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Non-selfadjoint Differential Operators Associated Sectorial Forms and its Applications

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Abstract: In the present paper, first we establish a new form and apply it to find an non-selfadjoint differential operator corresponding the proposed form utilizing the famous representation theorem. At the end, the resolvent of the derived operator using a new technique is given.

Keywords: Resolvent, Sectorial forms, Non-selfadjoint differential operators.

1 Introduction

Dute to the wide application of differential operators, in particular non-selfadjoint differential operators in mathematics and the other sciences. The author [6] discovered that the closed sectorial forms are very good tools to construct $m-$sectorial differential operators and introduced some special cases of such $m-$sectorial differential operators using the sectorial forms. For more results on this topic see [5], [8,9] and [12] . In this note, we consider the form Q on the Hilbert space $H = L_2(0,1)$ as follows:

$$Q[u,v] = \int_0^1 k(t)\mu(t)u'(t)\overline{v'(t)}dt.$$

Moreover, also let the form Q satisfy the following conditions:

$$k(t) \in C^1(0,1) \tag{1}$$

is a locally summable non-negative function,i.e., weight.

$$\mu(t) \in C^2[0,1], \quad \mu(t) \in \Phi_\theta, \tag{2}$$

where

$$\Phi_\theta = \left\{ z \in \mathbb{C} : |argz| \leq \theta,\ 0 < \theta < \frac{\pi}{2} \right\}.$$

The main goal of this paper is to find an operator corresponding the presented form and to establish some spectral properties for the obtained operator using a new technique.

The results of this note generalize the obtained results by author in [12].

We need the following definitions in our arguments.

Definition 1.[6] *Let N be a subspace of separable Hilbert space H. The complex-valued function $b : N \times N \to \mathbb{C}$ is said to be a sesquilinear form, if it be linear and semi-linear in the first argument and the second argument, respectively.*

We recall that N is the domain of the form b and denote by $D(b)$. Moreover, $\overline{N} = \overline{D(b)} = H$.

Remark.[6] A form b is said to be symmetric if $b[u,v] = \overline{b[v,u]}(u,v \in D(b))$.

Definition 2.[6] *$\Theta\ (b)$ denotes the numerical range of b and is defined as*

$$\Theta\ (b) = \{b[u,u]\ :\ u \in D(b) = N \quad : \quad \|u\| = 1\}.$$

Definition 3.[6] *A form b is said to be sectorial if $\Theta(b)$ is a subset of a sector of the form*

$$S = \left\{ z \in \mathbb{C} :\ |\arg(z-\gamma)| \leq \theta; \quad 0 \leq \theta < \frac{\pi}{2}\ , \gamma \in \mathbb{R} \right\},$$

where γ and θ are a vertex and a semi-angle of the form b respectively.

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Hereinafter the symbols $(,)$ and $\|.\|$ are used to define the scalar product and the norm in the space H, respectively.

Definition 4.[6] *A form b is said to be closed, if $D(b)$ is complete with respect to the following norm*

$$\|u\| = \left(Re(b(u,u)) + \frac{\delta}{M}\|u\|^2 \right)^{\frac{1}{2}}. \qquad (3)$$

Definition 5.[6] *An operator T in H is said to accretive, if*

$$Re\Theta(T) = Re(Tu,u) \geq 0,$$

for all $u \in D(T)$.

In addition, an operator T in H is said to m-accretive, if $\|T+\lambda I\| \leq (Re\lambda)^{-1}$ for $Re\lambda > 0$.

Definition 6.[6] *An operator T is said to be sectorial, if $\Theta\ (T)$ satisfies the following condition:*

$$\Theta(T) \subset S = \{z \in \mathbb{C} : |\arg(z-\gamma)| \leq \theta\}, 0 \leq \theta < \frac{\pi}{2}, \gamma \in R.$$

Again we recall that γ and θ denote a vertex and a semi-angle of the sectorial operator T respectively, for example see [6].

Definition 7.[6] *A operator T is said to be m- sectorial, if it is both sectorial and m-accretive operators.*

We denotes the C^*-algebra of all bounded linear operators on the complex Hilbert space $H = L_2(0,1)$ by $B(H)$. Let $T \in B(H)$ be compact. The Hilbert-Schmidt norm and the Kernel norm of T are defined by

$$\|T\|_2 = \sqrt{\sum_{j=1}^{\infty} s_j^2(T)}, \quad \|T\|_1 = \sum_{j=1}^{\infty} s_j(T),$$

respectively, where $s_1(T) \geq s_2(T) \geq \dots\dots$ are the singular values of T, that is, the eigenvalues of the positive operator $|T| = (T^*T)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

2 The main results

In this section, we offer a new form. Also we apply this form to find an sectorial operator. Then, we estimate the resolvent of the obtained operator using a new method.

Definition 8.*Assume that $k(t)$ satisfies (1). By $\mathscr{H}_s = W^1_{2,k(t)}(0,1)$ we introduce the class of all complex-valued functions $u(t)$ defined on $(0,1)$ with the following Sobolev norm:*

$$|u|_s = \left(\int_0^1 k(t)|u'(t)|^2_{\mathbb{C}}dt + \int_0^1 |u(t)|^2_{\mathbb{C}}dt \right)^{1/2}. \qquad (4)$$

The symbol $\overset{\circ}{\mathscr{H}}_s$ denotes the closure of linear manifold $C_0^\infty(0,1)$ in the space $\mathscr{H}_s$ with respect to the Sobolev norm. Here, $C_0^\infty(0,1)$ denotes the class of all infinitely differentiable functions with compact support in $(0,1)$.

Define the sesquilinear form Q on the space $H = L_2(0,1)$ as follows:

$$Q[u,v] = \int_0^1 k(t)\mu(t)u'(t)\overline{v'(t)}dt.$$

We need the following crucial Lemma to prove our main result:

Lemma 1.*Assume that the form Q be as above. Moreover, suppose that the function $\mu(t)$ holds in condition (1), then there exist a operator T on H such that $(Tu,v) = Q[u,v]$ and the domain of the operator T consist of the class of the vector functions $u(t) \in \overset{\circ}{\mathscr{H}}_s \cap W^1_{2,loc}(0,1)$ such that $g = -(k(t)\mu(t)u'(t))' \in H$. At that, $g = Tu$.*

*Proof.*To prove the assertion of Lemma (1), by [6] we need to extend its domain to the closed set $D(Q) = \overset{\circ}{\mathscr{H}}_s = (\overline{C_0^\infty(0,1),|.|_s})$. As the Sobolev norm and the norm in (3) are equivalent, by applying Definition(4), we conclude that the form Q is closed. Moreover, the condition (1), together with Definition(3), ensures that the form Q is sectorial. Now, according to [6], there exists an operator T such that $(Tu,v) = Q[u,v]$ and $D(T) \subset D(Q)$ for $u \in D(T)$ and $v \in D(Q)$. Now we show

$$D(T) = \left\{ u \in \overset{\circ}{\mathscr{H}}_s \cap W^1_{2,loc}(0,1) : Tu = -\left(k(t)\mu(t)u'(t)\right)' \in H \right\}.$$

Taking $u \in \overset{\circ}{H}_s \cap W^1_{2,loc}(0,1)$ and $g \in H$. Let $v \in C_0^\infty(0,1)$. By integrating by parts, we verify in a straightforward manner that

$$\begin{aligned}(g,v) &= \left(-(k(t)\mu(t)u')', v \right) \\ &= \int_0^1 -\left(k(t)\mu(t)u'(t)\right)' \overline{v(t)}dt \\ &= \int_0^1 k(t)\mu(t)u'(t)\overline{v'(t)}dt \\ &= Q[u,v].\end{aligned}$$

Now we let $v \in \overset{\circ}{\mathscr{H}}_s$. From $(\overline{C_0^\infty(0,1),|.|_s}) = \overset{\circ}{\mathscr{H}}_s$, making use of continuity of inner product, it follows that

$$(g,v) = lim_{n\to\infty}(g,v_n) = Q[u,v].$$

Therefore, $u \in D(Q)$ and $g = Tu$. And vice versa, let $u \in D(T)$ and $g_1 = Tu$. For every $v \in C_0^\infty(0,1)$, we have

$$(g_1, v) = (Tu, v) = \int_0^1 k(t)\mu(t)u(t)'\overline{v'(t)}dt.$$

Clearly, the above equality is an extension of the function

$$g_2 = -\left(k(t)\mu(t)u'\right)'.$$

So, it is simple to see that $g_1 = g_2$. Utilizing the general theory of elliptic equations, we obtain $u \in W_{2,loc}^1(0,1)$. The proof of Lemma1 is proved.

As application of Lemma1, we give the following Theorems:

Theorem 1. *Let T be the obtained operator in Lemma1, then for all $z \in \Phi_\psi, |z| > 1$, the operator $T - zI$ has a continuous inverse and the following inequality holds:*

$$\left\|(T - zI)^{-1}\right\| \le M_{\Phi_\psi}|z|^{-1}, \qquad \left(z \in \Phi_\psi,\ |z| > 1\right).$$

Here, M_{Φ_ψ} is a sufficiently large and positive number depending on Φ_ψ.

Proof. By virtue of the derived operator from Lemma1, one can write

$$\|(T-zI)u\|^2 = \|T(u)\|^2 + |z|^2\|u\|^2 - 2Re\left\{z\left(k^{\frac{1}{2}}(t)\overline{\mu}u', k^{\frac{1}{2}}(t)u'\right)\right\}.$$

Using condition(2) and Lemma1, we conclude that $(Tu, u) \in \Phi_\theta$. As $z \in \Phi_\psi$, it immedaitely follows that $Re\{z(Tu,u)\} \le 0$. The condition $Re\{z(Tu,u)\} \le 0$ and $|(Tu,u)| \le \frac{1}{2}\left(\|Tu\|^2 + \|u\|^2\right)$ imply that

$$(1-\chi)\left(\|T(u)\|^2 + |z|^2\|u\|^2\right) \le \|(T-zI)u\|^2,$$

where $\chi = \chi(\Phi_\theta, \Phi_\psi) < 1$. This completes our proof.

Theorem 2. *Let T be the derived operator in Lemma 1. Then, for $0 \le \theta < \frac{\pi}{2}$, we have*

$$N(\eta) = card\{j : |z_j(T)| \le \eta, |arg z_j(T)| \le \theta\} \le M(1+\eta)^{\frac{1}{2}}.$$

Proof. Corresponding to the form Q as in Lemma1, we consider the real part of the form Q with Q' (i.e., $Q' =$ Real Q) and we define it as follows:

$$Q'u, v] = \int_0^1 k(t)\mu_1(t)u'(t)\overline{v'(t)}dt,$$

where $\mu_1(t) = Re\mu(t)$ and $D(Q') = \overset{\circ}{\mathscr{H}}_s$. Analogous to Lemma1, there exists an operator T' such that $Q'(u,v) = (T'u, v)$. With the aid of the forms Q and Q' and the non-negative number z (that is $z \ge 0$), we can define the forms Q_z and Q'_z as follows:

$$Q_z[u,v] = Q[u,v] + z(u,v), \quad D(Q_z) = D(Q)$$

and

$$Q'_z[u,v] = Q'[u,v] + z(u,v), \quad D(Q'_z) = D(Q').$$

Applying Lemma1, one obtains two $m-$sectorial operators T_z and T'_z such that

$$T_z = T + zI, \quad T'_z = T' + zI.$$

In veiw of [6] there exists an symmetric operator $B \in B(H)$ such that $\|B\| \le tan\theta$ and

$$(T + zI) = (T' + zI)^{\frac{1}{2}}(I + iB(z))(T' + zI)^{\frac{1}{2}}, \tag{5}$$

where $z \ge 0$. From $B(z) = B(z^*)$ is a bounded operator, it follows that for every $u \in L_2(0,1)$

$$\|(I + iB(z))(u)\|^2 = \|u\|^2 + \|B(z))\|^2 \ge \|u\|^2,$$

which implies that

$$\|(I + iB(z))\|^{-1} \le 1.$$

Using the latter inequality and the relation (5), we get

$$(T + zI)^{-1} = (T' + zI)^{-\frac{1}{2}}Y(z)(T' + zI)^{-\frac{1}{2}}, \quad \|Y(z)\| \le 1, \quad z > 0. \tag{6}$$

In result, the operator $(T + zI)^{-1}$ is compact, and then it has countable spectrum. Therefore, the eigenvalues of the operator are as follows:

$$(z_1(T) + z)^{-1}, (z_2(T) + zI)^{-1}, \ldots.$$

Thus, we conclude that

$$\sum_{i=1}^{\infty} |(z_i(T) + z)^{-1}| \le \left|(T + zI)^{-1}\right|_1 \le \left|(T' + zI)^{-1}\right|_2^2,$$

where, $|.{}_2|$ is Hilbert Schmidt norm. As for every $u \in D(T)$, $|\arg(Tu,u)| \le \theta$, it follows that $|\arg z_i(T)| \le \theta$, for $i = 1, 2, \ldots$. This means that

$$(|z_i(T)| + z)^{-1} \le M_\theta\left|(z_i(T) + z)^{-1}\right|.$$

Now, we show that for $\eta > 0$

$$N(\eta) = card\{j : |z_j(T')| \le \eta\} \le M(1+\eta)^{\frac{1}{2}}.$$

From what has been discussed above, we obtain the following inequality:

$$\begin{aligned} N(\eta) &= \int_0^{\eta} d\,N(s) \\ &\leq 2\eta \int_0^{\eta} (s+z)^{-1} d\,N(s) \\ &\leq 2\eta \int_0^{\infty} (s+z)^{-1} d\,N(s) \\ &= 2\eta \sum_{i=1}^{\infty} (z_i(T)+\eta)^{-1} \\ &\leq 2\eta\, M_\theta \,.\left|(T^{'}+\eta\, I)^{-1}\right|_2^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \left|(T^{'}+\eta\, I)^{-1}\right|_2^2 &= \sum_{i=1}^{\infty} \left(z_i(T^{'}+\eta)^{-1}\right)^2 \\ &= \sum_{i=1}^{\infty} \left|(z_i(T^{'})+\eta)^{-2}\right|_2^2 \\ &= \int_0^{\infty} \frac{dn(s)}{(\eta+s)^2} = 2\int_0^{\infty} \frac{n(s)ds}{(\eta+s)^3} \\ &\leq 2\int_0^{\infty} \frac{(1+s)^{\frac{1}{2}}ds}{(\eta+s)^3} \leq 2\,M\,.(1+\eta)^{\frac{-3}{2}}, \end{aligned}$$

which implies that

$$N(\eta) = card\{j : |z_j(T)| \leq \eta, |argz_j(T)| \leq \theta\} \leq M(1+\eta)^{\frac{1}{2}}.$$

This completes the proof.

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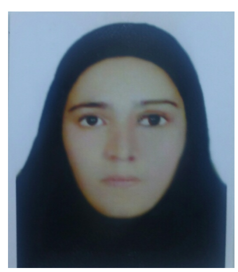

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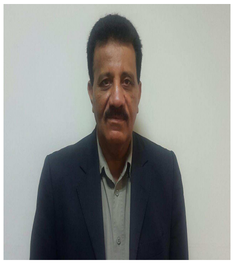

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