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A Multilevel Control Iterative Method for Nonlinear Partial Different Equations

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Abstract: The multilevel iterative method consists of the iterative systems associated with different levels, the solution in coarse level is in essence used as the initial guess of the iterative system in refined level. Since the different iterative targets, one doesn't guarantee that an excellent coarse level solution provides better starting point than a bad one for the refined level system. To decrease computational cost a natural idea is to control the iterative times in coarse levels. The objective of this paper is to propose a multilevel iterative method for nonlinear partial different equations. In which a controlling parameter that can be used to choose the iterative times is introduced based on multiresolution error analysis, and a strategy of selecting iterative times is also proposed. Finally, an example is illustrated to show the effectiveness of our method given here

Keywords: nonlinear, partial differential equation, iterative method, multilevel control, computational expense.

1 Introduction

As the chief means of providing mathematical models, partial differential equations (PDE) have numerous applications in science, engineering and other fields. Generally these models have to be solved numerically. In recent years, there has been a growing interest in developing multilevel numerical solvers for PDEs. The existing multilevel numerical algorithm can be roughly classified as either adaptive Galerkin method or adaptive collocation method, and these methods have been shown to be very effective for PDEs, especially for linear PDEs[1,2,3,4,5,6,7,8,9,10,11].

This paper focuses on the multilevel iterative method (MIM). The MIM method uses an ascending sequence of nested finite dimensional spaces generated by special multiscale bases to approximate the target space, and then uses iterative techniques to solve the systems resulting from the multilevel discretization of PDEs. Many researches have shown that this method can provide an efficient, stable and accurate solver for PDEs, and its early version can be traced back to 1980s[12,13,14].

Multilevel iterative method that is generally used to approximate the solution of nonlinear PDEs often involves high computational cost. How to improve the computational efficiency of MIM gains much attention in recent years, not only in the engineering but also in the mathematics community. A challenging issue for solving nonlinear PDEs is the treatment of the nonlinear term, and this problem has long been recognized and discussed in many researches, see e,g,[15,16,17]. As an example, consider wavelet-based Galerkin discretization of the PDEs including the term $f(u)$, where $f(u)$ is a smooth but nonlinear function. It requires evaluating quantities of the inner $(f(u_J), \psi_I)$, where I, J are possibly different index sets. A transformation of u_J into single scale representation can completely waste complexity reduction gained by the sparse approximation in wavelet space, and a naive application of quadrature to the quantities $(f(u_J), \psi_I)$ would also severely spoil complexity gains, because some of the quadrature domain are comparable to the whole domain so that sufficient accuracy would require a computational expense of the order of the size of the problem, please see [18,19] for details.

Different from these existing researches, this paper tries to optimize the multilevel iterative algorithm through taking full advantage of multilevel approximation properties. Assume that X_J is the target space satisfying the accuracy requirement of the numerical solution, the solution u_j^* in $X_j\,(j < J)$ is only viewed as the

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approximation of the target solution u_j^*, the iterative solution $u_j^{(s_j)}$ associated with the iterative target solution u_j^* in coarse level X_j is in essence used as the initial guess of the iterative target solution u_{j+1}^* in refined level X_{j+1}. Since the different iterative targets in the multilevel iterative methods, one doesn't guarantee that an excellent coarse level solution provides better starting point than a bad one for the refined level iteration system. To improve computational efficiency, this paper tries to control the iterative times of iterative procedure by introducing a controlling parameter. Based on the related theoretical analysis, the scheme of the iterative time selection is proposed.

2 Multilevel iterative framework

To introduce ideas, consider the following operator equation

$$Lu + f(u) = g, \tag{1}$$

where $L : X \subset Y \to Y$ is a densely defined linear operator, and $f : X \to Y$ is a nonlinear operator. Usually Y is the certain function space equipped with the standard inner product $(\cdot)$ which naturally induces an associated norm $\|\cdot\|$. And X is a Hilbert space with respect to another inner product $(\cdot)_H$ with a stronger norm $\|\cdot\|_H$, i.e. there exists $c_0 > 0$ such that

$$\|u\| \le c_0\|u\|_H, \forall u \in X.$$

Define a bilinear form on $X \times X$:

$$a(u, v) := (Lu, v), u, v \in X,$$

then the weak form of the (1)

$$a(u, v) + (f(u), v) = (g, v), \forall v \in X. \tag{2}$$

Assume that there exists a nested finite- dimensional subspace sequence X_j satisfying

$$X_j \subset X_{j+1}, j \in N, \tag{3}$$

and

$$\overline{\sum_j X_j} = X. \tag{4}$$

Define for each $j \in N$ subspaces $W_{j+1} \subset X_{j+1}$ such that

$$X_{j+1} = X_j \oplus W_{j+1}, \tag{5}$$

where for subsets S_1 and S_2 of X, $S_1 \oplus S_2$ denotes the direct sum of S_1 and S_2 satisfying that for all $c \in S_1 \oplus S_2$, there exists $a \in S_1, b \in S_2$ such that $c = a + b$. Thus by setting $W_0 := X_0$, we have the multilevel space decomposition that for any $j \in N^+$

$$X_j = W_0 \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_j. \tag{6}$$

In this paper, we let N_j and n_j denote the dimension of X_j and W_j respectively, and ψ_{j,k_j} denote the basis of W_j. So $u_j \in X_j$ can be written as

$$u_j = \sum_{i=0}^{j} \sum_{k_i=1}^{n_i} u_{i,k_i}\psi_{i,k_i}(x). \tag{7}$$

Then for any $j \in N$, we obtain a nonlinear system which is the approximate form of (2)

$$a(u_j, v) + (f(u_j), v) = (g, v), \forall v \in X_j. \tag{8}$$

Note that this paper doesn't concerned with the specific details of (8), and always assume that certain conditions will ensure the existence and uniqueness of the solution of (8).

Let u^* and u_j^* respectively denote the exact solutions of (2) and (8). For each j, u_j^* is an approximate solution of u^*, and the solutions u_j^* is increasingly good approximation of u^* as j increases. Without loss of generality, assume that u_j^* is approximated by the following nonlinear iterative formulation

$$u_j^{(k+1)} = F_j(u_j^{(k)}), k \in N. \tag{9}$$

Then the multilevel iterative procedure can be described as follows

Step 1: Starting from an initial guess $u_0^{(0)} \in X_0$, the approximate solution $u_0^{(s_0)}$ of u_0^* is computed from (9) with iteration times s_0.

Step 2: for each $j \in N^+$, let $u_j^{(0)} = u_{j-1}^{(s_{j-1})}$, the approximate solution $u_j^{(s_j)}$ of u_j^* is computed from (9) with iteration times s_j.

The dimension of the space X_j increases geometrically while the scale j increase, this leads to that the computational expense of the nonlinear system (9) increases geometrically. Let X_J be the target space, then an important issue is to determine iteration times: $s_0, s_1, \cdots, s_J$.

Consider the nonlinear iterative formulation (9). Let

$$\delta_j = \|u_j^{(s_j)} - u_j^*\|_H, \tag{10}$$

generally, the error δ_j decreases while the iterative times s_j increases. It follows from (10) that

$$\|u_j^{(s_j)} - u^*\|_H \le \delta_j + \|u_j^* - u^*\|_H. \tag{11}$$

If the $u_j^{(s_j)}$ is used as the approximate solution of u^*, a small δ_j is generally preferred. Meanwhile it is easy to see that

$$\|u_{j+1}^0 - u_{j+1}^*\|_H = \|u_j^{(s_j)} - u_{j+1}^*\|_H \ge \|u_j^* - u_{j+1}^*\|_H - \delta_j. \tag{12}$$

Obviously, a smaller δ_j maybe lead to a larger error $\|u_{j+1}^0 - u_{j+1}^*\|_H$, this is why we try to control the iterative times $s_0, s_1, \cdots, s_J$. Our target is to avoid the unnecessary computation by controlling the iterative times.

3 Multilevel control iterative method

This section focuses on the choice of the iterative times of multilevel iteration method. Our scheme is based on two basic suppositions i.e. the inequalities (13) and (14), which are typical situations arising in multilevel numerical method.

$$N_j = \lambda N_{j-1},\ \lambda > 1,\ j \in N^+, \tag{13}$$

and

$$\|u_j^* - u^*\|_H \le C_1 N_j^{-q}, \tag{14}$$

where q is a positive number and can be viewed as the optimal error bound determined by the properties of multilevel analysis and equation (1).

As mentioned in the previous section, $u_j^{(s_j)}$ is used as the initial guess of the iterative target solution u_{j+1}^* and a small s_j is preferred. A naive idea is to control iterative times s_j by the following error inequality

$$\|u_j^{(s_j)} - u_j^*\|_H \le C_1 \varepsilon_j N_j^{-q},\ j \in N, \tag{15}$$

where $\varepsilon_j (\ge 1)$ is called the error controlling parameter. We expect to choose ε_j as big as possible such that s_j is as small as possible. However, this method would be very difficult for applications because so many parameters $\{\varepsilon_j\}$ need to be evaluated.

Now introduce a controlling parameter γ which satisfies

$$\frac{1}{\lambda^q} \le \gamma < 1, \tag{16}$$

and let

$$\varepsilon_0 = 1,\ \varepsilon_j = \gamma \lambda^q \varepsilon_{j-1},\ j \in N^+, \tag{17}$$

then (14) can be written as

$$\|u_j^* - u^*\|_H \le M_0 \gamma^j, \tag{18}$$

where $M_0 = C_1 N_0^{-q}$.

Theorem 3.1 Let

$$\mu = \frac{\gamma}{2+\gamma}. \tag{19}$$

If

$$\|u_0^{(s_0)} - u_0^*\| \le M_0 \tag{20}$$

and

$$\|u_j^{(s_j)} - u_j^*\|_H \le \mu \|u_j^{(0)} - u_j^*\|_H, \tag{21}$$

then for any $j \in N^+$

$$\|u_j^{(s_j)} - u_j^*\|_H \le M_0 \gamma^j. \tag{22}$$

Proof. Let $e_j = \|u_j^{(s_j)} - u_j^*\|_H$, we have from (18) and (21) that

$$\begin{aligned} e_j &\le \mu \|u_j^{(0)} - u_j^*\|_H \\ &\le \mu [\|u_{j-1}^{(s_{j-1})} - u_{j-1}^*\|_H + \|u_{j-1}^* - u^*\|_H + \|u_j^* - u^*\|_H] \\ &\le \mu [e_{j-1} + M_0(\gamma^{j-1} + \gamma^j)]. \end{aligned}$$

If $e_{j-1} \le M_0 \gamma^{j-1}$, we have from (19) that

$$\begin{aligned} e_j &\le \mu M_0 \gamma^{j-1}(2+\gamma) \\ &= M_0 \gamma^j. \end{aligned}$$

By utilizing the mathematical induction, we have that

$$e_j \le M_0 \gamma^j,\ \forall j \in N.$$

Thus, the inequality (22) holds, and the proof is completed.

For given the controlling parameter $\gamma \in (\lambda^{-q}, 1)$, the μ value in theorem 3.1 is immediately gained from (19). A key issue for multilevel control iterative method is to evaluate $\|u_j^{(s_j)} - u_j^*\|_H$ and find the iterative times s_j such that (21) holds, here s_j should be as small as possible.

Firstly, similar to the proof of Theorem 3.1, we have the following corollary:

Corollary 3.1 Under the suppositions of theorem 3.1

$$\|u_j^{(0)} - u_j^*\|_H \le \frac{M_0}{\mu} \gamma^j, \tag{23}$$

where $j \in N^+$ and $u_j^{(0)} = u_{j-1}^{s_{j-1}}$.

Assume the iterative formulation (9) has the p-th convergence order,i.e, there exists a positive constant α such that for any $k \in N$

$$\|u_j^{(k+1)} - u_j^*\|_H \le \alpha \|u_j^{(k)} - u_j^*\|_H^p. \tag{24}$$

It follows from (24) that

$$\|u_j^{(s)} - u_j^*\|_H \le \alpha^{\frac{p^s-1}{p-1}} \|u_j^{(0)} - u_j^*\|_H^{p^s}, s \in N^+. \tag{25}$$

From (23) to (25), we have that

$$\begin{aligned} \|u_j^{(s)} - u_j^*\|_H &\le \alpha^{\frac{p^s-1}{p-1}} \|u_j^{(0)} - u_j^*\|_H^{p^s-1} \cdot \|u_j^{(0)} - u_j^*\|_H \\ &\le \alpha^{\frac{p^s-1}{p-1}} (\frac{M_0}{\mu} \gamma^j)^{p^s-1} \cdot \|u_j^{(0)} - u_j^*\|_H. \end{aligned}$$

Thus, we have following theorem

Theorem 3.2 Under the suppositions of theorem 3.1, if the inequality (24) holds, then

$$\|u_j^{(s)} - u_j^*\|_H \le \widetilde{\mu}(j,\ s) \|u_j^{(0)} - u_j^*\|_H, \tag{26}$$

where $s \in N^+$ and

$$\widetilde{\mu}(j,\ s) = \alpha^{\frac{p^s-1}{p-1}} (\frac{M_0}{\mu} \gamma^j)^{p^s-1}. \tag{27}$$

From Theorem 3.1 and 3.2, one can determined the iterative time s_j by the following method

$$s_j = min\{s \in N^+ | \widetilde{\mu}(j,\ s) \le \mu\} \tag{28}$$

An interesting fact is that $\widetilde{\mu}(j,\, s)$ is monotonically decreasing with respect to the variable j, and the limit of $\widetilde{\mu}(j,\, s)$ is zero as j approaches ∞. From this fact we gain the following corollary:

Corollary 3.2 Assume the $s_j,\, j \in N^+$ are determined by (28), then

(i) $s_1 \geq s_2 \geq s_3 \cdots \geq s_j \cdots$.

(ii) there exists $M > 0$ such that $s_j = 1$ for any $j > M$.

The key issue of the multilevel control iterative method is to evaluate $\widetilde{\mu}(j, s)$. In (27), the parameters $\gamma,\, \mu$ and p are given, and in general $\alpha,\, M_0$ can be approximately evaluated according to the properties of the parameters. For example from (20), $\alpha,\, M_0$ can be approximated by

$$\alpha \approx \frac{\|u_j^s - u_j^{(s-1)}\|_H}{\|u_j^{(s)} - u_j^{(s-2)}\|_H^p}$$

and

$$M_0 \approx \|u_0^{(s_0)} - u_0^{(s_0-1)}\|_H.$$

Especially from the geometric meaning of the inequality (21), $\widetilde{\mu}(j, s)$ can be immediately approximated by

$$\widetilde{\mu}(j,s) \approx \frac{\|u_j^s - u_j^{s-1}\|_H}{\sum_{k=1}^{s} \|u_j^k - u_j^{k-1}\|_H}. \tag{29}$$

4 Numerical example

In this section, we demonstrate the performance of the proposed multilevel control iterative method by solving the following problem

$$\begin{cases} -\Delta u + u^2 = g(x,y), x \in \Omega, \\ u(x) = 0,\, x \in \partial\Omega, \end{cases} \quad \Omega = [0,\, 1] \times [0,\, 1], \tag{30}$$

where Δ is the 2th order Laplace operator, and g(x,y) is determined such that the (30) admits $u^*(x,\, y) = sin(\pi x) sin(\pi y)$ as the exact solution.

(30) will be solved by using the multilevel analysis of the space $H_0^1(\Omega)$. Consider quadrangular mesh partition of Ω with equidistant interval $1/2^{j+1}$. Let V_j denote the inner knots of the partition and ψ_p be the bilinear finite element basic function on the p point,then $V_j \subset V_{j+1}$. Let

$$X_j = span\{\psi_p : p \in V_j\}, \tag{31}$$

then $\{X_j\}$ is a nested finite-dimensional subspace sequence i.e.

$$X_0 \subset X_1 \subset \cdots \subset X_j \subset \cdots \subset H_0^1(\Omega). \tag{32}$$

Further let

$$W_j = span\{\psi_p : p \in V_j \setminus V_{j-1}\}, \tag{33}$$

we have that

$$X_j = W_0 \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_j, \tag{34}$$

Thus, we gain a multilevel analysis of the space $H_0^1(\Omega)$, in which the basic function of W_0 is as

$$\psi_0(x,y) = \begin{cases} 4xy,\, (x,\, y) \in [0, 1/2] \times [0.1/2], \\ 4(1-x)y,\, (x,\, y) \in [1/2, 1] \times [0.1/2], \\ 4(1-x)(1-y),\, (x,\, y) \in [1/2, 1] \times [1/2, 1], \\ 4x(1-y),\, (x,\, y) \in [0, 1/2] \times [1/2, 1]. \end{cases}$$

Similar to the classical wavelet analysis, the basis functions of $W_j (j > 0)$ are obtained by the dilation and translation of $\psi_0(x, y)$. Figure 1 gives the schematic diagram of basis functions of W_1.

Thus for given $j \in N$, the nonlinear system associated with (30) is obtained from the discrete scheme presented in section 2. The classical Newton iterative formulation is employed to solve these nonlinear systems, the program is run under Matlab 7.6.

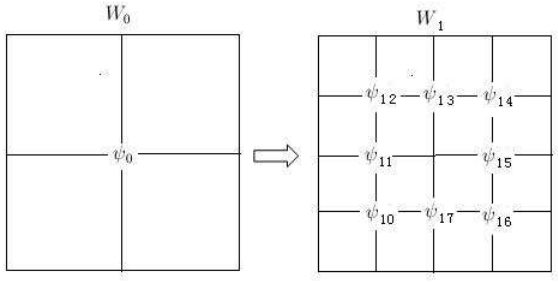

Figure 1: The schematic diagram of basis functions of W_1.

With the initial guess $u_1^{(0)} = 0$, we start the iterative procedure until $\|u_1^{(s)} - u_1^{(s-1)}\|_H \leq 1.0\, E-04$. We have that

$$\|u_1^{(6)} - u_1^{(5)}\|_H \cong 2.0573E-06,$$

where $\|\cdot\|_H$ is the $H^1(\Omega)$ norm. Note that here X_1 is viewed as the initial space W_0.

In what follows, start with $u_2^{(0)} = u_1^{(6)}$, the multilevel iterative procedures are respectively performed in three different control levels $\gamma = 0.95, \gamma = 0.7$ and $\gamma = 0.5$. Note that $s_{j+1} = 1$ woul be immediately chosen when $s_j = 2$, and the error is computed from

$$Error = \frac{\|u_j^{(s)} - u^*\|_H}{\|u^*\|_H},$$

where $u_j^{(s)}$ and u^* are the approximate solution and exact solution of (30), respectively. The results are shown in Table 1.

From Table 1, there doesn't exist significant difference in errors of three control levels. However, the

computational cost at $\gamma = 0.95$ or 0.7 is much less than at $\gamma = 0.5$. This shows the efficiency of the proposed method.

Table 1: The relative results when the multilevel iteration scheme with different controlling levels applied to (30).

γ	μ	Iter. Num.	Error	Time(s)
0.95	0.322	s_2=4,s_3=3, s_4=2,s_5=1	1.713E-04	128
0.70	0.259	s_2=5,s_3=3, s_4=2,s_5=1	1.313E-04	135
0.50	0.20	s_2=6,s_3=4, s_4=3,s_5=2	7.325E-05	239

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