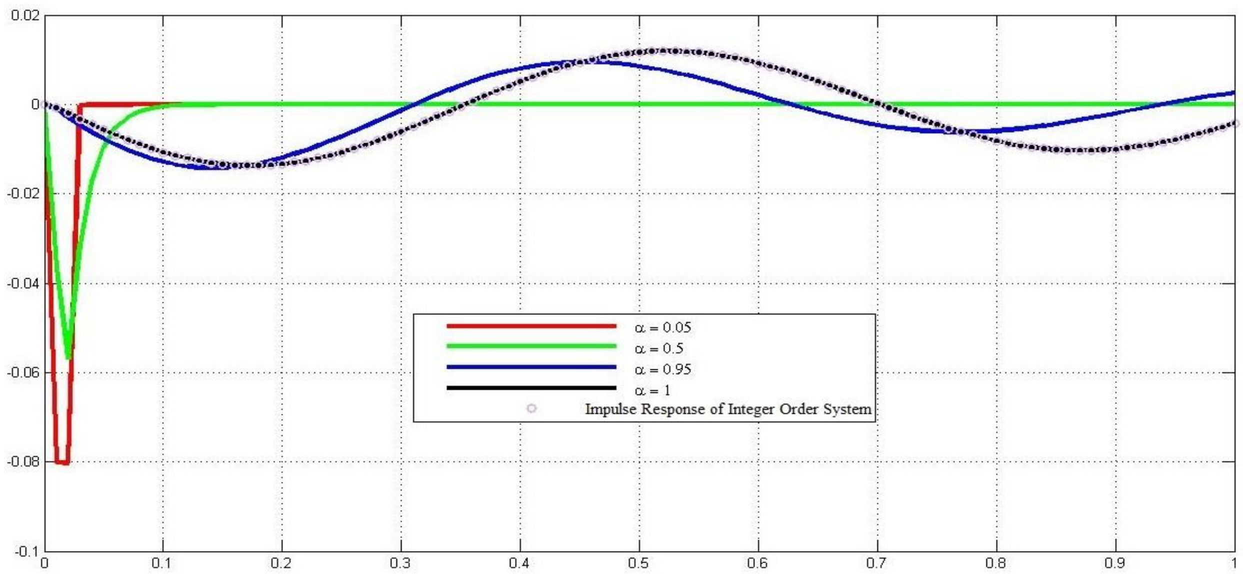

Fig.3. Fractional and integer order impulse response of the system

In Figure 3 the system response is shown for integer order system and for fractional order system. To understand the nature of the fractional order response we consider four values of the fractional order, $\alpha = 0.05, \alpha = 0.5, \alpha = 0.95$ and $\alpha = 1$. The initial rate of decrease of fractional order impulse response for $\alpha = 0.05$ is highest. As α is increased, the rate of decrease slows down gradually. The plot clearly shows that for the values of α tending to 1 the impulse response of the fractional order system approaches more and more towards that of the integer order system. For the value of $\alpha = 1$, both the responses overlap each other.

4 Hamiltonian equations of motion

The Fractional Lagrangian of equation (10) is written using fractional embedding technique used in [16] as:

$$Ł = \frac{1}{2}(M+m)\left[(D_{t+}^{\alpha}x)^2 + \left(D_{t-}^{\beta}x\right)^2\right] + \frac{1}{2}(ml^2)\left[(D_{t+}^{\alpha}\theta)^2 + \left(D_{t-}^{\beta}\theta\right)^2\right]$$
$$+(D_{t+}^{\alpha}x + D_{t-}^{\beta}x)(D_{t+}^{\alpha}\theta + D_{t-}^{\beta}\theta)ml\,cos\theta - Mgl - mgl + mgl\,cos\theta. \quad (26)$$

We define the canonical momentum [17] p_1, p_2, p_3 and p_4 as follows:

$$p_1 = \frac{\partial Ł}{\partial D_{t-}^{\beta}x} = (M+m)D_{t-}^{\beta}x + (D_{t+}^{\alpha}\theta + D_{t-}^{\beta}\theta)\,ml\,cos\theta, \quad (27)$$

$$p_2 = \frac{\partial Ł}{\partial D_{t+}^{\alpha}x} = (M+m)D_{t+}^{\alpha}x + (D_{t+}^{\alpha}\theta + D_{t-}^{\beta}\theta)\,ml\,cos\theta, \quad (28)$$

$$p_3 = \frac{\partial Ł}{\partial D_{t-}^{\beta}\theta} = ml^2 D_{t-}^{\beta}\theta + (D_{t+}^{\alpha}x + D_{t-}^{\beta}x)\,ml\,cos\theta, \quad (29)$$

$$p_4 = \frac{\partial Ł}{\partial D_{t+}^{\alpha}\theta} = ml^2 D_{t+}^{\alpha}\theta + (D_{t+}^{\alpha}x + D_{t-}^{\beta}x)\,ml\,cos\theta. \quad (30)$$

Therefore, the fractional canonical Hamiltonian will be:

$$H = p_1 D_{t-}^{\beta}x + p_2 D_{t+}^{\alpha}x + p_3 D_{t-}^{\beta}\theta + p_4 D_{t+}^{\alpha}\theta - Ł. \quad (31)$$

Taking the total differential of above equation we get,

$$\mathrm{dH} = \mathrm{dp_1 D_{t-}^{\beta}x + dp_2 D_{t+}^{\alpha}x + dp_3 D_{t-}^{\beta}\theta + dp_4 D_{t+}^{\alpha}\theta} - \left[\frac{\partial Ł}{\partial x}\mathrm{dx} + \frac{\partial Ł}{\partial \theta}\mathrm{d\theta}\right] - \frac{\partial Ł}{\partial t}\mathrm{dt}. \quad (32)$$

Consider the fractional Euler-Lagrangian equations [17] as:

$$\frac{\partial Ł}{\partial x} + D_{t+}^{\alpha}\left(\frac{\partial Ł}{\partial D_{t-}^{\beta}x}\right) + D_{t-}^{\beta}\left(\frac{\partial Ł}{\partial D_{t+}^{\alpha}x}\right) = u, \quad (33)$$

$$\frac{\partial Ł}{\partial \theta} + D_{t+}^{\alpha}\left(\frac{\partial Ł}{\partial D_{t-}^{\beta}\theta}\right) + D_{t-}^{\beta}\left(\frac{\partial Ł}{\partial D_{t+}^{\alpha}\theta}\right) = 0. \quad (34)$$

Therefore, we can write equation (31) using above two equations as follows:

$$\mathrm{dH} = \mathrm{dp_1 D_{t-}^{\beta}x + dp_2 D_{t+}^{\alpha}x + dp_3 D_{t-}^{\beta}\theta + dp_4 D_{t+}^{\alpha}\theta}$$
$$+\left[D_{t+}^{\alpha}\left(\frac{\partial Ł}{\partial D_{t-}^{\beta}x}\right) + D_{t-}^{\beta}\left(\frac{\partial Ł}{\partial D_{t+}^{\alpha}x}\right) - u\right]\mathrm{dx}$$
$$+\left[D_{t+}^{\alpha}\left(\frac{\partial Ł}{\partial D_{t-}^{\beta}\theta}\right) + D_{t-}^{\beta}\left(\frac{\partial Ł}{\partial D_{t+}^{\alpha}\theta}\right)\right]\mathrm{d\theta} - \frac{\partial Ł}{\partial t}\mathrm{dt}. \quad (35)$$

It can be observed that the Hamiltonian is the function of,

$$H = H(t, p_1, p_2, p_3, p_4, x, \theta). \tag{36}$$

Then total differentiation of equation (36) will be:

$$dH = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial p_1}p_1 + \frac{\partial H}{\partial p_2}p_2 + \frac{\partial H}{\partial p_3}p_3 + \frac{\partial H}{\partial p_4}p_4 + \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial \theta}d\theta. \tag{37}$$

Finally comparing equation (35) and (37), the Hamiltonian equations can be written as follows:

$$\frac{\partial H}{\partial t} = -\frac{\partial Ł}{\partial t}, \tag{38}$$

$$\frac{\partial H}{\partial p_1} = D_{t-}^{\beta}x, \tag{39}$$

$$\frac{\partial H}{\partial p_2} = D_{t+}^{\alpha}x, \tag{40}$$

$$\frac{\partial H}{\partial p_3} = D_{t-}^{\beta}\theta, \tag{41}$$

$$\frac{\partial H}{\partial p_4} = D_{t+}^{\alpha}\theta, \tag{42}$$

$$\frac{\partial H}{\partial x} = D_{t+}^{\alpha}\left(\frac{\partial Ł}{\partial D_{t-}^{\beta}x}\right) + D_{t-}^{\beta}\left(\frac{\partial Ł}{\partial D_{t+}^{\alpha}x}\right) - u, \tag{43}$$

$$\frac{\partial H}{\partial \theta} = D_{t+}^{\alpha}\left(\frac{\partial Ł}{\partial D_{t-}^{\beta}\theta}\right) + D_{t-}^{\beta}\left(\frac{\partial Ł}{\partial D_{t+}^{\alpha}\theta}\right). \tag{44}$$

The above equations are known as Hamiltonian equations of motion corresponding to the overhead crane system.

5 Conclusions

Modeling in Control Systems is a matter of judgment. This judgment is developed by developing models and learning from other models. In this paper we have modeled the overhead crane system into its fractional equivalent model and the Hamiltonian equations of motion for the considered system. Fractional modeling will be very useful in the study and/or analysis of the non-integer behavior of any mechanical system.

References

[1] C.A. Monje, On fractional PI λ controllers: some tuning rules for robustness to plant uncertainties, *Nonlinear Dyn.* **38.1-4**, 369-381 (2004).

[2] A. P. Singh, S.K. Faruk , M.S. Navdeep and P. Srivastava., PIαDβ controller design for underactuated mechanical systems, In Control Automation Robotics & Visio;n (ICARCV), 2012, 12th International Conference. *IEEE*, 1654-1658 (2012).

[3] S. Pritesh, S. D. Agashe and A. P. Singh, Fractional order modelling using state space theory, *Int. J. Eng. Techn.* **5.3**, 2891-2894 (2013).

[4] I. Podlubny, Fractional differential equations, Academic Press, San Diego, CA, 1999.

[5] P.S.V. Nataraj, Lecture notes on Fractional calculus and fractional differential equations with SCILAB, Systems and Control Engineering Indian Institute of Technology Bombay, 2010.

[6] K. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, *John Wiley and Sons*, New York, 1993.

[7] K. Miller and B. Ross, Fractional Green's functions, *Indian J. Pure Appl. Math.* **22.9**, 763-767 (1991).

[8] M. D. Ortigueira, Introduction to fractional linear systems. Part 1:Continuous-time case, *IEEE Proc.Vision, Image Sign.Process.* **147(1)**, 62-70 (2000).

[9] G. B. Bohannan, Analog fractional order controller in a temperature control application, *Fractional Differentiation and*

its Applications **2(1)**, 40-45 (2006).

[10] J. P. Clerc et al., AC response of fractal networks, *J. Phys. Lett.* **45.19**, 913-924 (1984).

[11] P. Srivastava et al., Observer design for non-linear systems, *Int. J. Appl. Eng. Res.* **8.8**, 957-967 (2013).

[12] I. Podlubny, Fractional-order systems and PI/sup/spl lambda//D/sup/spl mu//-controllers, Automatic Control, *IEEE Transactions*. **44.1**, 208-214 (1999).

[13] R. Olfati-Saber, Nonlinear control and reduction of underactuated systems with symmetry. III. Input coupling case, In Decision and Control, 2001. *Proceedings of the 40th IEEE Conference* **4**, 3778-3783 (2001).

[14] A. P. Singh, F. Kazi, N. M. Singh and V. Vyawahare, Fractional Order controller design for underactuated mechanical systems, *The 5th IFAC Symposium on Fractional Differentiation and its Applications-FDA (*2012).

[15] T. Hartley and C. F. Lorenzo, Fractional-order system identification based on continuous order-distributions, *Signal Process.* **83.11**, 2287-2300 (2003).

[16] S. I. Muslih and D. Baleanu, Formulation of Hamiltonian equations for fractional variational problems, *Czech. J. Phys.* **55(6)**, 633-642 (2005).

[17] D. Baleanu and O. P. Agrawal, Fractional Hamilton formalism within Caputo's derivative, *Czech. J. Phys.* **56.10-11,** 1087-1092 (2006).

[18] M. I. Solihin, Sensorless anti-swing control strategy for automatic gantry crane system: Soft sensor approach, In Intelligent and Advanced Systems, *ICIAS IEEE,* 992-996 (2007).

[19] F. Asa, Modal synthesis when modeling damping by use of fractional derivatives, *AIAA Journal* **34.5**, 1051-1058 (1996).

[20] H. Sheng et al., A physical experimental study of variable-order fractional integrator and differentiator, *Eur. Phys. Journ. S.T.***193.1**, 93-104 (2011).

[21] H.Wang and M. Junhai, Chaos control and synchronization of a fractional-order autonomous system, *WSEAS Trans. Math.***11.8**, 700-711 (2012).

[22] C. Li and Y. Tong, Adaptive control and synchronization of a fractional-order chaotic system, *Pramana* **80.4**, 583-592 (2013).

[23] H. Arpacia and Ö. F. Özgüvenb, ANFIS & PIλDμ controller design and comparison for overhead cranes, *Indian J. Eng. Mat. Sci.* **18**, 191-203 (2011).