Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/040303

Translations and Multiplications of Cubic Subalgebras and Cubic Ideals of BCK/BCI-algebras

A. K. Dutta[1], *S. R. Barbhuiya*[2,*] *and K. Dutta Choudhury*[3]

[1] Department of Mathematics, D.H.S.K. College, Dibrugarh, Assam- 786001, India.
[2] Department of Mathematics, Srikishan Sarda College, Hailakandi-788151, Assam, India.
[3] Department of Mathematics, Assam University, Silchar, Assam-, India.

Received: 27 Sep. 2016, Revised: 26 May 2017, Accepted: 29 May 2017
Published online: 1 Sep. 2017

Abstract: In this paper, we introduced the concept of translation and multiplication of cubic subalgebras and cubic ideals of BCK/BCI-algebras and investigated some of their basic properties. The notion of cubic extension of cubic subalgebras and cubic extension cubic ideals are introduced and several related properties are investigated.

Keywords: BCK/BCI-algebra, Interval-valued fuzzy set, Cubic set, Translation, Multiplication, Extension.

1 Introduction

After the introduction of fuzzy set by Zadeh ([19]), the theory of fuzzy sets, a newly developing subject in mathematics, which makes inroads into different disciplines of mathematics. Among various branches of pure mathematics, algebra is one of the subjects where the notion of fuzzy set is applied. Rosenfeld ([15]) first fuzzified the algebraic concept of ‘Group’ in to fuzzy subgroup and opened up a new insight in the field of pure mathematics. Since then, a host of mathematicians have been engrossed in extending the concepts and results of abstract algebra to boarder framework of fuzzy setting. Imai and Iseki ([3]) introduced BCK-algebra as a generalization of notion of the concept of set theoretic difference and propositional calculus and in the same year Iseki ([4]) introduced the notion of BCI-algebra. Xi([17]) applied the concept of fuzzy set to BCK-algebra and discussed some properties and also introduced fuzzy subalgebra and fuzzy ideals in BCK-algebra. After that a huge number of literature has been produced on the theory of fuzzy BCK/BCI-algebras. In particular, emphasis seems to have been put on the subalgebra and ideal theory of fuzzy BCK/BCI-algebras.

In ([20]) Zadeh generalized the concept of fuzzy set by an interval-valued fuzzy set. In traditional fuzzy logic, the expert’s degree of certainty in different statements represent by numbers from the interval $[0;1]$. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Saeid ([16]) defined interval valued fuzzy BG-algebras. In ([2,14,6]) different authors applied interval-valued fuzzy sets in various algebraic structures.

Using a fuzzy set and an interval-valued fuzzy set, Jun et al. ([8]) introduced a new type of fuzzy sets called a cubic set, and investigated several properties. Jun et al. applied the cubic set theory to BCK/BCI-algebras ([9]). Subsequently the theory of cubic sets attracted by several mathematicians. Jun et al. ([8,9,10,11,12]) studied the theory of cubic sets in different algebraic structures. Yaqoob and others investigated some properties of cubic KU-ideals of KU-algebras ([18]). Lee et al. ([13]) and Jun([7]) discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras and ideals in BCK/BCI-algebras. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. In ([1]) Barbhuiya introduced interval valued fuzzy translation and interval valued multiplication. In this paper, we introduced the concept of translation, multiplication and extension of cubic subalgebras and cubic ideals of BCK/BCI-algebras.

* Corresponding author e-mail: saidurbarbhuiya@gmail.com

2 Preliminaries

In this section, we will recall some concepts related to BCK/BCI-algebra, interval-valued fuzzy sets and cubic fuzzy sets.

Definition 1([6,7]) *An algebra $(X,*,0)$ of type (2, 0) is called a BCK-algebra if it satisfies the following axioms:*
(i) $((x*y)*(x*z))*(z*y)=0$
(ii) $(x*(x*y))*y=0$
(iii) $x*x=0$
(iv) $0*x=0$
(v) $x*y=0$ *and* $y*x=0 \Rightarrow x=y$ *for all* $x,y,z\in X$
We can define a partial ordering "$\leq$" on X by $x\leq y$ iff $x*y=0$. A BCK-algebra X is said to be commutative if it satisfies the identity $x\wedge y=y\wedge x$ where $x\wedge y=y*(y*x)$ $\forall x,y\in X$. In a commutative BCK-algebra, it is known that $x\wedge y$ is the greatest lower bound of x and y.
In a BCK-algebra X, the following hold:
(i) $x*0=x$
(ii) $(x*y)*z=(x*z)*y$
(iii) $x*y\leq x$
(iv) $(x*y)*z\leq(x*z)*(y*z)$
(v) $x\leq y$ *implies* $x*z\leq y*z$ *and* $z*y\leq z*x$.

A BCK-algebra X is said to be associative if it satisfies the identity $(x*y)*z=x*(y*z)$ $\forall x,y,z\in X$. A non-empty subset S of a BCK-algebra X is called a subalgebra of X if x ∗ y ∈ S, for all x, y ∈ S. A nonempty subset I of a *BCK*-algebra X is called an ideal ([5,7]) of X if (I_1) $0\in I$, (I_2) $x*y\in I$ and $y\in I\Rightarrow x\in I$, for all $x,y\in X$.

Definition 2*A fuzzy subset μ of a BCK-algebra X is called a fuzzy subalgebra of X if $\mu(x*y)\geq min\{\mu(x),\mu(y)\}$, for all x, y ∈ X.*

Definition 3([5,7]) *A fuzzy set μ in BCK-algebra X is called a fuzzy ideal of X if it satisfies the following axioms:*
(i) $\mu(0)\geq\mu(x)$
(ii) $\mu(x)\geq min\{\mu(x*y),\mu(y)\}$, *for all* $x,y\in X$.

The notion of interval-valued fuzzy set was introduced by Zadeh([20]). To consider the notion of interval-valued fuzzy sets, we need the following definitions. Let I be a closed unit interval, i.e., $I=[0,\ 1]$. An interval number on I denoted by $\tilde{a}$, is defined as the closed sub interval of I, where $\tilde{a}=[\underline{a},\overline{a}]$,satisfying $0\leq\underline{a}\leq\overline{a}\leq 1$. Let $D[0,\ 1]$ denote the set of all such interval numbers on I and also denote the interval numbers $[0,\ 0]$ and $[1,\ 1]$ by $\tilde{0}$ and $\tilde{1}$ respectively.
Now consider two intervals $\tilde{a_1}=[\underline{a_1},\overline{a_1}],\tilde{a_2}=[\underline{a_2},\overline{a_2}]\in D[0,\ 1]$ then we define refine minimum as rmin $rmin(\tilde{a_1},\tilde{a_2})=[min(\underline{a_1},\underline{a_2}),min(\overline{a_1},\overline{a_2})]$ and refine maximum as rmax $rmax(\tilde{a_1},\tilde{a_2})=[max(\underline{a_1},\underline{a_2}),max(\overline{a_1},\overline{a_2})]$ generally if $\tilde{a_i}=[\underline{a_i},\overline{a_i}],\tilde{b_i}=[\underline{b_i},\overline{b_i}]\in D[0,\ 1]$ for i= 1,2,3,...then we define $rmax(\tilde{a_i},\tilde{b_i})=[max(\underline{a_i},\underline{b_i}),max(\overline{a_i},\overline{b_i})]$ and $rmin(\tilde{a_i},\tilde{b_i})=[min(\underline{a_i},\underline{b_i}),min(\overline{a_i},\overline{b_i})]$ and $rinf_i(\tilde{a_i})=[\wedge_i\underline{a_i},\wedge_i\overline{a_i}]$ and $rsup_i(\tilde{a_i})=[\vee_i\underline{a_i},\vee_i\overline{a_i}]$ $(D[0,\ 1],\preceq)$ is a complete lattice with $\wedge=rmin$, $\vee=rmax$, $\tilde{0}=[0,\ 0]$ and $\tilde{1}=[1,\ 1]$ being the least and the greatest element respectively.

Let $\tilde{a_1}=[\underline{a_1},\ \overline{a_1}]$ and $\tilde{a_2}=[\underline{a_2},\ \overline{a_2}]\in D[0,\ 1]$. Define on D[0, 1] the relations $\preceq,=,\prec,+,.$ by

1.$\tilde{a_1}\preceq\tilde{a_2}\Leftrightarrow\underline{a_1}\leq\underline{a_2}$ and $\overline{a_1}\leq\overline{a_2}$
2.$\tilde{a_1}=\tilde{a_2}\Leftrightarrow\underline{a_1}=\underline{a_2}$ and $\overline{a_1}=\overline{a_2}$
3.$\tilde{a_1}\prec\tilde{a_2}\Leftrightarrow\underline{a_1}<\underline{a_2}$ and $\overline{a_1}<\overline{a_2}$
4.$\tilde{a_1}+\tilde{a_2}\Leftrightarrow[\underline{a_1}+\underline{a_2},\ \overline{a_1}+\overline{a_2}]$
5.$\tilde{a_1}.\tilde{a_2}\Leftrightarrow[min(\underline{a_1}\underline{a_2},\underline{a_1}\overline{a_2},\overline{a_1}\underline{a_2},\overline{a_1}\overline{a_2}),max(\underline{a_1}\underline{a_2},\underline{a_1}\overline{a_2},\overline{a_1}\underline{a_2},\overline{a_1}\overline{a_2})]=[\underline{a_1}\underline{a_2},\overline{a_1}\overline{a_2}]$
6.$k\tilde{a}=[k\underline{a},k\overline{a}]$ where $0\leq k\leq 1$.

Definition 4*An interval-valued fuzzy set defined on a non empty set X is an objects having the form $\tilde{\mu}=\{(x,[\underline{\mu}(x),\overline{\mu}(x)])|x\in X\}$, where $\underline{\mu}$ and $\overline{\mu}$ are two fuzzy sets in X such that $\underline{\mu}(x)\leq\overline{\mu}(x)$ for all $x\in X$. Let $\tilde{\mu}(x)=[\underline{\mu}(x),\overline{\mu}(x)],\forall x\in X$ is called degree of membership of an element x to $\tilde{\mu}$, in which $\underline{\mu}(x)$ and $\overline{\mu}(x)$ are refereed to as the lower and upper degrees respectively of membership x to $\tilde{\mu}$. Then $\tilde{\mu}(x)\in D[0\ 1],\forall x\in X$.*
If $\tilde{\mu}$ and $\tilde{\nu}$ be two interval-valued fuzzy sets in X, then we define

–$\tilde{\mu}\subset\tilde{\nu}\Leftrightarrow$ *for all* $x\in X,\underline{\mu}(x)\leq\underline{\nu}(x)$ *and* $\overline{\mu}(x)\leq\overline{\nu}(x)$.
–$\tilde{\mu}=\tilde{\nu}\Leftrightarrow$ *for all* $x\in X,\underline{\mu}(x)=\underline{\nu}(x)$ *and* $\overline{\mu}(x)=\overline{\nu}(x)$.
–$(\tilde{\mu}\tilde{\cup}\tilde{\nu})(x)=\hat{\mu}(x)\vee\hat{\nu}(x)=[max\{\underline{\mu}(x),\underline{\nu}(x)\},max\{\overline{\mu}(x),\overline{\nu}(x)\}]$.
–$(\tilde{\mu}\tilde{\cap}\tilde{\nu})(x)=\hat{\mu}(x)\wedge\hat{\nu}(x)=[min\{\underline{\mu}(x),\underline{\nu}(x)\},min\{\overline{\mu}(x),\overline{\nu}(x)\}]$.
–$(\tilde{\mu}\times\tilde{\nu})(x,y)=\hat{\mu}(x)\wedge\hat{\nu}(y)=[min\{\underline{\mu}(x),\underline{\nu}(y)\},min\{\overline{\mu}(x),\overline{\nu}(y)\}]$.
–$\tilde{\mu}^c(x)=[1-\overline{\mu}(x),1-\underline{\mu}(x)]$.

Definition 5*Let $\tilde{\mu}$ be an interval-valued fuzzy set in X. Then for every $[0,\ 0]\prec\tilde{t}\preceq[1,\ 1]$, the crisp set $\tilde{\mu}_t=\{x\in X|\tilde{\mu}(x)\succeq\tilde{t}\}$ is called the level subset of $\tilde{\mu}$.*

Definition 6*An interval-valued fuzzy set $\tilde{\mu}$ in BCK/BCI-algebra X is called an interval-valued fuzzy subalgebra of X if $\tilde{\mu}(x*y)\succeq rmin\{\tilde{\mu}(x),\tilde{\mu}(y)\}$, for all $x,y\in X$.*

Definition 7*An interval-valued fuzzy set $\tilde{\mu}$ in BCK/BCI-algebra X is called an interval-valued fuzzy ideal of X if*
(i) $\tilde{\mu}(0)\succeq\tilde{\mu}(x)$
(ii) $\tilde{\mu}(x)\succeq rmin\{\tilde{\mu}(x*y),\tilde{\mu}(y)\}$, *for all* $x,y\in X$.

Combining the notion of interval-valued fuzzy set and fuzzy set Jun's introduced cubic set in ([8]) and is defined as follows:

Definition 8 *Let X be a nonempty set. A cubic set A in X is a structure*

$$A = \{< x, \tilde{\mu}_A(x), \nu_A(x) > | x \in X\},$$

which is briefly denoted by $A = (\tilde{\mu}_A, \nu_A)$ *where* $\tilde{\mu}_A = [\underline{\mu}_A, \overline{\mu}_A]$ *is an IVFS in X and* ν_A *is a fuzzy set in X and it is written as* $A =< \tilde{\mu}_A, \nu_A >$.
For two cubic sets $A =< \tilde{\mu}_A, \nu_A >$ *and* $B =< \tilde{\mu}_B, \nu_B >$ *in X, we define*

– $A \sqsubseteq B \Leftrightarrow \tilde{\mu}_A \preceq \tilde{\mu}_B$ *and* $\nu_A \geq \nu_B$
– $A = B \Leftrightarrow$ *for all* $x \in X$, $\tilde{\mu}_A(x) = \tilde{\mu}_B(x)$ *and* $\nu_A(x) = \nu_B(x)$
– $A \sqcap B = \{< x | (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(x), (\nu_A \cup \nu_B)(x) > | x \in X\}$ *Where* $(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(x) = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ *and* $(\nu_A \cup \nu_B)(x) = max\{\nu_A(x), \nu_B(x)\}$
– $A \sqcup B = \{< x | (\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x), (\nu_A \cap \nu_B)(x) > | x \in X\}$ *Where* $(\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x) = rmax\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ *and* $(\nu_A \cap \nu_B)(x) = min\{\nu_A(x), \nu_B(x)\}$.

Definition 9 *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be cubic set in X , where X is a* BCK/BCI*-subalgebra, then the set A is cubic* BCK/BCI*-subalgebra over the binary operator* $*$ *if it satisfies the following conditions:*

(i) $\tilde{\mu}_A(x * y) \succeq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$
(ii) $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$, *for all* $x, y \in X$.

Example 1 *Consider BCK-algebra* $X = \{0, a, b, c, d\}$ *with the following cayley table.*

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	b	d	0

Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ *in X by*

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.6, 0.7] & [0.15, 0.25] & [0.4, 0.5] & [0.3, 0.4] & [0.5, 0.65] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.2 & 0.7 & 0.4 & 0.6 & 0.3 \end{pmatrix}$$

Then it is easy to verify that $A = (\tilde{\mu}_A, \nu_A)$ *is a cubic subalgebra of X.*

Definition 10 *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be cubic set in X , where X is a* BCK/BCI*-subalgebra, then the set A is called cubic ideal of* BCK/BCI*-algebra X over the binary operator* $'*'$ *if it satisfies the following conditions:*

(i) $\tilde{\mu}_A(0) \succeq \tilde{\mu}_A(x)$
(ii) $\nu_A(0) \leq \nu_A(x)$
(iii) $\tilde{\mu}_A(x) \succeq rmin\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$
(iv) $\nu_A(x) \leq max\{\nu_A(x * y), \nu_A(y)\}$ *for all* $x, y \in X$.

Example 2 *Consider BCK-algebra X as in Example 1. Define a cubic set* $A = (\tilde{\mu}_A, \nu_A)$ *in X by*

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.7, 0.8] & [0.4, 0.6] & [0.1, 0.3] & [0.4, 0.6] & [0.1, 0.3] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.7 & 0.3 & 0.2 & 0.3 & 0.2 \end{pmatrix}$$

Then it is easy to verify that $A = (\tilde{\mu}_A, \nu_A)$ *is a cubic ideal of X.*

3 Translation and Multiplication of Cubic Subalgebras and Ideals of BCK/BCI-Algebras

In what follows, let X denotes a BCK/BCI-algebra and for any cubic set $A = (\tilde{\mu}_A, \nu_A)$ of X, let $\tau = inf\{\nu_A(x) | x \in X\}$ and $\tilde{T} = (\underline{T}, \overline{T})$ $(\underline{T} \leq \overline{T}.)$ where $\overline{T} = 1 - sup\{\overline{\mu_A}(x) \; | x \in X\}$.

Definition 11 *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be cubic subset of X and* $0 < \overline{\alpha} \leq \overline{T}$ *where* $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in D[0, \overline{T}]$ *and* $\beta \in [0, \tau]$*. An object of the form* $A^T_{(\tilde{\alpha}, \beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ *is called an cubic* $(\tilde{\alpha}, \beta)$*-translation of A if it satisfies* $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha}$ *and* $(\nu_A)^T_\beta(x) = \nu_A(x) - \beta$ *for all* $x \in X$.

Example 3 *Consider BCK-algebra X as in Example 1 and a cubic set* $A = (\tilde{\mu}_A, \nu_A)$ *in X by*

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.6, 0.7] & [0.15, 0.25] & [0.4, 0.5] & [0.3, 0.4] & [0.5, 0.65] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.2 & 0.7 & 0.4 & 0.6 & 0.3 \end{pmatrix}$$

Here $\overline{T} = 1 - sup\{\overline{\mu}_A(x) \; | x \in X.\} = 1 - 0.7 = 0.3$ *and* $\tau = inf\{\nu_A(x) | x \in X\} = 0.2$*. Let* $\tilde{\alpha} = [0.15, \; 0.25] \in D[0, \overline{T}]$ *and* $\beta = 0.1 \in [0, \tau]$*. Then the* $(\tilde{\alpha}, \beta)$*-translation* $((\tilde{\mu}_A)^T_{[0.15, \; 0.25]}, (\nu_A)^T_{0.1})$ *of cubic set* $A = (\tilde{\mu}_A, \nu_A)$ *is given by*

$$(\tilde{\mu}_A)^T_{[0.15, \, 0.25]} = \begin{pmatrix} 0 & a & b & c & d \\ [0.75, 0.95] & [0.3, 0.5] & [0.55, 0.75] & [0.45, 0.65] & [0.65, 0.9] \end{pmatrix}$$

and

$$(\nu_A)^T_{0.1} = \begin{pmatrix} 0 & a & b & c & d \\ 0.1 & 0.6 & 0.3 & 0.5 & 0.2 \end{pmatrix}$$

Definition 12 *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be a cubic fuzzy subset of X and* $\gamma \in (0, 1]$*. An object having the form* $A^M_\gamma = ((\tilde{\mu}_A)^M_\gamma, (\nu_A)^M_\gamma)$ *is called a cubic* γ*-multiplication of A if* $(\tilde{\mu}_A)^M_\gamma(x) = \tilde{\mu}_A.\gamma$ *and* $(\nu_A)^M_\gamma = \nu_A.\gamma$*, for all* $x \in X$.

Example 4*Consider a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X as in Example 1, Let $\gamma = 0.6 \in (0,1]$. Then the γ-multiplication of $((\tilde{\mu}_A)^M_{0.6}, (\nu_A)^M_{0.6})$ cubic set $A = (\tilde{\mu}_A, \nu_A)$ is given by*

$$(\tilde{\mu}_A)^M_{0.6} = \begin{pmatrix} 0 & a & b & c & d \\ [0.36, 0.42] & [0.09, 0.15] & [0.24, 0.30] & [0.18, 0.24] & [0.30, 0.39] \end{pmatrix}$$

and

$$(\nu_A)^M_{0.6} = \begin{pmatrix} 0 & a & b & c & d \\ 0.12 & 0.42 & 0.24 & 0.36 & 0.18 \end{pmatrix}$$

Definition 13*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of X and $0 \le \overline{\alpha} \le \overline{T}$ where $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in D[0, \overline{T}]$, $\beta \in [0, \tau]$ and $\gamma \in (0,1]$. A mapping $A^{MT}_{(\tilde{\alpha},\beta;\gamma)} = ((\tilde{\mu}_A)^{MT}_{\tilde{\alpha};\gamma}, (\nu_A)^{MT}_{\beta;\gamma}) : X \longrightarrow [D[0,\ 1], [0,1]]$ is said to be a cubic magnified $(\tilde{\alpha}, \beta; \gamma)$ translation of $A = (\tilde{\mu}_A, \nu_A)$ if it satisfies $(\tilde{\mu}_A)^{MT}_{\tilde{\alpha};\gamma}(x) = \gamma \tilde{\mu}_A(x) + \tilde{\alpha}$ and $(\nu_A)^{MT}_{\beta;\gamma}(x) = \gamma \nu_A(x) - \gamma\beta$ for all $x \in X$.*

Example 5*Consider a cubic fuzzy set $A = (\tilde{\mu}_A, \nu_A)$ in X as in Example 1, For this cubic set $\overline{T} = 1 - sup\{\overline{\mu}_A(x)\ |x \in X.\} = 1 - 0.7 = 0.3$ and $\tau = inf\{\nu_A(x)|x \in X\} = 0.2$. Let $\tilde{\alpha} = [0.1,\ 0.2] \in D[0, \overline{T}]$, $\beta = 0.1 \in [0, \tau]$ and $\gamma = 0.5 \in (0,1]$. Then the cubic magnified $(\tilde{\alpha}, \beta; \gamma)$ translation $A^{MT}_{([0.1,\ 0.2], 0.1; 0.5)}$ of cubic set $A = (\tilde{\mu}_A, \nu_A)$ is given by*

$$(\tilde{\mu}_A)^{MT}_{([0.1,\ 0.2];0.5)} = \begin{pmatrix} 0 & a & b & c & d \\ [0.4, 0.55] & [0.175, 0.325] & [0.3, 0.45] & [0.25, 0.40] & [0.35, 0.525] \end{pmatrix}$$

and

$$(\nu_A)^{MT}_{(0.1;0.5)} = \begin{pmatrix} 0 & a & b & c & d \\ 0.05 & 0.30 & 0.15 & 0.25 & 0.10 \end{pmatrix}$$

4 Translations of Cubic Subalgebras

Theorem 1*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X and let $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$. Then the cubic $(\tilde{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X.*

*Proof.*Here $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ and let $x, y \in X$. Then

$$\begin{aligned}
(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x * y) &= \tilde{\mu_A}(x * y) + \tilde{\alpha} \\
&\succeq rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\} + \tilde{\alpha} \\
&= rmin\{\tilde{\mu_A}(x) + \tilde{\alpha}, \tilde{\mu_A}(y) + \tilde{\alpha}\} \\
&= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
(\nu_A)^T_{\beta}(x * y) &= \nu_A(x * y) - \beta \\
&\le max\{\nu_A(x), \nu_A(y)\} - \beta \\
&= max\{\nu_A(x) - \beta, \nu_A(y) - \beta\} \\
&= max\{(\nu_A)^T_{\beta}(x), (\nu_A)^T_{\beta}(y)\} \quad \text{for all} \quad x, y \in X.
\end{aligned}$$

Therefore the cubic $(\tilde{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X.

Theorem 2*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of X such that the cubic $(\hat{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X for some $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$ Then $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X.*

*Proof.*Here $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ is a cubic subalgebra of X and let $x, y \in X$, we have

$$\begin{aligned}
\tilde{\mu_A}(x * y) + \tilde{\alpha} &= (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x * y) \\
&\succeq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
&= rmin\{\tilde{\mu_A}(x) + \tilde{\alpha}, \tilde{\mu_A}(y) + \tilde{\alpha}\} \\
&= rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\} + \tilde{\alpha} \\
\nu_A(x * y) - \beta &= (\nu_A)^T_{\beta}(x * y) \\
&\le max\{(\nu_A)^T_{\beta}(x), (\nu_A)^T_{\beta}(y)\} \\
&= max\{\nu_A(x) - \beta, \nu_A(y) - \beta\} \\
&= max\{\nu_A(x), \nu_A(y)\} - \beta \quad \text{for all} \quad x, y \in X.
\end{aligned}$$

Which implies $\tilde{\mu_A}(x * y) \succeq rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(x)\}$ and $\nu_A(x * y) \le max\{\nu_A(x), \nu_A(x)\}$. Hence $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X.

Theorem 3*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of X and $0 \le \overline{\alpha} \le \overline{T}$ where $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in D[0, \overline{T}]$, $\beta \in [0, \tau]$ and $\gamma \in (0,1]$. Then $A = (\tilde{\mu}_A, \nu_A)$ is cubic subalgebra of X iff $A^{MT}_{(\tilde{\alpha},\beta;\gamma)}$is cubic subalgebra of X.*

*Proof.*It follows from Theorem 1 and Theorem 2.

Theorem 4*Intersection and union of any two $(\tilde{\alpha}, \beta)$-translations of a cubic subalgebra of X is also a cubic subalgebra of X.*

*Proof.*Let $A^T_{(\tilde{\alpha},\beta)}$ and $A^T_{(\tilde{\gamma},\delta)}$ be two fuzzy translatons a cubic subalgebra $A = (\tilde{\mu}_A, \nu_A)$ of X. Where $\tilde{\alpha}, \tilde{\gamma} \in D[0, \overline{T}], \beta, \delta \in [0, \tau]$ Assume that $\tilde{\alpha} \preceq \tilde{\gamma}$ and $\beta \ge \delta$ By Theorem 1 $A^T_{(\tilde{\alpha},\beta)}$ and $A^T_{(\tilde{\gamma},\delta)}$ are cubic subalgebras of X. Now

$$\begin{aligned}
((\tilde{\mu}_A)^T_{\tilde{\alpha}} \tilde{\cup} (\tilde{\mu}_A)^T_{\tilde{\gamma}})(x) &= rmax\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\gamma}}(x)\} \\
&= rmax\{\tilde{\mu_A}(x) + \tilde{\alpha}, \tilde{\mu_A}(x) + \tilde{\gamma}\} \\
&= rmax\{[\underline{\mu_A}(x) + \underline{\alpha}, \overline{\mu_A}(x) + \overline{\alpha}], [\underline{\mu_A}(x) + \underline{\gamma}, \overline{\mu_A}(x) + \overline{\gamma}]\} \\
&= [max\{\underline{\mu_A}(x) + \underline{\alpha}, \underline{\mu_A}(x) + \underline{\gamma}\}, max\{\overline{\mu_A}(x) + \overline{\alpha}, \overline{\mu_A}(x) + \overline{\gamma}\}] \\
&= [\underline{\mu_A}(x) + \underline{\gamma}, \overline{\mu_A}(x) + \overline{\gamma}] \\
&= \tilde{\mu}_A(x) + \tilde{\gamma} = (\tilde{\mu}_A)^T_{\tilde{\gamma}}(x). \\
((\nu_A)^T_{\beta} \cup (\nu_A)^T_{\delta})(x) &= min\{(\nu_A)^T_{\beta}(x), (\nu_A)^T_{\delta}(x)\} \\
&= min\{\nu_A(x) - \beta, \nu_A(x) - \delta\} \\
&= \nu_A(x) - \beta \\
&= (\nu_A)^T_{\beta}(x)
\end{aligned}$$

Also

$$\begin{aligned}
((\tilde{\mu}_A)^T_{\tilde{\alpha}} \tilde{\cap} (\tilde{\mu}_A)^T_{\tilde{\gamma}})(x) &= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\gamma}}(x)\} \\
&= rmin\{\tilde{\mu_A}(x)+\tilde{\alpha}, \tilde{\mu_A}(x)+\tilde{\gamma}\} \\
&= rmin\{[\underline{\mu_A}(x)+\underline{\alpha}, \overline{\mu_A}(x)+\overline{\alpha}], [\underline{\mu_A}(x)+\underline{\gamma}, \overline{\mu_A}(x)+\overline{\gamma}]\} \\
&= [min\{\underline{\mu_A}(x)+\underline{\alpha}, \underline{\mu_A}(x)+\underline{\gamma}\}, min\{\overline{\mu_A}(x)+\overline{\alpha}, \overline{\mu_A}(x)+\overline{\gamma}\}] \\
&= [\underline{\mu_A}(x)+\underline{\alpha}, \overline{\mu_A}(x)+\overline{\alpha}] \\
&= \tilde{\mu}_A(x)+\tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \\
((\nu_A)^T_{\beta} \cap (\nu_A)^T_{\delta})(x) &= max\{(\nu_A)^T_{\beta}(x), (\nu_A)^T_{\delta}(x)\} \\
&= max\{\nu_A(x)-\beta, \nu_A(x)-\delta\} \\
&= \nu_A(x)-\delta \\
&= (\nu_A)^T_{\delta}(x)
\end{aligned}$$

Definition 14 *For two cubic sets $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ of X. We define $A \sqsubseteq B \Longleftrightarrow \tilde{\mu}_A \preceq \tilde{\mu}_B, \nu_A \geq \nu_B$ for all $x \in X$. Then we say that B is an cubic extension of A.*

Definition 15 *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be two cubic subsets of X. Then, B is called a cubic S-extension(subalgebra-extension) of A if the following assertions are valid:*
(i) B is a cubic extension of A.
(ii) If A is a cubic subalgebra of X, then B is a cubic subalgebra of X.

Theorem 5 *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of X and $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$ Then the the cubic $(\tilde{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic S-extension of A.*

Proof. Here $A = (\tilde{\mu}_A, \nu_A)$ and $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$
Now, $\tilde{\mu}_A(x) + \tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$. Which implies $\tilde{\mu_A}(x) \preceq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$.
Again $\nu_A(x) - \beta = (\nu_A)^T_{\beta}(x)$. Which implies $\nu_A(x) \geq (\nu_A)^T_{\beta}(x)$.
Therefore $A \sqsubseteq A^T_{(\tilde{\alpha},\beta)}$. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic extension of A.
Since A is a cubic subalgebra of X. Therefore $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic S-extension of A.

Remark 1 *The converse of above theorem is not true as seen in the following Example.*

Example 6 *Consider BCK-algebra X as in Example 1. Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X by*

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.6, 0.7] & [0.15, 0.25] & [0.4, 0.5] & [0.3, 0.4] & [0.5, 0.65] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.2 & 0.7 & 0.4 & 0.6 & 0.3 \end{pmatrix}$$

Then it is easy to verify that A is cubic subalgebra of X. let $B = (\tilde{\mu}_B, \nu_B)$ be an cubic subset of X defined by

$$\tilde{\mu}_B = \begin{pmatrix} 0 & a & b & c & d \\ [0.62, 0.8] & [0.18, 0.3] & [0.48, 0.52] & [0.33, 0.43] & [0.54, 0.7] \end{pmatrix}$$

and

$$\nu_B = \begin{pmatrix} 0 & a & b & c & d \\ 0.11 & 0.61 & 0.38 & 0.56 & 0.24 \end{pmatrix}$$

Then $B = (\tilde{\mu}_B, \nu_B)$ is an cubic S-extension of A. But it is not the cubic $(\tilde{\alpha}, \beta)$-translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ of A for all $\tilde{\alpha} \in D[0, \overline{T}], \beta \in [0, \tau]$.

Theorem 6 *Intersection of any two cubic S-extensions of a cubic subalgebra of X is also a cubic S-extension of X.*

Proof. Let $S = (\tilde{\mu}_S, \nu_S)$ and $T = (\tilde{\mu}_T, \nu_T)$ be two cubic S-extensions of a cubic subalgebra $A = (\tilde{\mu}_A, \nu_A)$. Therefore we have $A \sqsubseteq S \Longleftrightarrow \tilde{\mu}_A \preceq \tilde{\mu}_S, \nu_A \geq \nu_S$ for all $x \in X$. and $A \sqsubseteq T \Longleftrightarrow \tilde{\mu}_A \preceq \tilde{\mu}_T, \nu_A \geq \nu_T$ for all $x \in X$. Since

$$(\tilde{\mu}_S \tilde{\cap} \tilde{\mu}_T)(x) = rmin\{\tilde{\mu}_S(x), \tilde{\mu}_T(x)\} = \begin{cases} \tilde{\mu}_T(x) & \text{if} \quad \tilde{\mu}_S(x) \succeq \tilde{\mu}_T(x) \\ \tilde{\mu}_S(x) & \text{if} \quad \tilde{\mu}_T(x) \succeq \tilde{\mu}_S(x) \end{cases}$$

$$(\nu_S \cap \nu_T)(x) = max\{\nu_S(x), \nu_T(x)\} = \begin{cases} \nu_S(x) & \text{if} \quad \nu_S(x) \geq \nu_T(x) \\ \nu_T(x) & \text{if} \quad \nu_T(x) \geq \nu_S(x) \end{cases}$$

Hence $\tilde{\mu}_A(x) \preceq (\tilde{\mu}_S \tilde{\cap} \tilde{\mu}_T)(x)$ and $\nu_A(x) \geq (\nu_S \cap \nu_T)(x)$ for all $x \in X$.
Therefore $S \sqcap T$ is a cubic S-extension of X.

Remark 2 *Union of two cubic S-extensions of a cubic subalgebra of X may not a cubic S-extension of X as shown in following example.*

Example 7 *Consider BCK-algebra $X = \{0, a, b, c, d\}$ with the following cayley table.*

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	d	a	0

Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X by

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.7, 0.8] & [0.5, 0.6] & [0.2, 0.4] & [0.5, 0.6] & [0.3, 0.5] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.2 & 0.5 & 0.7 & 0.5 & 0.6 \end{pmatrix}$$

Then it is easy to verify that A is cubic fuzzy subalgebra of X. Let $S = (\tilde{\mu}_S, \nu_S)$ and $T = (\tilde{\mu}_T, \nu_T)$ be two cubic subsets of X defined by

$$\tilde{\mu}_S = \begin{pmatrix} 0 & a & b & c & d \\ [0.75, 0.85] & [0.6, 0.7] & [0.25, 0.45] & [0.6, 0.7] & [0.7, 0.8] \end{pmatrix}$$

and

$$\nu_S = \begin{pmatrix} 0 & a & b & c & d \\ 0.15 & 0.4 & 0.6 & 0.4 & 0.3 \end{pmatrix}$$

$$\tilde{\mu}_T = \begin{pmatrix} 0 & a & b & c & d \\ [0.8, 0.9] & [0.65, 0.77] & [0.3, 0.4] & [0.62, 0.7] & [0.5, 0.6] \end{pmatrix}$$

and

$$\nu_T = \begin{pmatrix} 0 & a & b & c & d \\ 0.1 & 0.3 & 0.5 & 0.35 & 0.4 \end{pmatrix}$$

Now $S \sqcup T = (\tilde{\mu}_{(S\tilde{\cup}T)}, \nu_{(S\cup T)})$, *where*

$$\tilde{\mu}_{(S\tilde{\cup}T)} = \begin{pmatrix} 0 & a & b & c & d \\ [0.8,0.9] & [0.65,0.77] & [0.3,0.4] & [0.62,0.7] & [0.7,0.8] \end{pmatrix}$$

and

$$\nu_{(S\cup T)} = \begin{pmatrix} 0 & a & b & c & d \\ 0.1 & 0.3 & 0.5 & 0.35 & 0.3 \end{pmatrix}$$

Then S and T both are cubic S-extension of A. Also $S \sqcup T$ *is cubic extension of A but* $S \sqcup T$ *is not cubic S-extension of A. Since* $\tilde{\mu}_{(S\tilde{\cup}T)}(d * a) = \tilde{\mu}_{(S\tilde{\cup}T)}(c) = [0.62,0.7] \not\succeq [0.65,0.77] = rmin\{[0.65,0.77],[[0.7,0.8]\} = rmin\{\tilde{\mu}_{(S\tilde{\cup}T)}(a), \tilde{\mu}_{(S\tilde{\cup}T)}(d)\}$ *and* $\nu_{(S\cup T)}(d * a) = \nu_{(S\cup T)}(c) = 0.35 \not\leq 0.3 = max\{\nu_{(S\cup T)}(a), \nu_{(S\cup T)}(d)\}$.

Definition 16 *For a cubic subset* $A = (\tilde{\mu}_A, \nu_A)$ *of X. Let* $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$ *and with* $\tilde{t} \succeq \tilde{\alpha}$. *Let*

$$U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t}) = \{x \in X | \tilde{\mu}_A(x) \succeq \tilde{t} - \tilde{\alpha}\}$$
$$L_\beta(\nu_A;s) = \{x \in X | \nu_A(x) \leq s + \beta\}$$

Theorem 7 *If A is a cubic subalgebra of X, then* $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ *and* $L_\beta(\nu_A;s)$ *are subalgebras of X for all* $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$.

Proof. Let $x,y \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Therefore $\tilde{\mu}_A(x) \succeq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(y) \succeq \tilde{t} - \tilde{\alpha}$. Now

$$\begin{aligned} \tilde{\mu}_A(x*y) &\succeq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \\ &\succeq rmin\{\tilde{t} - \tilde{\alpha}, \tilde{t} - \tilde{\alpha}\} \\ &= \tilde{t} - \tilde{\alpha} \\ \Rightarrow \tilde{\mu}_A(x*y) &\succeq \tilde{t} - \tilde{\alpha}. \end{aligned}$$

Which implies $x*y \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$.
Let $x,y \in L_\beta(\nu_A;s)$. Therefore $\nu_A(x) \leq s+\beta$ and $\nu_A(y) \leq s+\beta$. Now

$$\begin{aligned} \nu_A(x*y) &\leq max\{\nu_A(x), \nu_A(y)\} \\ &\leq max\{s+\beta, s+\beta\} = s+\beta \\ \Rightarrow \nu_A(x*y) &\leq s+\beta. \end{aligned}$$

Which implies $x*y \in L_\beta(\nu_A;s)$.

Remark 3 *In above Theorem if A is not a cubic subalgebra of X, then* $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ *and* $L_\beta(\nu_A;s)$ *are not subalgebras of X as seen in the following example.*

Example 8 *Consider BCK-algebra X as in Example 1. Define a cubic set* $A = (\tilde{\mu}_A, \nu_A)$ *in X by*

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.2,0.4] & [0.5,0.7] & [0.6,0.75] & [0.7,0.8] & [0.3,0.5] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.6 & 0.3 & 0.4 & 0.5 & 0.8 \end{pmatrix}$$

Since $\tilde{\mu}_A(a * c) = \tilde{\mu}_A(0) = [0.2,0.4] \not\succeq [0.5,0.7] = rmin\{\tilde{\mu}_A(a), \tilde{\mu}_A(c)\}$ *and* $\nu_A(b * b) = \nu_A(0) = 0.6 \not\leq 0.4 = max\{\nu_A(b), \nu_A(b)\}$. *Therefore* $A = (\tilde{\mu}_A, \nu_A)$ *is not a cubic subalgebra of X. Let* $\tilde{\alpha} = [0.1,0.2] \in D[0,0.2], \beta = 0.25 \in [0,0.3]$ *and* $\tilde{t} = [0.5,0.8], s = 0.3,$. *Then* $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t}) = \{a,b,c\}$ *and* $L_\beta(\nu_A;s) = \{a,b,c\}$. *Since* $a * c = 0 \notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ *and* $a * c = 0 \notin L_\beta(\nu_A;s)$. *Therefore both* $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ *and* $L_\beta(\nu_A;s)$ *are not a subalgebras of X.*

Theorem 8 *For* $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$ *let* $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ *be the cubic* $(\hat{\alpha},\beta)$ *translation of* $A = (\tilde{\mu}_A, \nu_A)$. *Then, the following assertions are equivalent:*
(i) $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ *is a cubic subalgebra of X.*
(ii) $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ *and* $L_\beta(\nu_A;s)$ *are subalgebra of X for* $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$ *with* $\tilde{t} \succeq \tilde{\alpha}$.

Proof. Assume that $A^T_{(\tilde{\alpha},\beta)}$ is a cubic subalgebra of X. Then, $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy subalgebra of X and $(\nu_A)^T_\beta$ is a doubt fuzzy subalgebra of X. Let $x,y \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $\tilde{t} \in Im(\tilde{\mu}_A)$ with $\tilde{t} \succeq \hat{\alpha}$. Then $\tilde{\mu}_A(x) \succeq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(y) \succeq \tilde{t} - \tilde{\alpha}$. That is $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha} \succeq \tilde{t}$ and $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y) = \tilde{\mu}_A(y) + \tilde{\alpha} \succeq \tilde{t}$. Since $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy subalgebra of X, therefore, we have $\tilde{\mu}_A(x * y) + \tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x * y) \succeq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \succeq \tilde{t}$. which implies $\tilde{\mu}_A(x*y) \succeq \tilde{t} - \tilde{\alpha}$ so that $x*y \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$.

Again let $x,y \in X$ such that $x,y \in L_\beta(\nu_A;s)$ and $s \in Im(\nu_A)$. Then $\nu_A(x) \leq s+\beta$ and $\nu_A(y) \leq s+\beta$ i.e.,$(\nu_A)^T_\beta(x) = \nu_A(x) - \beta \leq s$ and $(\nu_A)^T_\beta(y) = \nu_A(x) - \beta \leq s$. Since $(\nu_A)^T_\beta$ is a doubt fuzzy subalgebra of X, it follows that

$$\nu_A(x*y) - \beta = (\nu_A)^T_\beta(x*y) \leq max\{(\nu_A)^T_\beta(x), (\nu_A)^T_\beta(y)\} \leq s$$

That is $\nu_A(x*y) \leq s+\beta$. So that $x*y \in L_\beta(\nu_A;s)$. Therefore $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are subalgebras of X.

Conversely, suppose that $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L(\nu_A;s)$ are subalgebras of X for $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$ with $\tilde{t} \geq \hat{\alpha}$. If $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ is not a cubic subalgebra of X. Then there exists some $a,b,c,d \in X$ such that at least one of $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a * b) \prec rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(b)\}$ and $(\nu_A)^T_\beta(c * d) > max\{(\nu_A)^T_\beta(c), (\nu_A)^T_\beta(d)\}$ hold. Suppose $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a * b) \prec rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(b)\}$ holds. choose $\tilde{t} \in D[0;1]$ such that

$$(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a*b) \prec \tilde{t} \preceq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(a), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(b)\}$$

then $\tilde{\mu}_A(a) \succeq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(y) \succeq \tilde{t} - \tilde{\alpha}$ but $\tilde{\mu}_A(a*b) \leq \tilde{t} - \tilde{\alpha}$ This shows that $a \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $b \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ but $a*b \notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ This is a contradiction, and therefore $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y) \succeq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\}$ for all $x,y \in X$.

Again if $(\nu_A)^T_\beta(c * d) > max\{(\nu_A)^T_\beta(c), (\nu_A)^T_\beta(d)\}$ holds. Then choose $\delta \in (0,1]$ such that

$(\nu_A)^T_\beta(c*d) > \delta \geq max\{(\nu_A)^T_\beta(c),(\nu_A)^T_\beta(d)\}$. Then $\nu_A(c) \leq \delta+\beta$ and $\nu_A(d) \leq \delta+\beta$ but $\nu_A(c*d) \geq \delta+\beta$. Hence $c \in L(\nu_A;s)$ and $d \in L_\beta(\nu_A;s)$ but $c*d \notin L_\beta(\nu_A;s)$. This is impossible and therefore $(\nu_A)^T_\beta(x*y) \leq max\{(\nu_A)^T_\beta(x),(\nu_A)^T_\beta(y)\}$. for all $x,y \in X$. Consequently $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ is a cubic subalgebra of X.

Theorem 9*Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of X and let $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$ If $\tilde{\alpha} \succeq \tilde{\gamma}, \beta \geq \delta$ then the cubic $(\tilde{\alpha},\beta)$ translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of A is a cubic S-extension of the cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)} = ((\tilde{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of A.*

*Proof.*Straightforward.

Theorem 10*Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of X and let $\hat{\gamma} \in D[0,\overline{T}], \delta \in [0,\tau]$. For every cubic S-extension $B=(\tilde{\mu}_B,\nu_B)$ of the cubic $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)} = ((\tilde{\mu}_A)^T_{\tilde{\gamma}},(\nu_A)^T_\delta)$ of A., there exists $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$. such that $\tilde{\alpha} \succeq \tilde{\gamma}, \beta \geq \delta$ and B is a cubic S-extension of the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)} = ((\hat{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of A.*

Example 9Illustration of Theorem 10
Consider a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X as in Example 1,

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c & d \\ [0.6,0.7] & [0.15,0.25] & [0.4,0.5] & [0.3,0.4] & [0.5,0.65] \end{pmatrix}$$

and

$$\nu_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.2 & 0.7 & 0.4 & 0.6 & 0.3 \end{pmatrix}$$

Then it is easy to verify that $A=(\tilde{\mu}_A,\nu_A)$ is a cubic subalgebra of X. Here $\overline{T} = 1 - sup\{\overline{\mu}_A(x) \ |x \in X.\} = 1-0.7 = 0.3$ and $\tau = inf\{\nu_A(x)|x \in X\} = 0.2$. Let $\tilde{\gamma} = [0.05,\ 0.1] \in D[0,\overline{T}]$ and $\delta = 0.1 \in [0,\tau]$. Then the $(\tilde{\gamma},\delta)$-translation $((\tilde{\mu}_A)^T_{[0.05,\ 0.1]},(\nu_A)^T_{0.1})$ of cubic set $A=(\tilde{\mu}_A,\nu_A)$ is given by

$$(\tilde{\mu}_A)^T_{[0.05,\ 0.1]} = \begin{pmatrix} 0 & a & b & c & d \\ [0.65,0.80] & [0.2,0.35] & [0.45,0.6] & [0.35,0.5] & [0.55,0.75] \end{pmatrix}$$

and

$$(\nu_A)^T_{0.1} = \begin{pmatrix} 0 & a & b & c & d \\ 0.1 & 0.6 & 0.3 & 0.5 & 0.2 \end{pmatrix}$$

Let $B=(\tilde{\mu}_B,\nu_B)$ be a cubic subset of X defined by

$$\tilde{\mu}_B = \begin{pmatrix} 0 & a & b & c & d \\ [0.75,0.91] & [0.35,0.50] & [0.51,0.75] & [0.45,0.61] & [0.65,0.86] \end{pmatrix}$$

and

$$\nu_B = \begin{pmatrix} 0 & a & b & c & d \\ 0.01 & 0.52 & 0.2 & 0.42 & 0.14 \end{pmatrix}$$

Then B is a cubic S-extension of $(\tilde{\gamma},\delta)$-translation $A^T_{(\tilde{\gamma},\delta)}$ of A and it can be easily verified that B is a cubic subalgebra of X. But B is not a cubic $(\tilde{\alpha},\beta)$-translation of A for all $\hat{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$. If we take $\hat{\alpha} = [0.1,0.2]$ and $\beta = 0.15$ then $\tilde{\alpha} \succeq \tilde{\gamma}, \beta \geq \delta$ and the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_\beta)$ of A is given by

$$(\tilde{\mu}_A)^T_{[0.1,\ 0.2]} = \begin{pmatrix} 0 & a & b & c & d \\ [0.7,0.9] & [0.25,0.45] & [0.5,0.7] & [0.4,0.6] & [0.6,0.85] \end{pmatrix}$$

and

$$(\nu_A)^T_{0.15} = \begin{pmatrix} 0 & a & b & c & d \\ 0.05 & 0.55 & 0.25 & 0.45 & 0.15 \end{pmatrix}$$

Since $\tilde{\mu}_B(x) \succeq (\hat{\mu}_A)^T_{\tilde{\alpha}}(x)$ and $\nu_B(x) \leq (\nu_A)^T_\beta)$ Therefore $A^T_{(\tilde{\alpha},\beta)} \sqsubseteq B$ i.e., B is an cubic S-extension of the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A.

Theorem 11*$A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of X, then the following assertions are equivalent:*
(i) A is a cubic subalgebra of X.
(ii) For all $\gamma \in (0,1]$, A^M_γ is a cubic subalgebra of X.

*Proof.*Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of X. Therefore $\tilde{\mu}_A(x*y) \succeq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and $\nu_A(x*y) \leq max\{\nu_A(x),\nu_A(y)\}$, for all $x,y \in X$

$$\begin{aligned}(\tilde{\mu}_A)^M_\gamma(x*y) = \tilde{\mu}_A(x*y).\gamma &\succeq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}.\gamma \\ &= rmin\{\tilde{\mu}_A(x).\gamma,\tilde{\mu}_A(y).\gamma\} \\ &= rmin\{(\tilde{\mu}_A)^M_\gamma(x),(\tilde{\mu}_A)^M_\gamma(y)\} \\ (\nu_A)^M_\gamma(x*y) = \nu_A(x*y).\gamma &\leq max\{\nu_A(x),\nu_A(y)\}.\gamma \\ &= max\{\nu_A(x).\gamma,\nu_A(y).\gamma\} \\ &= max\{\nu_A(x).\gamma,\nu_A(y).\gamma\} \\ &= max\{(\nu_A)^M_\gamma(x),(\nu_A)^M_\gamma(y)\}\end{aligned}$$

Hence A^M_γ is a cubic subalgebra of X.
Conversely let $\gamma \in (0,1]$ be such that $A^M_\gamma = ((\tilde{\mu}_A)^M_\gamma,(\nu_A)^M_\gamma)$ is a cubic subalgebra of X. Then for all $x,y \in X$, we have

$$\begin{aligned}\tilde{\mu}_A(x*y).\gamma = (\tilde{\mu}_A)^M_\gamma(x*y) &\succeq rmin\{(\tilde{\mu}_A)^M_\gamma(x),(\tilde{\mu}_A)^M_\gamma(y)\} \\ &= rmin\{\tilde{\mu}_A(x).\gamma,\tilde{\mu}_A(y).\gamma\} \\ &= rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}.\gamma \\ \nu_A(x*y).\gamma = (\nu_A)^M_\gamma(x*y) &\leq max\{(\nu_A)^M_\gamma(x),(\nu_A)^M_\gamma(y)\} \\ &= max\{\nu_A(x).\gamma,\nu_A(y).\gamma\} \\ &= max\{\nu_A(x),\nu_A(y)\}.\gamma\end{aligned}$$

Therefore $\tilde{\mu}_A(x*y) \succeq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and $\nu_A(x*y) \leq max\{\nu_A(x),\nu_A(y)\}$ for all $x,y \in X$ since $\gamma \neq 0$. Hence, $A=(\tilde{\mu}_A,\nu_A)$ be a cubic subalgebra of X.

5 Translations of Cubic Ideals

Theorem 12*Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic ideal of X and let $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$. Then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic ideal of X.*

*Proof.*Here $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ and let $x, y \in X$. Then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(0) = \tilde{\mu}_A(0) + \tilde{\alpha} \succeq \tilde{\mu}_A(x) + \tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$ and $(\nu_A)^T_{\beta})(0) = \nu_A(0) - \beta \leq \nu_A(x) - \beta = (\nu_A)^T_{\beta})(x)$ for all $x \in X$. Also

$$\begin{aligned}
(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) &= \tilde{\mu}_A(x) + \tilde{\alpha} \\
&\succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\} + \tilde{\alpha} \\
&= rmin\{\tilde{\mu}_A(x*y) + \tilde{\alpha}, \tilde{\mu}_A(y) + \tilde{\alpha}\} \\
&= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
(\nu_A)^T_{\beta}(x) &= \nu_A(x) - \beta \\
&\leq max\{\nu_A(x*y), \nu_A(y)\} - \beta \\
&= max\{\nu_A(x*y) - \beta, \nu_A(y) - \beta\} \\
&= max\{(\nu_A)^T_{\beta}(x*y), (\nu_A)^T_{\beta}(y)\} \quad \text{for all} \quad x, y \in X.
\end{aligned}$$

Therefore the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic ideal of X.

Theorem 13*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of X such that the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic ideal of X for some $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$. Then $A = (\tilde{\mu}_A, \nu_A)$ is a cubic ideal of X.*

*Proof.*We have $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ and let $x, y \in X$, we have

$$\begin{aligned}
\tilde{\mu_A}(0) + \tilde{\alpha} &= (\tilde{\mu}_A)^T_{\tilde{\alpha}}(0) \\
&\succeq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \\
&= \tilde{\mu}_A(x) + \tilde{\alpha} \\
\nu_A(0) - \beta &= (\nu_A)^T_{\beta}(0) \\
&\leq (\nu_A)^T_{\beta}(x) \\
&= \nu_A(x) - \beta, \quad \text{for all} \quad x, y \in X.
\end{aligned}$$

Which implies $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\nu_A(0) \leq \nu_A(x)$

$$\begin{aligned}
\tilde{\mu}_A(x) + \tilde{\alpha} &= (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \\
&= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
&= rmin\{\hat{\mu}_A(x*y) + \hat{\alpha}, \tilde{\mu}_A(y) + \hat{\alpha}\} \\
&\succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\} + \tilde{\alpha} \\
\nu_A(x) - \beta &= (\nu_A)^T_{\beta})(x) \\
&\leq max\{(\nu_A)^T_{\beta}(x*y), (\nu_A)^T_{\beta}(y)\} \\
&= max\{\nu_A(x*y) - \beta, \nu_A(y) - \beta\} \\
&= max\{\nu_A(x*y), \nu_A(y)\} - \beta, \quad \text{for all} \quad x, y \in X.
\end{aligned}$$

Which implies $\tilde{\mu}_A(x) \succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\nu_A(x) \leq max\{\nu_A(x*y), \nu_A(y)\}$. Hence $A = (\tilde{\mu}_A, \nu_A)$ is a cubic ideal of X.

Theorem 14*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subset of X and $0 \leq \overline{\alpha} \leq \overline{T}$ where $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in D[0,\overline{T}]$, $\beta \in [0,\tau]$ and $\gamma \in (0,1]$. Then $A = (\tilde{\mu}_A, \nu_A)$ is cubic ideal of X iff $A^{MT}_{(\tilde{\alpha},\beta;\gamma)}$is cubic ideal of X.*

*Proof.*It follows from Theorem 12 and Theorem 13.

Lemma 1*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. If $x \leq y$, then $\tilde{\mu}_A(x) \succeq \tilde{\mu}_A(y)$ and $\nu_A(x) \leq \nu_A(y)$ that is $\tilde{\mu}_A$ is order-reversing and ν_A is order preserving.*

*Proof.*Let $x, y \in X$ be such that $x \leq y$. Then $x*y = 0$ and so

$$\tilde{\mu}_A(x) \succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\} = rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$$
$$\nu_A(x) \leq max\{\nu_A(x*y), \nu_A(y)\} = max\{\nu_A(0), \nu_A(y)\} = \nu_A(y)$$

This completes the proof.

Theorem 15*let $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$ and $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. If X is a BCK-algebra, then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X.*

*Proof.*Since $x*y \leq x$ for all $x, y \in X$, then by lemma 1 we have $\tilde{\mu}_A(x*y) \succeq \tilde{\mu}_A(x)$ and $\nu_A(x*y) \leq \nu_A(x)$. Now

$$\begin{aligned}
(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y) &= \tilde{\mu}_A(x*y) + \tilde{\alpha} \\
&\succeq \tilde{\mu}_A(x) + \tilde{\alpha} \\
&\succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\} + \tilde{\alpha} \\
&\succeq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} + \tilde{\alpha} \\
&= rmin\{\tilde{\mu}_A(x) + \tilde{\alpha}, \tilde{\mu}_A(y) + \tilde{\alpha}\} \\
&= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
(\nu_A)^T_{\beta}(x*y) &= \nu_A(x*y) - \beta \\
&\leq \nu_A(x) - \beta \\
&\leq max\{\nu_A(x*y), \nu_A(y)\} - \beta \\
&\leq max\{\nu_A(x), \nu_A(y)\} - \beta \\
&= max\{\nu_A(x) - \beta, \nu_A(y) - \beta\} \\
&= max\{(\nu_A)^T_{\beta}(x), (\nu_A)^T_{\beta}(y)\} \quad \text{for all} \quad x, y \in X.
\end{aligned}$$

Hence the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic subalgebra of X.

Theorem 16*Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X such that the cubic fuzzy $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_{\beta})$ of A is a cubic ideals of X for all $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$. If $(x*y)*z = 0$ for all $x, y, z \in X$, then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \succeq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)\}$ and $(\nu_A)^T_{\beta}(x) \leq max\{(\nu_A)^T_{\beta}(y), (\nu_A)^T_{\beta}(z)\}$.*

*Proof.*Let $x, y, z \in X$, be such that $(x*y)*z = 0$. Then,

$$\begin{aligned}
(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) &\succeq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
&\succeq rmin\{rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}((x*y)*z), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)\}(\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)\} \\
&= rmin\{rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(0), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)\}, (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \\
&= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(z), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \quad \text{Since}(\tilde{\mu}_A)^T_{\tilde{\alpha}}(0) \succeq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y) \\
&= rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(z)\}
\end{aligned}$$

$$\begin{aligned}
(\nu_A)^T_{\beta}(x) &\leq max\{(\nu_A)^T_{\beta}(x*y), (\nu_A)^T_{\beta}(y)\} \\
&\leq max\{max\{(\nu_A)^T_{\beta}((x*y)*z), (\nu_A)^T_{\beta}(z)\}, (\nu_A)^T_{\beta}(y)\} \\
&\leq max\{max\{(\nu_A)^T_{\beta}(0), (\nu_A)^T_{\beta}(z)\}, (\nu_A)^T_{\beta}(y)\} \\
&= max\{(\nu_A)^T_{\beta}(z), (\nu_A)^T_{\beta}(y)\} \quad \text{Since}(\nu_A)^T_{\beta}(0) \leq (\nu_A)^T_{\beta}(y) \\
&= max\{(\nu_A)^T_{\beta}(y), (\nu_A)^T_{\beta}(z)\}
\end{aligned}$$

Theorem 17 *Intersection and union of any two $(\tilde{\alpha},\beta)$-translations of a cubic ideals of X is also a cubic ideal of X.*

Proof. Same as Theorem 4.

Definition 17 *Let $A=(\tilde{\mu}_A,\nu_A)$ and $B=(\tilde{\mu}_B,\nu_B)$ be two cubic subsets of X. Then B is called a cubic I-extension (ideal-extension) of A if the following assertions are valid:*
(i) B is a cubic extension of A.
(ii) If A is a cubic ideal of X, then B is a cubic ideal of X.

Theorem 18 *Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic ideal of X and $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. Then the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic I-extension of A.*

Proof. Here $A=(\tilde{\mu}_A,\nu_A)$ and $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$
Now $\tilde{\mu}_A(x)+\tilde{\alpha}=(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$. Which implies $\tilde{\mu}_A(x)\preceq(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$. Again $\nu_A(x)-\beta=(\nu_A)^T{}_{\beta}(x)$. Which implies $\nu_A(x)\geq(\nu_A)^T_{\beta}(x)$.
Therefore $A\sqsubseteq A^T_{(\tilde{\alpha},\beta)}$. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic extension of A.
Since A is a cubic ideal of X. Therefore $A^T_{(\tilde{\alpha},\beta)}$ of A is a cubic ideal of X. Hence $A^T_{(\tilde{\alpha},\beta)}$ is a cubic I-extension of A.

Remark 4 *The converse of above theorem is not true as seen in the following Example.*

Example 10 *Consider BCK-algebra X as in Example 1. Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X by*

$$\tilde{\mu}_A=\begin{pmatrix}0 & a & b & c & d\\ [0.7,0.8] & [0.4,0.6] & [0.1,0.3] & [0.4,0.6] & [0.1,0.3]\end{pmatrix}$$

and

$$\nu_A=\begin{pmatrix}0 & a & b & c & d\\ 0.7 & 0.3 & 0.2 & 0.3 & 0.2\end{pmatrix}$$

Then $A=(\tilde{\mu}_A,\nu_A)$ is a cubic ideal of X. let $B=(\hat{\mu}_B,\nu_B)$ be an cubic subset of X defined by

$$\tilde{\mu}_B=\begin{pmatrix}0 & a & b & c & d\\ [0.72,0.9] & [0.42,0.69] & [0.11,0.37] & [0.42,0.69] & [0.11,0.37]\end{pmatrix}$$

and

$$\nu_B=\begin{pmatrix}0 & a & b & c & d\\ 0.75 & 0.31 & 0.29 & 0.31 & 0.29\end{pmatrix}$$

Then $B=(\tilde{\mu}_B,\nu_B)$ is an cubic I-extension of A. But it is not the cubic $(\tilde{\alpha},\beta)$-translation $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ of A for all $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$.

Theorem 19 *Intersection of any two cubic I-extensions of a cubic ideals of X is also a cubic I-extension of X.*

Proof. Same as Theorem 6.

Remark 5 *Union of two cubic I-extensions of a cubic ideal of X may not a cubic ideal of X.*

Definition 18 *For a cubic fuzzy subset $A=(\tilde{\mu}_A,\nu_A)$ of X. Let $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$ and with $\tilde{t}\geq\tilde{\alpha}$. Let*

$$U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})=\{x\in X|\tilde{\mu}_A\succeq\tilde{t}-\tilde{\alpha}\}$$
$$L_{\beta}(\nu_A;s)=\{x\in X|\nu_A\leq s+\beta\}$$

Theorem 20 *If A is a cubic ideal of X, then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are ideals of X for all $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$.*

Proof. Let $x\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Therefore $\tilde{\mu}_A(x)\succeq\tilde{t}-\tilde{\alpha}$. Now $\tilde{\mu}_A(0)\succeq\tilde{\mu}_A(x)\succeq\tilde{t}-\tilde{\alpha}\Rightarrow\tilde{\mu}_A(0)\succeq\tilde{t}-\tilde{\alpha}$. Which implies $0\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Again let $x\in L_{\beta}(\nu_A;s)$. Therefore $\nu_A(x)\leq s+\beta$. Now $\nu_A(0)\leq\nu_A(x)\leq s+\beta\Rightarrow\nu_A(0)\leq s+\beta$. Which implies $0\in L_{\beta}(\nu_A;s)$. Let $x*y,y\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. Therefore $\mu_A(x*y)\succeq\tilde{t}-\tilde{\alpha}$ and $\mu_A(y)\succeq\tilde{t}-\tilde{\alpha}$. Now

$$\begin{aligned}\tilde{\mu}_A(x)&\succeq rmin\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}\\&\succeq rmin\{\tilde{t}-\tilde{\alpha},\tilde{t}-\tilde{\alpha}\}\\&=\tilde{t}-\tilde{\alpha}\\ \Rightarrow\tilde{\mu}_A(x)&\succeq\tilde{t}-\tilde{\alpha}\end{aligned}$$

Which implies $x\in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$.
Let $x*y,y\in L_{\beta}(\nu_A;s)$. Therefore $\nu_A(x*y)\leq s+\beta$ and $\nu_A(y)\leq s+\beta$. Now

$$\begin{aligned}\nu_A(x)&\leq max\{\nu_A(x*y),\nu_A(y)\}\\&\leq max\{s+\beta,s+\beta\}=s+\beta\\ \Rightarrow\nu_A(x)&\leq s+\beta\end{aligned}$$

Which implies $x\in L_{\beta}(\nu_A;s)$. Hence $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are ideals of X.

Remark 6 *In above Theorem if A is not a cubic ideal of X, then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are not an ideals of X as seen in the following example.*

Example 11 *Consider BCK-algebra X as in Example 1. Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X by*

$$\tilde{\mu}_A=\begin{pmatrix}0 & a & b & c & d\\ [0.2,0.4] & [0.5,0.7] & [0.6,0.75] & [0.7,0.8] & [0.5,0.8]\end{pmatrix}$$

and

$$\nu_A=\begin{pmatrix}0 & a & b & c & d\\ 0.6 & 0.3 & 0.4 & 0.5 & 0.3\end{pmatrix}$$

Since $\tilde{\mu}_A(a*c)=\tilde{\mu}_A(0)=[0.2,0.4]\not\succeq[0.5,0.7]=rmin\{\tilde{\mu}_A(a),\tilde{\mu}_A(c)\}$ and $\nu_A(b*b)=\nu_A(0)=0.6\not\leq 0.4=max\{\nu_A(b),\nu_A(b)\}$. Therefore $A=(\tilde{\mu}_A,\nu_A)$ is not a cubic ideal of X. Let $\tilde{\alpha}=[0.1,0.15]\in D[0,0.2],\beta=0.25\in[0,0.3]$ and $\tilde{t}=[0.4,0.6],s=0.3,$. Then $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})=\{a,b,c,d\}$ and $L_{\beta}(\nu_A;s)=\{a,b,c,d\}$. Since $a*c=0\notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $a*c=0\notin L_{\beta}(\nu_A;s)$. Therefore both $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_{\beta}(\nu_A;s)$ are not an ideals of X.

Theorem 21 *For $\tilde{\alpha}\in D[0,\overline{T}],\beta\in[0,\tau]$. let $A^T_{(\tilde{\alpha},\beta)}=((\tilde{\mu}_A)^T_{\tilde{\alpha}},(\nu_A)^T_{\beta})$ be the cubic $(\tilde{\alpha},\beta)$-translation of $A=(\tilde{\mu}_A,\nu_A)$. Then the following assertions are*

equivalent:
(i) $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ *is a cubic ideal of X.*
(ii) $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ *and* $L_\beta(\nu_A;s)$ *are ideals of X for* $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$ *with* $\tilde{t} \succeq \hat{\alpha}$.

Proof. Assume that $A^T_{(\tilde{\alpha},\beta)}$ is a cubic ideal of X. Then $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued ideal of X and $(\nu_A)^T_\beta$ is a doubt fuzzy ideal of X. Let $x \in X$ such that $x \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $\tilde{t} \in Im(\tilde{\mu}_A)$ with $\tilde{t} \succeq \tilde{\alpha}$. Then $\tilde{\mu}_A(x) \succeq \tilde{t} - \tilde{\alpha}$. That is $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha} \succeq \tilde{t}$. Since $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy ideal of X, therefore, we have $\tilde{\mu}_A(0) + \tilde{\alpha} \succeq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(0) \succeq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) = \tilde{\mu}_A(x) + \tilde{\alpha} \succeq \tilde{t}$ that is $\tilde{\mu}_A(0) \succeq \tilde{t} - \tilde{\alpha}$ so that $0 \in U_{\tilde{\alpha}}(\mu_A;\tilde{t})$.
Let $x,y \in X$ such that $x*y, y \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $\tilde{t} \in Im(\tilde{\mu}_A)$ with $\tilde{t} \geq \hat{\alpha}$. Then $\tilde{\mu}_A(x*y) \geq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(y) > \tilde{t} - \tilde{\alpha}$. That is $(\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(x*y) = \tilde{\mu}_A(x*y) + \tilde{\alpha} \preceq \tilde{t}$ and $(\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(y) = \tilde{\mu}_A(y) + \tilde{\alpha} \preceq \tilde{t}$. Since $(\tilde{\mu}_A)^T_{\tilde{\alpha}}$ is an interval valued fuzzy ideal of X, therefore, we have $\tilde{\mu}_A(x) + \tilde{\alpha} = (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x) \succeq rmin\{(\tilde{\mu}_A)^T_{\tilde{\alpha}}(x*y), (\tilde{\mu}_A)^T_{\tilde{\alpha}}(y)\} \succeq \tilde{t}$. That is, $\tilde{\mu}_A(x) \succeq \tilde{t} - \tilde{\alpha}$ so that $x \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$.
Again let $x \in X$ such that $x \in L_\beta(\nu_A;s)$ and $s \in Im(\nu_A)$. Then, $\nu_A(x) \leq s + \beta$ i.e., $(\nu_A)^T_\beta(x) = \nu_A(x) - \beta \leq s$. Since $(\nu_A)^T_\beta$ is a doubt fuzzy ideal of X it follows that $\nu_A(0) - \beta = (\nu_A)^T_\beta(0) \leq (\nu_A)^T_\beta(x) = \nu_A(x) - \beta \leq s$. That is $\nu_A(0) \leq s + \beta$. So that $0 \in L_\beta(\nu_A;s)$.
Again let $x,y \in X$ such that $x*y, y \in L_\beta(\nu_A;s)$ and $s \in Im(\nu_A)$ Then $\nu_A(x*y) \leq s + \beta$ and $\nu_A(y) \leq s + \beta$ i.e., $(\nu_A)^T_\beta(x*y) = \nu_A(x) - \beta \leq s$ and $(\nu_A)^T_\beta(y) = \nu_A(x) - \beta \leq s$. Since $(\nu_A)^T_\beta$ is a doubt fuzzy ideal of X, it follows that $\nu_A(x) - \beta = (\nu_A)^T_\beta(x) \leq max\{(\nu_A)^T_\beta(x*y), (\nu_A)^T_\beta(y)\} \leq s$. That is $\nu_A(x) \leq s + \beta$. So that $x*y \in L_\beta(\nu_A;s)$. Therefore $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are ideals of X.

Conversely, $U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $L_\beta(\nu_A;s)$ are ideals of X for $\tilde{t} \in Im(\tilde{\mu}_A), s \in Im(\nu_A)$ with $\tilde{t} \geq \hat{\alpha}$. If possible $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ is not a cubic ideal of X then there exists some $a \in X$ such that $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(0) \prec \tilde{t} \preceq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(a)$ i.e., $\tilde{\mu}_A(0) + \tilde{\alpha} \prec \tilde{t} \preceq \tilde{\mu}_A(a) + \tilde{\alpha}$ then $\tilde{\mu}_A(a) \succeq \tilde{t} - \tilde{\alpha}$ but $\tilde{\mu}_A(0) \prec \tilde{t} - \tilde{\alpha}$. This shows that $a \in U_\alpha(\mu_A;\tilde{t})$ but $0 \notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. This is a contradiction, and therefore we must have $(\tilde{\mu}_A)^T_{\tilde{\alpha}}(0) \succeq (\tilde{\mu}_A)^T_{\tilde{\alpha}}(x)$ for all $x,y \in X$. If there exists $a,b \in X$ such that $(\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(a) \prec \tilde{t} \preceq rmin\{(\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(a*b), (\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(b)\}$ then $\tilde{\mu}_A(a*b) \succeq \tilde{t} - \tilde{\alpha}$ and $\tilde{\mu}_A(b) \succeq \tilde{t} - \tilde{\alpha}$ but $\tilde{\mu}_A(a) \preceq \tilde{t} - \tilde{\alpha}$ This shows that $a*b \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ and $b \in U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$ but $a \notin U_{\tilde{\alpha}}(\tilde{\mu}_A;\tilde{t})$. This is a contradiction, and therefore $(\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(x) \succeq rmin\{(\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(x*y), (\tilde{\mu}_A)^T{}_{\tilde{\alpha}}(y)\}$, for all $x,y \in X$.

Again assume that there exist $c \in X$ such that $(\nu_A)^T_\beta(0) > \delta \geq (\nu_A)^T_\beta(c)$ Then $\nu_A(c) \leq \delta + \beta$ $\nu_A(0) \geq \delta + \beta$. Hence $c \in L_\beta(\nu_A;s)$ $0 \notin L_\beta(\nu_A;s)$. This is impossible and therefore $(\nu_A)^T_\beta(0) \leq (\nu_A)^T_\beta(x)$ for all $x,y \in X$.

Again assume that there exist $c,d \in X$ such that $(\nu_A)^T_\beta(c) > \delta \geq max\{(\nu_A)^T_\beta(c*d), (\nu_A)^T_\beta(d)\}$. Then $\nu_A(c*d) \leq \delta + \beta$ and $\nu_A(d) \leq \delta + \beta$ but $\nu_A(c) \geq \delta + \beta$ Hence $c*d \in L_\beta(\nu_A;s)$ and $d \in L_\beta(\nu_A;s)$ but $c \notin L_\beta(\nu_A;s)$. This is impossible and therefore $(\nu_A)^T_\beta(x) \leq max\{(\nu_A)^T_\beta(x*y), (\nu_A)^T_\beta(y)\}$ for all $x,y \in X$. Consequently $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ is a cubic ideal of X.

Theorem 22 *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be a cubic subalgebra of X and let* $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$ *If* $\tilde{\alpha} \succeq \tilde{\gamma}, \beta \geq \delta$ *then the cubic* $(\tilde{\alpha},\beta)$ *translation* $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ *of A is a cubic I-extension of the cubic* $(\hat{\gamma},\delta)$*-translation* $A^T_{(\tilde{\gamma},\delta)} = ((\hat{\mu}_A)^T_{\tilde{\gamma}}, (\nu_A)^T_\delta)$ *of A.*

Proof. It is straightforward.

Theorem 23 *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be a cubic ideal of X and let* $\tilde{\gamma} \in D[0,\overline{T}], \delta \in [0,\tau]$. *For every cubic I-extension* $B = (\tilde{\mu}_B, \nu_B)$ *of the cubic* $(\tilde{\gamma},\delta)$*-translation* $A^T_{(\tilde{\gamma},\delta)} = ((\hat{\mu}_A)^T_{\tilde{\gamma}}, (\nu_A)^T_\delta)$ *of A., there exists* $\tilde{\alpha} \in D[0,\overline{T}], \beta \in [0,\tau]$ *such that* $\tilde{\alpha} \succeq \tilde{\gamma}, \beta \geq \delta$ *and B is an cubic I-extension of the cubic* $(\tilde{\alpha},\beta)$*-translation* $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ *of A.*

Proof. It is straightforward.

Theorem 24 $A = (\tilde{\mu}_A, \nu_A)$ *be a cubic ideal of X, then the following assertions are equivalent:*
(i) A is a cubic ideal of X.
(ii) For all $\gamma \in [0,1]$, A^M_γ *is a cubic ideal of X.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. Therefore we have
$\tilde{\mu}_A(0) \succeq \tilde{\mu}_A(x)$, $\nu_A(0) \leq \nu_A(x)$, $\tilde{\mu}_A(x) \succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\nu_A(x) \leq max\{\nu_A(x*y), \nu_A(y)\}$ for all $x,y \in X$. Now

$$\begin{aligned}
(\tilde{\mu}_A)^M_\gamma(0) = \tilde{\mu}_A(0).\gamma &\succeq \tilde{\mu}_A(x).\gamma = (\tilde{\mu}_A)^M_\gamma(x)\\
(\nu_A)^M_\gamma(0) = \nu_A(0).\gamma &\leq \nu_A(x).\gamma = (\nu_A)^M_\gamma(x)\\
(\tilde{\mu}_A)^M_\gamma(x) = \tilde{\mu}_A(x).\gamma &\succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}.\gamma\\
&= rmin\{\tilde{\mu}_A(x*y).\gamma, \tilde{\mu}_A(y).\gamma\}\\
&= rmin\{(\tilde{\mu}_A)^M_\gamma(x*y), (\tilde{\mu}_A)^M_\gamma(y)\}\\
(\nu_A)^M_\gamma(x) = \nu_A(x).\gamma &\leq max\{\nu_A(x*y), \nu_A(y)\}.\gamma\\
&= max\{\nu_A(x*y).\gamma, \nu_A(y).\gamma\}\\
&= max\{(\nu_A)^M_\gamma(x*y), (\nu_A)^M_\gamma(y)\}
\end{aligned}$$

Converse part let $\gamma \in [0,1]$ be such that $A^M_\gamma = ((\tilde{\mu}_A)^M_\gamma, (\nu_A)^M_\gamma)$ is a cubic ideal of X. Then, for all $x,y \in X$, we have

$$\tilde{\mu}_A(0).\gamma = (\tilde{\mu}_A)^M_\gamma(0) \succeq (\tilde{\mu}_A)^M_\gamma(x) = \tilde{\mu}_A(x).\gamma$$

$$\nu_A(0).\gamma = (\nu_A)^M_\gamma(0) \le (\nu_A)^M_\gamma(x) = \nu_A(x).\gamma$$

Which implies $\tilde{\mu}_A(0) \succeq \tilde{\mu}_A(x)$ and $\nu_A(0) \le \nu_A(x)$ for all $x \in X$ since $\gamma \neq 0$.

$$\begin{aligned}
\tilde{\mu}_A(x).\gamma = (\tilde{\mu}_A)^M_\gamma(x) &\succeq rmin\{(\tilde{\mu}_A)^M_\gamma(x*y), (\tilde{\mu}_A)^M_\gamma(y)\} \\
&= rmin\{\tilde{\mu}_A(x*y).\gamma, \tilde{\mu}_A(y).\gamma\} \\
&= rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}.\gamma \\
\nu_A(x).\gamma = (\nu_A)^M_\gamma(x) &\le max\{(\nu_A)^M_\gamma(x*y), (\nu_A)^M_\gamma(y)\} \\
&= max\{\nu_A(x*y).\gamma, \nu_A(y).\gamma\} \\
&= max\{\nu_A(x*y), \nu_A(y)\}.\gamma
\end{aligned}$$

Therefore $\tilde{\mu}_A(x) \succeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\nu_A(x) \le max\{\nu_A(x*y), \nu_A(y)\}$ for all $x, y \in X$ since $\gamma \neq 0$. Hence $A = (\tilde{\mu}_A, \nu_A)$ is a cubic ideal of X.

6 Conclusion and Future Work

In this paper, we have introduced translation in cubic sets, particularly we studied translation of cubic subalgebras and cubic ideals in BCK/BCI-algebras. The relationships between cubic translations and cubic extensions of cubic subalgebras and cubic ideals are also discussed. The cubic translation $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}}, (\nu_A)^T_\beta)$ of a cubic set A can also be represenred by $A^T_{(\tilde{\alpha},\beta)} = ((\tilde{\mu}_A)^T_{\tilde{\alpha}+}, (\nu_A)^T_{\beta-})$, where $(\tilde{\mu}_A)^T_{\tilde{\alpha}+}(x) = rmin\{\tilde{\mu_A}(x) + \tilde{\alpha}, \tilde{1}\}$ and $(\nu_A)^T_{\beta-}(x) = max\{\nu_A(x) - \beta, 0\}$ for all $x \in X, \tilde{\alpha} \in D[0,1]$ and $\beta \in [0,1]$. In this case $A^T_{(\tilde{\alpha},\beta)}$ may be called as cubic translation operator and it has two components $(\tilde{\mu}_A)^T_{\tilde{\alpha}+}$ and $(\nu_A)^T_{\beta-}$ respectively called as $\tilde{\alpha}$-up and β-down operators. We can study the action of operator $A^T_{(\tilde{\alpha},\beta)}$ in any cubic structure. It is our hope that this work would other foundations for further study of the theory cubic sets. In our future study of cubic sets, we try to introduced cubic matrices and represent these operators in terms of cubic matrices.

References

[1] S. R. Barbhuiya, *Fuzzy translations and Fuzzy multiplications of Interval-Valued Doubt Fuzzy Ideals of BF-Algebras,* Journal of Information and Optimization Sciences, **37(1)**, (2016), 23-36.

[2] T. K. Dutta, S. Kar, and S. Purkait, *Interval-valued fuzzy prime and semiprime ideals of a hyper semiring,* Annals of Fuzzy Mathematics and Informatics, 9(2), (2015), 261-278.

[3] Y. Imai and K. Iseki, *On Axiom systems of Propositional calculi XIV,* Proc, Japan Academy, **42** (1966), 19-22.

[4] K. Iseki, *An algebra related with a propositional calculus, Proc. Japan Academy,* **42** (1966), 26-29.

[5] K. Iseki, *On some ideals in BCK-algebras, Math.Seminar Notes,* **3** (1975), 65-70.

[6] Y. B. Jun, *Interval-valued fuzzy subalgebras/ideals in BCK-algebras,* Scientiae Mathematicae, 3(3) (2000) 435-444.

[7] Y. B. Jun, *Translation of fuzzy ideals in BCK/BCI-Algebras,* Hacettepe Journal of Mathematics and Statistics, Volume **40(3)** (2011),349-358.

[8] Y. B. Jun, C. S. Kim and K.O. Yang, *Cubic sets,* Ann Fuzzy Math Inform 4(1) (2012), 83–98.

[9] Y. B. Jun, C. S. Kim and M.S. Kang, *Cubic subalgebras and ideals of BCK/BCI-algebras,* Far East J Math Sci 44 (2010),239–250.

[10] Y. B. Jun, K. J. Lee and M.S. Kang,*Cubic structures applied to ideals of BCI-algebras,* Comput Math Appl 62(9) (2011), 3334–3342.

[11] Y. B. Jun, C. S. Kim and J.G. Kang, *Cubic q-ideals of BCIalgebras,* Ann Fuzzy Math Inform 1(1) (2011), 25–34.

[12] Y. B. Jun, S. T. Jung and M.S. Kim, *Cubic subgroups,* Ann Fuzzy Math Inform 2(1) (2011), 9–15.

[13] K. J. Lee, Y. B. Yun, M. I. Doh , *Fuzzy translation and Fuzzy multiplications of BCK/BCI-algebras,* Commun Korean Math.Soc **24(3)** (2009) 353-360.

[14] Mostafa, S. M., Abd-Elnaby, M. A., Elgendy, O.R. *Interval-valued fuzzy KU-ideals in KU-algebras,* Int. Math. Forum 6(64),(2011), 3151-3159.

[15] A. Rosenfeld, *Fuzzy subgroups,* J Math Anal Appl. 35 (1971),512–517.

[16] A. B. Saeid, *Interval-valued fuzzy BG-algebras,* Kangweon-Kyungki Math.Jour, **14(2)** (2006), 203-215.

[17] O. G. Xi, *Fuzzy BCK algebras,* Math Japonica, 36,(1991) 935-942.

[18] N. Yaqoob, S. M. Mostafa and M. A. Ansari, *On cubic KU-ideals of KU-algebras,* ISRN Algebra, Article ID 935905 (2013) 10 pages.

[19] L. A. Zadeh, *Fuzzy sets,* Information and Control **8** (1965), 338-353.

[20] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning-I,* Information Science **8** (1975), 199-249.

Ananta Kumar Dutta is an Associate Professor Department of Mathematics, Dibrugarh Hanumnabax Surajmall Kanio College, Dibrugarh, Assam (India). He received his M. Sc. degree from Gauhati University, Guwahati, Assam (India) . His research interests include Fuzzy sets, Fuzzy algebraic structures and their applications.

Saidur Rahman Barbhuiya is an Assistant Professor and Head Department of Mathematics , Srikishan Sarda College, Hailakandi, Assam (India). He received his M. Sc. and Ph. D. degree from Assam University, Silchar, Assam (India) . He is referee and Editorial board member of several international journals. His research interests include Fuzzy sets, Fuzzy algebraic structures and their applications. In his credit, he has a number of papers published in various national and international journals and conferences. Mr. Barbhuiya is a life member of Association of Mathematics Teachers of India (AMTI), Chennai, life member of Assam Science Society and life member of Barak Education Society, Silchar.

Karabi Dutta Choudhury is working as a Professor in the Department of Mathematics, Assam University, Silchar(a Central University), India. Prof. Dutta Choudhury has been awarded Msc. and Ph.D degree from Jadavpur University, Jadavpur, India. She has 26 years of teaching experience and 25 years of research experience. Prof. Dutta Choudhury has published more than twenty research papers. She has written and edited no. of books. Prof. Dutta Choudhury is a regular reviewer of many National and International Journal. She is also acting as a member of the editorial board of different journals. Three scholars have been awarded Ph. D degree and six scholars are doing researches under the supervision of Prof. K. Dutta Choudhury.