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k-Generalized Space-Time Fractional Ultra-Hyperbolic Diffusion-Wave Equation with the Prabhakar Integral Operator

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Abstract: The objective of the present work is to study a generalization of the ultra-hyperbolic diffusion-wave equation introduced in [1] using the k-Prabhakar derivative introduced in [2] for the time variable and a fractional power of ultra-hyperbolic operator on the space variable.

Keywords: Fractional power, ultra-hyperbolic operator, partial fractional differential equations, integrals transforms, Mittag-Leffler type function, fractional derivative, Prabhakar operators.

1 Introduction and preliminaries

The ultra-hyperbolic diffusion-wave equation was introduced in [1] replacing the first and second-order derivative in the temporal variable by a fractional derivative in the Hilfer sense in the equations introduced and studied in [3] and [4]. The name ultra-hyperbolic is due to the Laplacian operator is replaced by the ultra-hyperbolic operator. The ultra-hyperbolic diffusion-wave equation turns out to be an interesting generalization of the equations mentioned. Furthermore, by means of a suitable choice of parameters, it also generalizes the fractional wave-diffusion equation studied by many authors (cf. [5]-[15] and the references cited therein).

In this paper, we study a new generalization of this important equation. In this case, we replace the Hilfer fractional derivative by another so-called k-Prabhakar fractional derivative introduced in [2]. This derivative generalizes the Prabhakar derivative (cf. [16]) as well as the classical Riemann-Liouville fractional derivative. The k-Prabhakar fractional derivative has been generalized in [17], where a new Hilfer type derivative was defined. In the space variable, we replace the Laplacian operator by a power of the ultra-hyperbolic operator following the definition suggested by Samko in [18]. The work ends with the exposition of interesting particular cases.

For the reader's convenience, we begin the work by making a brief review of the definitions and properties to use.

Definition 1. *Let $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is defined by*

$$\mathfrak{F}\{f(x)\}(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx, \tag{1}$$

where $\xi = (\xi_1, \xi_2, ..., \xi_n)$, $x = (x_1, x_2, ...x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + ... + \xi_n x_n$ and $dx = dx_1 dx_2 ... dx_n$. The inverse Fourier transform is given by

$$\mathfrak{F}^{-1}\{\widehat{f}(\xi)\}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)d\xi. \tag{2}$$

It is known (see for example [18]) that the Fourier transform of the ultra-hyperbolic operator defined by

$$\Box = \left(\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2} \right), \qquad p + q = n. \tag{3}$$

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it is given by

$$\mathfrak{F}\{-\Box f(x)\}(\xi) = Q(\xi)\mathfrak{F}\{f(x)\}(\xi), \tag{4}$$

where $Q(\xi)$

$$Q(\xi) = \xi_1^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \ldots - \xi_{p+q}^2.$$

To define the fractional power of the ultrahyperbolic operator (3), the Fourier transform of certain generalized functions introduced by Gelfand in [19] is used. Although here we use the following particular case (see for example in [20]):

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0^+} (P \pm i\varepsilon \|x\|^2)^\lambda = \begin{cases} |P(x)|^\lambda, & \text{si } P(x) > 0; \\ e^{\pm\lambda\pi i}|P(x)|^\lambda, & \text{si } P(x) \leq 0, \end{cases} \tag{5}$$

where $P = P(x)$

$$P(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2, \tag{6}$$

and λ is a complex variable.

Introducing the functions

$$P_+^\lambda(x) = \begin{cases} |P(x)|^\lambda, & \text{si } P(x) > 0; \\ 0, & \text{si } P(x) \leq 0. \end{cases} \tag{7}$$

$$P_-^\lambda(x) = \begin{cases} |P(x)|^\lambda, & \text{si } P(x) < 0; \\ 0, & \text{si } P(x) \geq 0. \end{cases} \tag{8}$$

(5) can be expressed as

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm\lambda\pi i}P_-^\lambda. \tag{9}$$

From the above equation it follows that $(P \pm i0)^\lambda$ and P^λ coincide for $\lambda \in \mathbb{N}$. Here we only consider $\lambda \in \mathbb{R}^+$.

Remark. Note that for $q = 0$, that is $n = p$, from (5), it turns out that for $x \in \mathbb{R}^n$

$$P(x) = x_1^2 + \ldots + x_p^2 = \|x\|^2. \tag{10}$$

The fractional power of the ultra-hyperbolic operator (3) that will be used here is defined by (cf.[18]):

$$(-\Box)^\beta \varphi(x) = \mathscr{F}^{-1}\{(Q \mp i0)^\beta \hat{\varphi}(\xi)\} \tag{11}$$

taking into account

$$\mathscr{F}\left\{(P \pm i0)^\lambda\right\} = \frac{e^{\mp i\frac{q}{2}\pi}\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+n/2)}{\Gamma(-\lambda)}(Q \mp i0)^{-\lambda-n/2}, \tag{12}$$

result due to Gelfand [19].

1.1 Fractional calculus operators

Definition 2. *Let $f \in L^1_{loc}[a,b]$ where $-\infty \leq a < t < b \leq \infty$. The Riemann-Liouville integral of order α is defined as*

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau \qquad \alpha > 0. \tag{13}$$

Definition 3. *Let $f \in L^1[a,b]$, $-\infty \le a < t < b \le \infty$ and $I^{n-\alpha} f(t) \in W^{n,1}[a,b]$, $n = [\alpha]+1$, $\alpha > 0$*
The Riemann-Liouville derivative of order ν, is given by

$$D^{\alpha} f(t) := \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t), \tag{14}$$

where $W^{n,1}[a,b] = \{f \in L^1[a,b] : f^{(n)} \in L^1[a,b]\}$.

Definition 4. *[21] Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, $f \in L^1([0,\infty))$. The k-Riemann-Liouville fractional integral of f is*

$$I_k^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau \; t > 0, \tag{15}$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \; k > 0 \tag{16}$$

is the k-Gamma function introduced in [22] and whose relationship with the classical Gamma function is

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right). \tag{17}$$

The k-Riemann-Liouville fractional integral (15) also satisfies the semigroup property

Proposition 1. *[21] Let $\alpha, \beta \in \mathbb{R}^+$, $f \in L^1([0,\infty))$ and $k > 0$, then*

$$I_k^{\alpha} I_k^{\beta} f(t) = I_k^{\alpha+\beta} f(t) = I_k^{\beta} I_k^{\alpha} f(t). \tag{18}$$

Definition 5. *[23] Let $k, \alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n = [\frac{\alpha}{k}]+1$, $f \in L^1([0,\infty))$ and $I_k^{nk-\alpha} f(t) \in W^{n,1}[0,\infty)$; the k-Riemann-Liouville fractional derivative is given by*

$${}_k\mathfrak{D}_{RL}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n k^n I_k^{nk-\alpha} f(t). \tag{19}$$

Remark. If $k = 1$, (19) coincides with the classical Riemann-Liouville fractional derivative.

The results presented below can be seen in [2]

Definition 6. **(k-Prabhakar integral)** *Let $\alpha, \beta, \omega, \gamma, \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\alpha) > 0$; $\Re(\beta) > 0$ and $\varphi \in L^1([0,b])$, $(0 < x < b \le \infty)$. The k-Prabhakar integral operator is given by*

$$\begin{aligned}({}_k\mathbf{P}_{\alpha,\beta,\omega}^{\gamma}\varphi)(x) &= \int_0^x \frac{(x-t)^{\frac{\beta}{k}-1}}{k} E_{k,\alpha,\beta}^{\gamma}[\omega(x-t)^{\frac{\alpha}{k}}]\varphi(t)dt, \quad (x>0) \\ &= \left({}_k\mathscr{E}_{\alpha,\beta,\omega}^{\gamma} * f\right)(x),\end{aligned} \tag{20}$$

where

$${}_k\mathscr{E}_{\alpha,\beta,\omega}^{\gamma}(t) = \begin{cases} \frac{t^{\frac{\beta}{k}-1}}{k} E_{k,\alpha,\beta}^{\gamma}(\omega t^{\frac{\alpha}{k}}), & t > 0; \\ 0, & t \le 0 \end{cases} \tag{21}$$

and $*$ mean the usual convolution product for causal functions.

Remark. If $\gamma = 0$ we have

$$({}_k\mathbf{P}_{\alpha,\beta,\omega}^{0}\varphi)(t) = (I_k^{\beta}\varphi)(t). \tag{22}$$

Remark. If we put $k = 1$ $(\gamma \neq 0)$, the operator coincides with the Prabhakar operator (cf.[24],[25])

$$({}_1\mathbf{P}_{\alpha,\beta,\omega}^{\gamma}\varphi)(t) = (\mathbf{E}_{\alpha,\beta,\omega;0^+}^{\gamma}\varphi)(t). \tag{23}$$

Definition 7.(**k-Prabhakar fractional derivative**) *Given $k \in \mathbb{R}^+$, $\rho, \beta, \gamma, \omega \in \mathbb{C}$, $\Re(\rho) > 0, \Re(\beta) > 0$, $m = \left[\frac{\beta}{k}\right] + 1$ and $f \in L^1([0,b])$. We define the k-Prabhakar fractional derivative*

$$ {}_k\mathbf{D}^{\gamma}_{\rho,\beta,\omega} f(x) = \left(\frac{d}{dx}\right)^m k^m {}_k\mathbf{P}^{-\gamma}_{\rho,mk-\beta,\omega} f(x). \tag{24}$$

*Remark.*If $k = 1$ the k-Prabhakar fractional derivative coincide with the Prabhakar fractional derivative defined in [26].

*Remark.*If $\gamma = 0$ in (24) the k-Prabhakar fractional derivate coincide with the k-Riemann-Liouville fractional derivative given by [23].

*Remark.*If $k = 1$ and $\gamma = 0$, the k-Prabhakar fractional derivative coincide with the classical Riemann-Liouville fractional derivative.

Lemma 1(**[17]**). *The Laplace transform of the Prabhakar fractional derivative for $m = \left[\frac{\beta}{k}\right] + 1$, is given by*

$$\begin{aligned}\mathscr{L}\left\{{}_k\mathbf{D}^{\gamma}_{\rho,\beta,\omega} y(x)\right\} &= (ks)^{\frac{\beta}{k}} \left(1 - \omega k (ks)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}} \mathscr{L}\{y(x)\}(s) \\ &\quad - \sum_{j=0}^{m-1} k^m s^{m-j-1} \left(\frac{d^j}{dx^j} {}_k\mathbf{P}^{-\gamma}_{\rho,mk-\beta,\omega} y\right)(0)\end{aligned} \tag{25}$$

provided that $\left|\omega k (ks)^{-\frac{\rho}{k}}\right| < 1$.

2 Generalized space-time fractional ultra-hyperbolic diffusion-wave equation

Here, we generalize the ultra-hyperbolic diffusion-wave equation studied in [1] by replacing the Hilfer derivative of order α and type r by the k-Prabhakar fractional derivative (24) of order β in the time variable and considering the fractional power of the ultra-hyperbolic operator (3) defined in [18] in the space variable. That is, we study the following problem:

$$\begin{cases} {}_k\mathbf{D}^{\gamma}_{\rho,\beta,\omega} u(x,t) + c^2(-\Box)^{\lambda} u(x,t) = 0, \; t > 0; \, x \in \mathbb{R}^n \\ {}_k\mathbf{P}^{-\gamma}_{\rho,2k-\beta,\omega} u(x,t)|_{t=0} = f(x); \\ \frac{\partial}{\partial t} {}_k\mathbf{P}^{-\gamma}_{\rho,2k-\beta,\omega} u(x,t)|_{t=0} = g(x), \end{cases} \tag{26}$$

where $f(x)$ and $g(x)$ are functions belonging to the space $\mathscr{S}$, the Schwartz space on functions that is invariant by Fourier Transform.

To solve it, first, we apply Fourier transform with respect to the space variable and then we apply Laplace transform with respect to the time variable. Finally, using the initial conditions

$$(ks)^{\beta/k} \left(1 - wk(ks)^{-\rho/k}\right)^{\gamma/k} \tilde{\hat{u}}(\xi, s) - k(ks)\hat{f}(\xi) - k^2 \hat{g}(\xi) + c^2 (Q \mp i0)^{\lambda} \tilde{\hat{u}}(\xi, s) = 0, \tag{27}$$

$$\tilde{\hat{u}}(\xi, s) \left[(ks)^{\beta/k} \left(1 - wk(ks)^{-\rho/k}\right)^{\gamma/k} + c^2 (Q \mp i0)^{\lambda}\right] = k^2 \hat{g}(\xi) + k(ks)\hat{f}(\xi), \tag{28}$$

$$\begin{aligned}\tilde{\hat{u}}(\xi, s) &= \frac{k^2}{(ks)^{\beta/k} \left(1 - wk(ks)^{-\rho/k}\right)^{\gamma/k} + c^2 (Q \mp i0)^{\lambda}} \hat{g}(\xi) \\ &+ \frac{k(ks)}{(ks)^{\beta/k} \left(1 - wk(ks)^{-\rho/k}\right)^{\gamma/k} + c^2 (Q \mp i0)^{\lambda}} \hat{f}(\xi).\end{aligned} \tag{29}$$

In each term of the sum, we have

$$\begin{aligned}\frac{1}{(ks)^{\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{\gamma/k}+c^2(Q\mp i0)^{\lambda}} &= \frac{1}{(ks)^{\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{\gamma/k}} \\ &\times \frac{1}{1-\frac{-c^2(Q\mp i0)^{\lambda}}{(ks)^{\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{\gamma/k}}} \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j c^{2j}(Q\mp i0)^{\lambda j}}{(ks)^{(j+1)\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{(j+1)\gamma/k}}\end{aligned} \tag{30}$$

provided that

$$\left|\frac{-c^2(Q\mp i0)^{\lambda}}{(ks)^{\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{\gamma/k}}\right|<1. \tag{31}$$

Therefore,

$$\tilde{u}(\xi,s)=\hat{g}(\xi)\sum_{j=0}^{\infty}\frac{k^2(-1)^j c^{2j}(Q\mp i0)^{\lambda j}}{(ks)^{(j+1)\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{(j+1)\gamma/k}} \tag{32}$$

$$+\hat{f}(\xi)\sum_{j=0}^{\infty}\frac{k(ks)(-1)^j c^{2j}(Q\mp i0)^{\lambda j}}{(ks)^{(j+1)\beta/k}\left(1-wk(ks)^{-\rho/k}\right)^{(j+1)\gamma/k}}. \tag{33}$$

Now, applying the inverse Laplace transform and using the Laplace transform of the function k-Mittag-Leffler (cf.[27]), we get

$$\begin{aligned}\hat{u}(\xi,t) &= k\sum_{j=0}^{\infty}(-1)^j c^{2j}(Q\mp i0)^{\lambda j}\hat{g}(\xi)t^{\frac{\beta}{k}(j+1)-1}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right) \\ &+\sum_{j=0}^{\infty}(-1)^j c^{2j}(Q\mp i0)^{\lambda j}\hat{f}(\xi)t^{\frac{\beta(j+1)-k}{k}-1}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)-k}\left(wt^{\frac{\rho}{k}}\right).\end{aligned} \tag{34}$$

Note here that in both terms of the sum (34), there is an iterated series whose convergence we will study for the case of the first term (since in the other case the procedure is similar).

$$\begin{aligned}&\sum_{j=0}^{\infty}(-1)^j c^{2j}(Q\mp i0)^{\lambda j}\hat{g}(\xi)t^{\frac{\beta}{k}(j+1)-1}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right) \\ &\qquad = \hat{g}(\xi)t^{\frac{\beta}{k}-1}\sum_{j=0}^{\infty}\frac{(-c^2(Q\mp i0)^{\lambda})^j t^{\frac{\beta}{k}j}}{\Gamma_k(\gamma(j+1))}\sum_{r=0}^{\infty}\frac{\Gamma_k(\gamma(j+1)+rk)(wt^{\frac{\rho}{k}})^r}{\Gamma_k(\rho r+\beta(j+1))r!}.\end{aligned} \tag{35}$$

To show that the (35) series converges uniformly, we follow a similar procedure used in [28] (Appendix C). We must demonstrate that both the series with respect to the columns (keeping j fixed and adding in m) and the series with respect to the rows (adding in j and keeping m fixed) are uniformly convergent series. In that case, the resulting function is continuous within the radius of convergence and can be integrated within the convergence interval (cf.[29]). Since the k-Mittag-Leffler function is entire (cf. [30] for the case $p=1$), to prove the absolute convergence of (35) it is sufficient to show that for each $r\in\mathbb{N}_0$ the series

$$\sum_{j=0}^{\infty}\frac{(-c^2(Q\mp i0)^{\lambda}t^{\frac{\beta}{k}})^j\Gamma_k(\gamma(j+1)+rk)}{\Gamma_k(\gamma(j+1))\Gamma_k(\rho r+\beta(j+1))} \tag{36}$$

is absolutely convergent.

We study the radius of convergence, for this we consider

$$c_j = \frac{\Gamma_k(\gamma(j+1)+rk)}{\Gamma_k(\gamma(j+1))\Gamma_k(\rho r+\beta(j+1))},$$

and taking into account

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right); \tag{37}$$

result

$$c_j = \frac{k^{r+1-\frac{\rho r+\beta(j+1)}{k}}\Gamma\left(\frac{\gamma}{k}(j+1)+r\right)}{\Gamma\left(\frac{\gamma}{k}(j+1)\right)\Gamma\left(\frac{\rho r+\beta(j+1)}{k}\right)}. \tag{38}$$

Therefore,

$$\left|\frac{c_j}{c_{j+1}}\right| = |k^{-\frac{\beta}{k}}|\left|\frac{\Gamma\left(\frac{\gamma}{k}j+\frac{\gamma}{k}+r\right)}{\Gamma\left(\frac{\gamma}{k}j+\frac{\gamma}{k}\right)}\right|\left|\frac{\Gamma\left(\frac{\gamma}{k}j+\frac{2\gamma}{k}\right)}{\Gamma\left(\frac{\gamma}{k}j+\frac{2\gamma}{k}+r\right)}\right|\left|\frac{\Gamma\left(\frac{\rho r+\beta j+2\beta}{k}\right)}{\Gamma\left(\frac{\rho r+\beta j+\beta}{k}\right)}\right|. \tag{39}$$

In each product we apply the formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1+O\left(\frac{1}{z}\right)\right], \ (|\arg(z+a)|<\pi,\ |z|\longrightarrow\infty). \tag{40}$$

For the cases:

1.$z=\frac{\gamma}{k}j$, $a=\frac{\gamma}{k}+r$ y $b=\frac{\gamma}{k}$,
2.$z=\frac{\gamma}{k}j$, $a=\frac{2}{\gamma}k$ y $b=\frac{2\gamma}{k}+r$,
3.$z=\frac{\beta}{k}j$, $a=\frac{\rho r+2\beta}{k}+r$ y $b=\frac{\rho r+\beta}{k}$,

then it turns out

$$\left|\frac{c_j}{c_{j+1}}\right| \approx |k^{-\frac{\beta}{k}}|\left|\left(\frac{\beta}{k}j\right)^{\frac{\beta}{k}}\right| = |k^{-\frac{\beta}{k}}|\left|\left(\frac{\beta}{k}\right)^{\frac{\beta}{k}}\right| j^{\Re(\beta)} \longrightarrow \infty, \text{cuando } j\longrightarrow\infty. \tag{41}$$

Therefore, the series is absolutely convergent.

Now, returning to (34) and applying inverse Fourier transform, we finally obtain that the solution to the problem (26) is given by

$$\begin{aligned} u(x,t) = {} & \frac{t^{\frac{\beta}{k}-1}}{(2\pi)^n}\int_{\mathbb{R}^n} k\sum_{j=0}^{\infty}(-1)^j c^{2j} t^{\frac{\beta}{k}j} E^{\gamma(j+1)}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right)(Q\mp i0)^{\lambda j}\hat{g}(\xi)e^{-i\xi x}dx \\ & + \frac{t^{\frac{\beta}{k}-2}}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}(-1)^j c^{2j} t^{\frac{\beta}{k}j} E^{\gamma(j+1)}_{k,\rho,\beta(j+1)-k}\left(wt^{\frac{\rho}{k}}\right)(Q\mp i0)^{\lambda j}\hat{f}(\xi)e^{-i\xi x}dx. \end{aligned} \tag{42}$$

An equivalent and usual way of expressing the solution is as follows:

$$u(x,t) = \int_{\mathbb{R}^n} {}_{\gamma}G^{\rho,\beta}_{n,k,1}(x-\tau,t)g(\tau)d\tau + \int_{\mathbb{R}^n} {}_{\gamma}G^{\rho,\beta}_{n,k,2}(x-\tau)f(\tau)d\tau, \tag{43}$$

where

$${}_{\gamma}G^{\rho,\beta}_{n,k,1}(x,t) = kt^{\frac{\beta}{k}-1}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx \tag{44}$$

and

$${}_{\gamma}G^{\rho,\beta}_{n,k,2}(x,t) = t^{\frac{\beta}{k}-2}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)-k}\left(wt^{\frac{\rho}{k}}\right)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx. \tag{45}$$

Remark. ; Note that we can also express the solution in a different way taking into account (11):

$$u(x,t) = kt^{\frac{\beta}{k}-1}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right)(-\Box)^{\lambda j}f(x)$$
$$+\, t^{\frac{\beta}{k}-2}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{\gamma(j+1)}_{k,\rho,\beta(j+1)-k}\left(wt^{\frac{\rho}{k}}\right)(-\Box)^{\lambda j}g(x). \tag{46}$$

2.0.1 Particulars cases.

1.Note here that if we take $\gamma = 0$ we obtain the solution of the equation in terms of the k-Riemann-Liouville fractional derivative (19).

$$\begin{cases} {}_k\mathfrak{D}^{\beta}_{RL}u(x,t)+c^2(-\Box)^{\lambda}u(x,t)=0,\ t>0;\ x\in\mathbb{R}^n \\ I_k^{2k-\beta}u(x,t)|_{t=0}=f(x); \\ \frac{\partial}{\partial t}I_k^{2k-\beta}u(x,t)|_{t=0}=g(x). \end{cases} \tag{47}$$

The solution is given by

$$u(x,t) = \int_{\mathbb{R}^n} {}_0G^{\rho,\beta}_{n,k,1}(x-\tau,t)g(\tau)d\tau + \int_{\mathbb{R}^n} {}_0G^{\rho,\beta}_{n,k,2}(x-\tau)f(\tau)d\tau, \tag{48}$$

where

$${}_0G^{\rho,\beta}_{n,k,1}(x,t) = kt^{\frac{\beta}{k}-1}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{0}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx$$
$$= kt^{\frac{\beta}{k}-1}\sum_{j=0}^{\infty}\frac{(-1)^j c^{2j}t^{\frac{\beta}{k}j}}{\Gamma_k(\beta(j+1))}\int_{\mathbb{R}^n}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx \tag{49}$$

and

$${}_0G^{\rho,\beta}_{n,k,2}(x,t) = t^{\frac{\beta}{k}-2}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{0}_{k,\rho,\beta(j+1)-k}\left(wt^{\frac{\rho}{k}}\right)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx$$
$$= t^{\frac{\beta}{k}-2}\sum_{j=0}^{\infty}\frac{(-1)^j c^{2j}t^{\frac{\beta}{k}j}}{\Gamma_k(\beta(j+1)-k)}\int_{\mathbb{R}^n}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx. \tag{50}$$

Furthermore, under Remark 2, we can formally express the solution as follows

$$u(x,t) = kt^{\frac{\beta}{k}-1}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{0}_{k,\rho,\beta(j+1)}\left(wt^{\frac{\rho}{k}}\right)(-\Box)^{\lambda j}f(x) \tag{51}$$
$$+\, t^{\frac{\beta}{k}-2}\sum_{j=0}^{\infty}(-c^2)^j t^{\frac{\beta}{k}j}E^{0}_{k,\rho,\beta(j+1)-k}\left(wt^{\frac{\rho}{k}}\right)(-\Box)^{\lambda j}g(x). \tag{52}$$

2.If, in addition to taking $\gamma = 0$, we make $k = 1$, the problem (26) is expressed in terms of the Riemann-Liouville fractional derivative

$$\begin{cases} D^{\beta}u(x,t)+c^2(-\Box)^{\lambda}u(x,t)=0,\ t>0;\ x\in\mathbb{R}^n \\ I^{2-\beta}u(x,t)|_{t=0}=f(x); \\ \frac{\partial}{\partial t}I^{2-\beta}u(x,t)|_{t=0}=g(x). \end{cases} \tag{53}$$

The solution can be expressed using (48), (49) and (50) for $k = 1$:

$$u(x,t) = \int_{\mathbb{R}^n} {}_0G^{\rho,\beta}_{n,1,1}(x-\tau,t)g(\tau)d\tau + \int_{\mathbb{R}^n} {}_0G^{\rho,\beta}_{n,1,2}(x-\tau)f(\tau)d\tau, \tag{54}$$

where

$$
\begin{aligned}
{}_0G^{\rho,\beta}_{n,1,1}(x,t) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} t^{\beta-1}\sum_{j=0}^{\infty}\frac{(-c^2)^j t^{\beta j}}{\Gamma(\beta j+\beta)}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx \\
&= \mathfrak{F}^{-1}\{t^{\beta-1}E_{\beta,\beta}(-c^2(Q\mp i0)^{\lambda}t^{\beta})\} \qquad (55)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_0G^{\rho,\beta}_{n,1,2}(x,t) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} t^{\beta-2}\sum_{j=0}^{\infty}\frac{(-c^2)^j t^{\beta j}}{\Gamma(\beta j+\beta-1)}(Q\mp i0)^{\lambda j}e^{-i\xi x}dx \\
&= \mathfrak{F}^{-1}\{t^{\beta-2}E_{\beta,\beta-1}(-c^2(Q\mp i0)^{\lambda}t^{\beta})\}. \qquad (56)
\end{aligned}
$$

But the inverse transforms (55) and (56) were calculated in [31] in terms of the Fox H-function and are given by

$$
\begin{aligned}
&\mathfrak{F}^{-1}\{t^{\beta-1}E_{\beta,\beta}(-c^2(Q\mp i0)^{\lambda}t^{\beta})\} \\
&= \frac{e^{\pm i\frac{q}{2}\pi}t^{\beta-1}}{\left(4^{\lambda}\pi c^{\frac{2}{\lambda}}t^{\frac{\beta}{\lambda}}\right)^{\frac{n}{2}}} H^{2,1}_{2,3}\left(\frac{(P\pm i0)^{\lambda}}{4^{\lambda}c^2t^{\beta}}\,\middle|\, \begin{array}{c}(1-\frac{n}{2\lambda},1);(\beta-\frac{n\beta}{2\lambda};\beta)\\ (1-\frac{n}{2\lambda};1),(0,\lambda);(1-\frac{n}{2};\lambda)\end{array}\right) \qquad (57)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{F}^{-1}\{t^{\beta-2}E_{\beta,\beta-1}(-c^2(Q\mp i0)^{\lambda}t^{\beta})\} \\
&= \frac{e^{\pm i\frac{q}{2}\pi}t^{\beta-2}}{\left(4^{\lambda}\pi c^{\frac{2}{\lambda}}t^{\frac{\beta}{\lambda}}\right)^{\frac{n}{2}}} H^{2,1}_{2,3}\left(\frac{(P\pm i0)^{\lambda}}{4^{\lambda}c^2t^{\beta}}\,\middle|\, \begin{array}{c}(1-\frac{n}{2\lambda},1);(\beta-1-\frac{n\beta}{2\lambda};\beta)\\ (1-\frac{n}{2\lambda};1),(0,\lambda);(1-\frac{n}{2};\lambda)\end{array}\right). \qquad (58)
\end{aligned}
$$

Finally the solution results from replacing (57) and (58) in (54)

Let us now consider two important particular cases of (53):

1. If we take $\lambda = 1$ and $P > 0$ the problem (26) coincides with the particular case 2 of Problem 3.1 given in [1]:

$$
\begin{cases}
D^{\beta}u(x,t)-c^2\Box u(x,t)=0,\ t>0;\ x\in\mathbb{R}^n\\
I^{2-\beta}u(x,t)|_{t=0}=f(x);\\
\frac{\partial}{\partial t}I^{2-\beta}u(x,t)|_{t=0}=g(x),
\end{cases} \qquad (59)
$$

and the solution can be expressed here according to the Remark 2 by the formal series

$$
\begin{aligned}
u(x,t) &= t^{\beta-1}\sum_{j=0}^{\infty}\frac{c^{2j}t^{\beta j}}{\Gamma(\beta j+\beta)}\Box^j f(x) \qquad (60)\\
&+ t^{\beta-2}\sum_{j=0}^{\infty}\frac{c^{2j}t^{\beta j}}{\Gamma(\beta j+\beta-1)}\Box^j g(x). \qquad (61)
\end{aligned}
$$

This is a new expression of the solution given in terms of the two-parameter Mittag-Leffler function. The solution obtained in [1] is given in terms of the Fox H-function.

2.Taking now $q=0$ it turns out that $p=n$ and then $\Box=\Delta$ and we obtain

$$\begin{cases} D^{\beta}u(x,t)-c^2\Delta u(x,t)=0,\ t>0;\ x\in\mathbb{R}^n \\ I^{2-\beta}u(x,t)|_{t=0}=f(x); \\ \frac{\partial}{\partial t}I^{2-\beta}u(x,t)|_{t=0}=g(x), \end{cases} \tag{62}$$

and the solution is given by

$$u(x,t)=t^{\beta-1}\sum_{j=0}^{\infty}\frac{c^{2j}t^{\beta j}}{\Gamma(\beta j+\beta)}\Delta^j f(x) \tag{63}$$

$$+\,t^{\beta-2}\sum_{j=0}^{\infty}\frac{c^{2j}t^{\beta j}}{\Gamma(\beta j+\beta-1)}\Delta^j g(x). \tag{64}$$

This problem has been studied in [16] for the case $m-1<\alpha\leq m$, $m\in\mathbb{N}$. It can be seen that taking $m=2$ both solutions coincide.

3 Conclusion

It can be seen that for a suitable choice of parameters, the equation (26) also generalizes other equations studied by other authors, including the one-dimensional case studied by Mainardi [32]. It can be seen that the equation presented here contains interesting particular cases and can provide a different point of view to interpret the solutions. The equation can also be generalized considering other fractional derivatives in the time variable and considering other initial conditions.

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