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A new Definition of Fractional Derivative without Singular Kernel

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Abstract: In the paper, we present a new definition of fractional derivative with a smooth kernel which takes on two different representations for the temporal and spatial variable. The first works on the time variables; thus it is suitable to use the Laplace transform. The second definition is related to the spatial variables, by a non-local fractional derivative, for which it is more convenient to work with the Fourier transform. The interest for this new approach with a regular kernel was born from the prospect that there is a class of non-local systems, which have the ability to describe the material heterogeneities and the fluctuations of different scales, which cannot be well described by classical local theories or by fractional models with singular kernel.

Keywords: Fractional derivative, fading memory, thermodynamics.

1 Introduction

In the last decades the fractional calculus had a remarkable development as shown by the many mathematical volumes dedicated to it (e.g. Baleanu et al. [1], Caponetto [2], Caputo [3], Diethelm [4], Hilfer [5], Jiao et al [6], Kilbas et al. [7], Kyriakova [8], Mainardi [9], McBride [10], Miller and Ross [11], Petras [12], Samko et al [13], Podlubny [14], Sabatier et al. [15], Torres and Malinowska [16], Ying and Chen [17]) and by the notable diffusion as shown by the many meetings dedicated to it and the plethora of articles appeared in mathematical (e.g. Kilbas and Marzan [18], Heinsalu et al [19], Luchko and Gorenflo [20]) and non mathematical journals.

The use of derivative of fractional order has also spread into many other fields of science besides mathematics and physics (e.g. Laskin [21], Naber [22], Baleanu et al. [23], Zavada [24], Baleanu et al. [25], Caputo and Fabrizio [26],[27]) such as biology (e.g. Cesarone et al. [28], Caputo and Cametti [29]), economy (e.g. Caputo [30]), demography (e.g. Jumarie [31]), geophysics (e.g. Iaffaldano [32]), medicine (e.g El Sahed [33]) and bioengineering(e.g. Magin [34]). However some complaint has been made for the somewhat cumbersome mathematical expression of its definition and the consequent complications in the solutions of the fractional order differential equations.

In this note we suggest a new definition of fractional derivative, which assumes two different representations for the temporal and spatial variable. The first representation works on time variables, where the real powers appearing in the solutions of the usual fractional derivative will turn into integer powers, with some simplifications in the formulae and computations. In this framework, it is suitable to use the Laplace transform. The second representation is related to the spatial variables, thus for this non-local fractional derivative it is more convenient to work with the Fourier transform.

The interest for this new approach is due to the necessity of using a model describing the behavior of classical viscoelastic materials, thermal media, electromagnetic systems, etc. In fact, the original definition of fractional derivative appears to be particularly convenient for those mechanical phenomena, related with plasticity, fatigue, damage and with electromagnetic hysteresis. When these effects are not present it seems more appropriate to use the new fractional derivative.

We have also proposed a new non-local fractional derivative able to describe material heterogeneities and structures with different scales, which cannot be well described by classical local theories. So that, we rely that this spatial fractional

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derivative can play a meaningful role in the study of the macroscopic behaviors of some materials, related with nonlocal interactions, which are prevalent in determining the properties of the material.

This work also contains some applications and simulations related to the behavior of these new derivatives, applied to classical functions such as trigonometric functions. These simulations show some similarities with the corresponding results by usual fractional derivative.

2 A new fractional time derivative

Let us recall the usual Caputo fractional time derivative (UFD$_t$) of order α, given by

$$D_t^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{\dot{f}(\tau)}{(t-\tau)^\alpha} d\tau \tag{2.1}$$

with $\alpha \in [0,1]$ and $a \in [-\infty, t)$, $f \in H^1(a,b)$, $b > a$. By changing the kernel $(t-\tau)^{-\alpha}$ with the function $exp(-\frac{\alpha}{1-\alpha}t)$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$, we obtain the following new definition of fractional time derivative NFD$_t$

$$\mathscr{D}_t^{(\alpha)} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \dot{f}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{2.2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. According to the definition (2.2), the NFD$_t$ is zero when $f(t)$ is constant, as in the UFD$_t$, but, contrary to the UFD$_t$, the kernel does not have singularity for $t = \tau$.

The new NFD$_t$ can also be applied to functions that do not belong to $H^1(a,b)$. Indeed, the definition (2.2) can be formulated also for $f \in L^1(-\infty, b)$ and for any $\alpha \in [0,1]$ as

$$\mathscr{D}_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_{-\infty}^t (f(t) - f(\tau)) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau$$

Now, it is worth to observe that if we put

$$\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty] \quad , \quad \alpha = \frac{1}{1+\sigma} \in [0,1]$$

the definition (2.2) of NFD$_t$ assumes the form

$$\tilde{\mathscr{D}}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t \dot{f}(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau \tag{2.3}$$

where $\sigma \in [0,\infty]$ and $N(\sigma)$ is the corresponding normalization term of $M(\alpha)$, such that $N(0) = N(\infty) = 1$. Moreover, because

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{(t-\tau)}{\sigma}\right] = \delta(t-\tau) \tag{2.4}$$

and for $\alpha \to 1$, we have $\sigma \to 0$. Then (see [35] and [36])

$$\begin{aligned} \lim_{\alpha \to 1} \mathscr{D}_t^{(\alpha)} f(t) &= \lim_{\alpha \to 1} \frac{M(\alpha)}{1-\alpha} \int_a^t \dot{f}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\ &= \lim_{\sigma \to 0} \frac{N(\sigma)}{\sigma} \int_a^t \dot{f}(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = \dot{f}(t). \end{aligned} \tag{2.5}$$

Otherwise, when $\alpha \to 0$, then $\sigma \to +\infty$. Hence,

$$\begin{aligned} \lim_{\alpha \to 0} \mathscr{D}_t^{(\alpha)} f(t) &= \lim_{\alpha \to 0} \frac{M(\alpha)}{1-\alpha} \int_a^t \dot{f}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\ &= \lim_{\sigma \to +\infty} \frac{N(\sigma)}{\sigma} \int_a^t \dot{f}(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = f(t) - f(a). \end{aligned} \tag{2.6}$$

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Let us consider, the NFD$_t$ of a particular function, as $f(t) = \sin\omega t$, for $\alpha = 0.66$, $a = -8$ and $\omega = 1$

$$\mathscr{D}_t^{(0.66)} \sin\omega t = \frac{M(0.66)}{0.33} \int_a^t \cos\tau \exp -2(t-\tau) d\tau. \tag{2.7}$$

The simulation of this derivative produces the following pictures

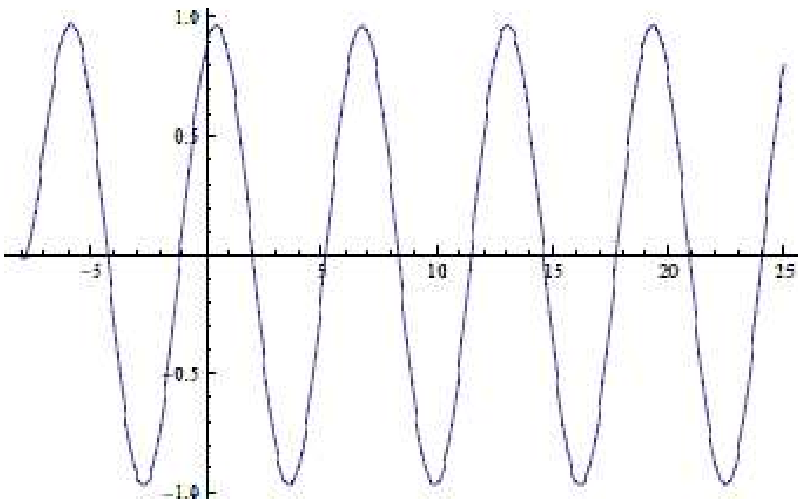

Fig.1. Simulation of NFD$_t$ (2.7), with $\alpha = 0.66$
in the time interval $[-8, 25]$

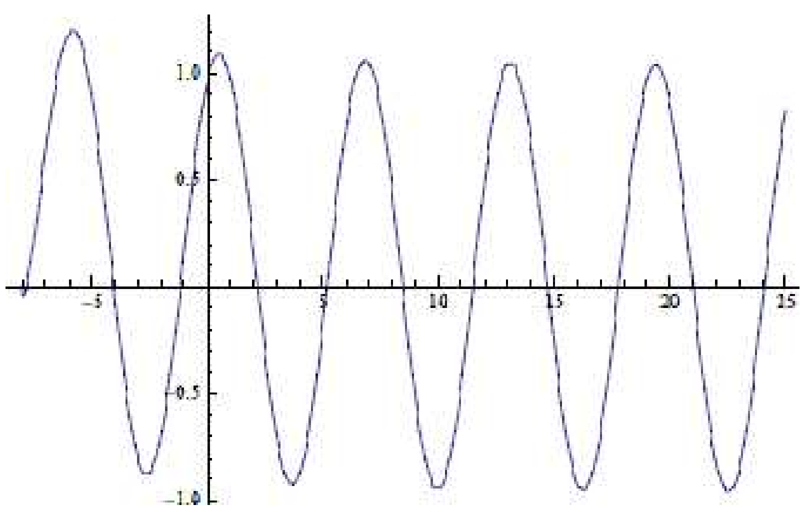

Fig.2. Simulation of UFD$_t$ (2.1), with $\alpha = 0.66$
in the time interval $[-8, 25]$

From these two simulations with $\alpha = 0.66$, it appears as the classical NFD$_t$ is very similar to the UFD$_t$.
Otherwise, when we study models with α close to 0, we see a different behavior

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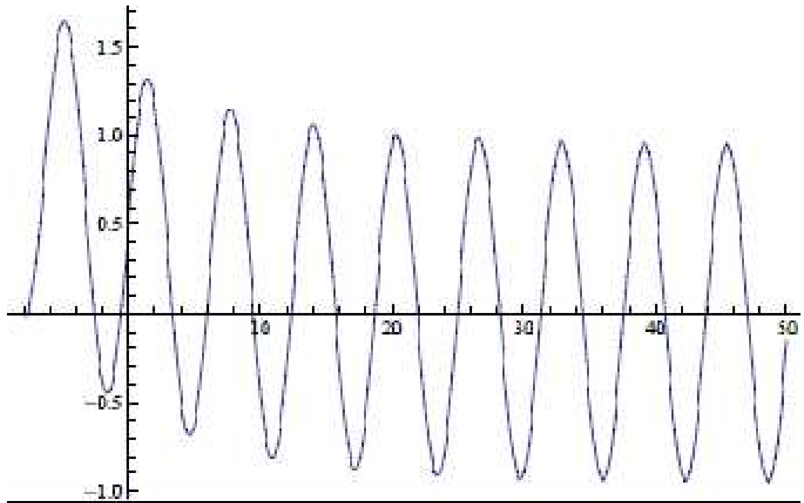

Fig.3. Simulation of NFD$_t$ (2.7) with $\alpha = 0.1$
in the time interval $[-8, 50]$

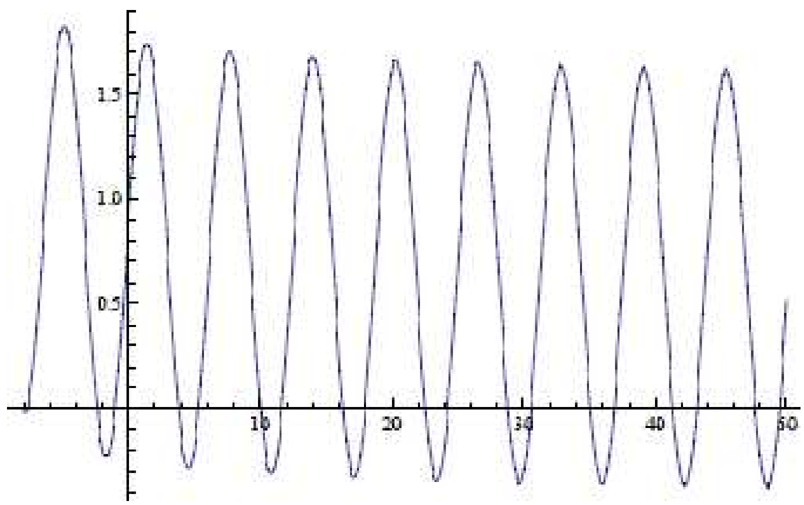

Fig.4. Simulation of UFD$_t$ (2.1), with $\alpha = 0.1$
in the time interval [-8, 50]

So that, for $\alpha = 0.1$ in Fig.3 and Fig. 4 we observe different actions between NFD$_t$ and UFD$_t$ simulations. In particular the classical UFD$_t$ is more affected by past, compared with the NFD$_t$ which show a rapid stabilization.

If $n \geq 1$, and $\alpha \in [0,1]$ the fractional time derivative $\mathscr{D}_t^{(\alpha+n)} f(t)$ of order $(n+\alpha)$ is defined by

$$\mathscr{D}_t^{(\alpha+n)} f(t) := D_t^{(\alpha)}(\mathscr{D}_t^{(n)} f(t)). \tag{2.8}$$

Theorem 1. *For NFD$_t$, if the function $f(t)$ is such that*

$$f^{(s)}(a) = 0\,, \quad s = 1,2,..,n$$

then, we have

$$\mathscr{D}_t^{(n)}(\mathscr{D}_t^{(\alpha)} f(t)) = \mathscr{D}_t^{(\alpha)}(\mathscr{D}_t^{(n)} f(t)) \tag{2.9}$$

Proof. We begin considering $n = 1$, then from definition (2.8) of $\mathscr{D}_t^{(\alpha+1)} f(t)$, we obtain

$$\mathscr{D}_t^{(\alpha)}\left(\mathscr{D}_t^{(1)} f(t)\right) = \frac{M(\alpha)}{1-\alpha}\int_a^t \ddot{f}(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau \tag{2.10}$$

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Hence, after an integration by parts and assuming $f'(a) = 0$, we have

$$\begin{aligned}\mathscr{D}_t^{(\alpha)}\left(\mathscr{D}_t^{(1)}f(t)\right) &= \frac{M(\alpha)}{(1-\alpha)}\int_a^t (\frac{d}{d\tau}\dot{f}(\tau))\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau = \\ &\frac{M(\alpha)}{(1-\alpha)}\left[\int_a^t \frac{d}{d\tau}(\dot{f}(\tau)\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right. \\ &\left.-\frac{\alpha}{1-\alpha}\int_a^t \dot{f}(\tau)\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right] \\ &= \frac{M(\alpha)}{(1-\alpha)}\left[\dot{f}(t)-\frac{\alpha}{1-\alpha}\int_a^t \dot{f}(\tau)\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right]\end{aligned} \tag{2.11}$$

otherwise

$$\begin{aligned}\mathscr{D}_t^{(1)}(\mathscr{D}_t^{(\alpha)}f(t)) &= \frac{d}{dt}(\frac{M(\alpha)}{1-\alpha}\int_a^t \dot{f}(\tau)\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau) = \\ &\frac{M(\alpha)}{1-\alpha}\left[\dot{f}(t)-\frac{\alpha}{1-\alpha}\int_a^t \dot{f}(\tau)\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right].\end{aligned} \tag{2.12}$$

It is easy to generalize the proof for any $n > 1$.

In the following, we suppose the function $M(\alpha) = 1$.

3 The Laplace transform of the NFD$_t$

In order to study the properties of the NFD$_t$, defined in equation (2.3) with $a = 0$, has priority the computation of its Laplace transform (LT) given with p variable

$$LT\left[\mathscr{D}_t^{(\alpha)}f(t)\right] = \frac{1}{(1-\alpha)}\int_0^\infty \exp-pt\int_0^t \dot{f}(\tau)\exp-\frac{\alpha(t-\tau)}{1-\alpha}d\tau dt$$

Hence, from the property of Laplace transform of a convolution, we have

$$\begin{aligned}LT\left[\mathscr{D}_t^{(\alpha)}f(t)\right] &= \frac{1}{(1-\alpha)}LT(\dot{f}(t))LT(\exp-\frac{\alpha t}{1-\alpha}) = \\ &\frac{(pLT(f(t)-f(0))}{p+\alpha(1-p)}\end{aligned}$$

Similarly,

$$\begin{aligned}LT\left[\mathscr{D}_t^{(\alpha+1)}f(t)\right] &= \frac{1}{(1-\alpha)}LT(\ddot{f}(t))LT(\exp-\frac{\alpha t}{1-\alpha}) = \\ &= \frac{(p^2LT[f(t)]-pf(0)-f'(0))}{p+\alpha(1-p)}\end{aligned}$$

Finally,

$$\begin{aligned}LT\left[\mathscr{D}_t^{(\alpha+n)}f(t)\right] &= \frac{1}{1-\alpha}LT\left[f^{(n+1)}(t)\right]LT\left[\exp-\frac{\alpha t}{1-\alpha}\right] = \\ &= \frac{p^{n+1}LT[f(t)]-p^nf(0)-p^{n-1}f'(0)\ldots-f^{(n)}(0)}{p+\alpha(1-p)}\end{aligned}$$

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4 Fractional gradient operator

In this section, we introduce a new notion of fractional gradient able to describe non-local dependence in constitutive equations (see [37] and [38]).

Let us consider a set $\Omega \in \mathbb{R}^3$ and a scalar function $u(\cdot):\Omega \to \mathbb{R}$, we define the fractional gradient of order $\alpha \in [0,1]$ as follows

$$\nabla^{(\alpha)}u(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \tag{4.1}$$

with $\mathbf{x},\mathbf{y} \in \Omega$.

It is simple to prove from definition (4.1) and by the property

$$\lim_{\alpha\to 1} \frac{\alpha}{(1-\alpha)\sqrt{\pi}} \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] = \delta(\mathbf{x}-\mathbf{y})$$

that

$$\nabla^{(1)}u(\mathbf{x}) = \nabla u(\mathbf{x}) \text{ and } \nabla^{(0)}u(\mathbf{x}) = \int_\Omega \nabla u(\mathbf{y}) d\mathbf{y}$$

So, when $\alpha = 1$, $\nabla^{(1)}u(\mathbf{x})$ loses the non-locality, otherwise $\nabla^{(0)}u(\mathbf{x})$ is related with the mean value of $\nabla u(\mathbf{y})$ on Ω.

In the case of a vector $\mathbf{u}(\mathbf{x})$, we define the fractional tensor by

$$\nabla^{(\alpha)}\mathbf{u}(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \tag{4.2}$$

Thus, a material with non-local property may be described by fractional constitutive equations. As an example we consider an elastic non-local material, defined by the following constitutive equation between the stress tensor $\mathbf{T}$ and $\nabla^{(\alpha)}\mathbf{u}(\mathbf{x})$

$$\mathbf{T}(\mathbf{x},t) = \mathbf{A}\nabla^{(\alpha)}\mathbf{u}(\mathbf{x},t)\ ,\ \alpha \in (0,1]$$

where $\mathbf{A}$ is a fourth order symmetric tensor, or in the integral form

$$\mathbf{T}(\mathbf{x},t) = \frac{\alpha \mathbf{A}}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}$$

Likewise, we introduce the fractional divergence, defined for a smooth $\mathbf{u}(\cdot):\Omega \to \mathbb{R}^3$ by

$$\nabla^{(\alpha)}\cdot\mathbf{u}(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla\cdot \mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \tag{4.3}$$

Theorem 2.*From definitions (4.1) and (4.3), we have for any $u(\mathbf{x}):\Omega \to \mathbb{R}$, such that*

$$\nabla u(\mathbf{x})\cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{4.4}$$

the following identity

$$\nabla \cdot \nabla^{(\alpha)}u(\mathbf{x}) = \nabla^{(\alpha)}\cdot \nabla u(\mathbf{x}) \tag{4.5}$$

*Proof.*By means of (4.1), we obtain

$$\begin{aligned}
\nabla\cdot\nabla^{(\alpha)}u(\mathbf{x}) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(\mathbf{y})\cdot\nabla_{\mathbf{x}} \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \\
&= -\frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(\mathbf{y})\cdot\nabla \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \\
&= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla\cdot\nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} - \\
&\quad -\frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_{\partial\Omega} \nabla u(\mathbf{y})\cdot\mathbf{n} \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}
\end{aligned} \tag{4.6}$$

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hence, from the boundary condition (4.4), the identity (4.5) is proved, because (4.6) coincides with

$$\nabla^{(\alpha)} \cdot \nabla u(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla \cdot \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}$$

5 Fourier transform of fractional gradient and divergence

For a smooth function $u(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}$ the Fourier transform (FT) of the fractional gradient is defined by

$$FT(\nabla^{(\alpha)} u(\mathbf{x}))(\xi) = \int_{\mathbb{R}^3} \nabla^{(\alpha)} u(\mathbf{x}) \exp\left[-2\pi i \xi \cdot \mathbf{x}\right] \, d\mathbf{x}$$

Thus, if we consider the gradient of (4.1), the Fourier transform is given by

$$FT(\nabla^{\alpha} u)(\xi) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT(\int_{\mathbb{R}^3} \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y})(\xi) =$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT(\nabla u)(\xi) FT(\exp\left[-\frac{\alpha^2 \mathbf{x}^2}{(1-\alpha)^2}\right])(\xi)$$

where

$$FT(\exp\left[-\frac{\alpha^2 \mathbf{x}^2}{(1-\alpha)^2}\right])(\xi) = \frac{(1-\alpha)\sqrt{\pi}}{\alpha} \exp\left[-\frac{\pi^2(1-\alpha)^2\xi^2}{\alpha^2}\right].$$

Thus, we obtain:

$$FT(\nabla^{\alpha} u)(\xi) = \sqrt{\pi^{1-\alpha}} FT(\nabla u)(\xi) \exp\left[-\frac{\pi^2(1-\alpha)^2\xi^2}{\alpha^2}\right]$$

The Fourier transform of fractional divergence is defined by

$$FT(\nabla^{\alpha} \cdot \mathbf{u})(\xi) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT(\int_{\Omega} \nabla \cdot \mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y})(\xi)$$

from which we have

$$FT(\nabla^{\alpha} \cdot \mathbf{u})(\xi) = \sqrt{\pi^{1-\alpha}} FT(\nabla \cdot \mathbf{u})(\xi) \exp\left[-\frac{\pi^2(1-\alpha)^2\xi^2}{\alpha^2}\right]$$

6 Fractional Laplacian

In the study of partial differential equations, there is a great interest on fractional Laplacian. Using the definitions of fractional gradient and divergence, we can suggest the representation of fractional Laplacian for a smooth function $f(\mathbf{x})$: $\Omega \to \mathbb{R}^3$, such that $\nabla f(\mathbf{x}) \cdot \mathbf{n}|_{\partial\Omega} = 0$, as

$$(\nabla^2)^{\alpha} f(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla \cdot \nabla f(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}$$

By Theorem 2.1, we have

$$(\nabla^2)^{\alpha} f(\mathbf{x}) = \nabla \cdot \nabla^{\alpha} f(x) = \nabla^{\alpha} \cdot \nabla f(x)$$

Now, we suppose that

$$f(x) = 0\,, \quad on\ \partial\Omega$$

then we extend the function $f(x) = 0$ on $\mathbb{R}^3 \backslash \Omega$. So, we consider the Fourier transform

$$\begin{aligned} FT((\nabla^2)^{\alpha} f(\mathbf{x})) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT(\int_{\mathbb{R}^3} \nabla^2 f(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y})(\xi) \\ &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT(\nabla \cdot \nabla f(\mathbf{x}))(\xi) FT(\exp\left[-\frac{\alpha^2\mathbf{x}^2}{(1-\alpha)^2}\right])(\xi) \\ &= 4\pi|\xi|^2 FT(f(\mathbf{x}))(\xi)\sqrt{\pi^{1-\alpha}} \exp\left[-\frac{(1-\alpha)^2\xi^2}{\alpha^2}\right] \end{aligned} \tag{6.1}$$

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Finally, if $\alpha = 1$ we obtain from (6.1)

$$FT(\nabla^2 f(\mathbf{x})) = -\lim_{\alpha \to 1} 4\pi |\xi|^2 FT(f(\mathbf{x}))(\xi)\sqrt{\pi^{1-\alpha}} \exp\left[-\frac{(1-\alpha)^2\xi^2}{\alpha^2}\right] =$$

$$= -4\pi |\xi|^2 LT(f(\mathbf{x}))(\xi)$$

7 The memory operators

The fractional derivatives are memory operators which usually represent dissipation of energy (see [39], [9], [14]) or damage (see [26]) in the medium as in the case of anelastic media or reassessment of the porosity in the diffusion in porous media. Moreover, in general they are in agreement with the Second principle of thermodynamics [39] and [40].

They are accepted not only because they represent appropriately a variety of phenomena, but also because they have the "elegant and rigorous property" that when the order of differentiation is integer, they coincide with the classic derivative of that order. However this property is not relevant to the effect they represent in the physical phenomena and one wonders if using other differential operators, possibly simpler but without this property, one may obtain the same results of the fractional derivatives.

7.1 The response to a linear trend

The effects of the fractional memory formalism (SMFP) by the new fractional derivative (NFD_t), compared with the Caputo derivative (UFD_t) on a linear trend, are readily obtained from their definitions applied to the simple linear function as in formulae (2.2) and (2.3). We find the results illustrated in the following figures.

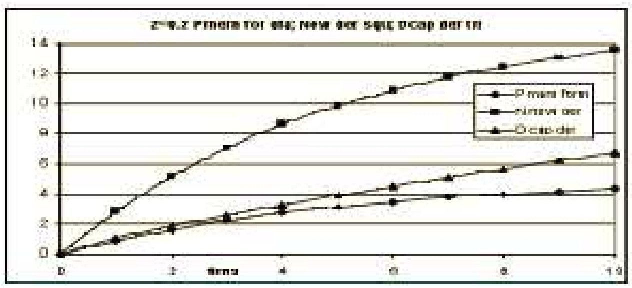

Fig.5. Deformation of a linear trend caused by the SMFP,
the NFD_t and the Caputo derivative in the case when the order of differentiation is 0.2.

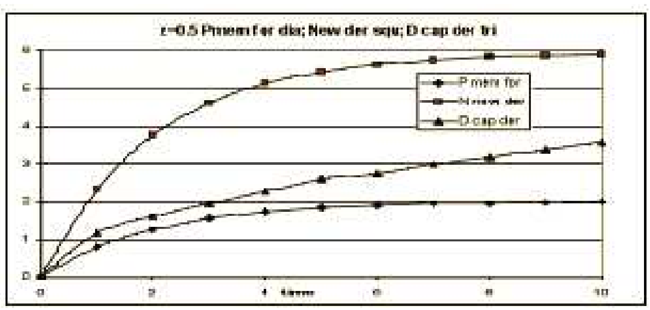

Fig.6. Deformation of a linear trend caused by the SMFP,
the NFD_t and the Caputo derivative in the case when the order of differentiation is 0.5.

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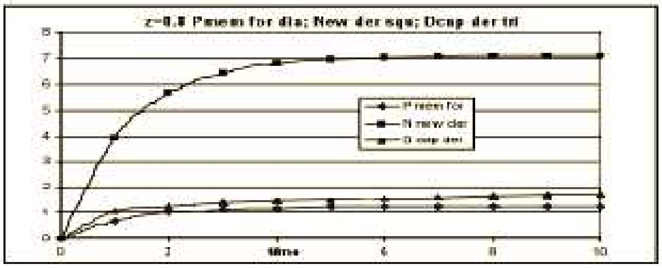

Fig.7. Deformation of a linear trend caused by the SMFP,
the NFD_t and the Caputo derivative in the case, when the order of differentiation is 0.8.

In the following Figs. 8, 9 and 10, we show the Fourier spectra, frequency response curves of the SMFP, the NFD_t and of the Caputo derivative, concerning the LT domain response functions.

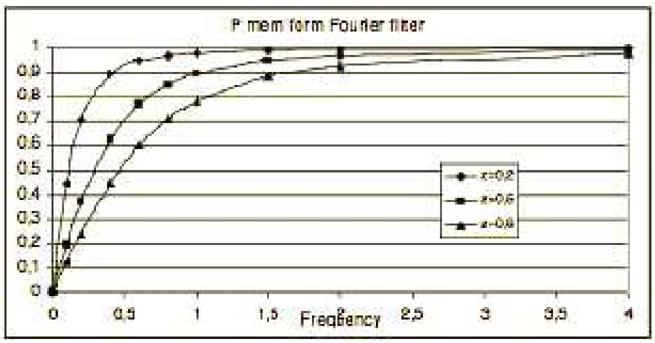

Fig.8.Laplace transform response of the memory formalism (SMFP)
for the orders of differentiation z = 0.2, z = 0.5 and z = 0.8.

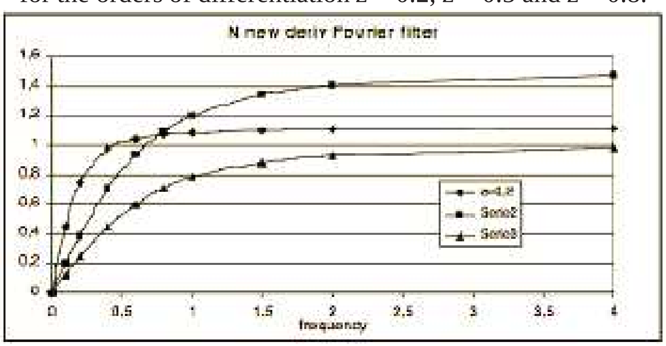

Fig.9. Laplace transform response of the memory formalism (NFD)
for the orders of differentiation z = 0.2, z = 0.5 and z = 0.8.

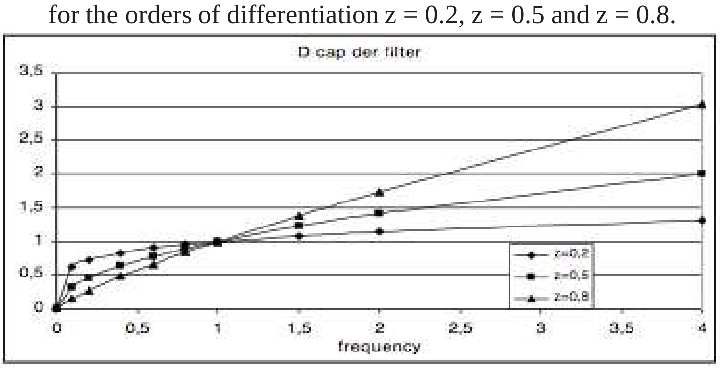

Fig.10. Laplace transform response of the memory formalism (UFD)
for the orders of differentiation z = 0.2, z = 0.5 and z = 0.8.

We note in the figures 8 and 9 the asymptotic behavior of the memory formalism and of the new derivative, while that the curves of the Caputo derivative, for the larger values of the variable, are linearly increasing and diverging.

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7.2 The distributed order of the memory operator SMFP $P^{(z)}$

The simple definition of the memory operator SMFP readily allows a definition of the distributed order fractional memory operator, which is simpler and easier to handle that the Caputo derivative (see [41])

The distributed order operator $P^{(z)}$ is defined for the fractional derivative of Caputo [3], [42], by

$$ {}_aP_b^{(\alpha)} f(t) =) \int_a^b g(\alpha) d\alpha \left[D_t^{(\alpha)} f(t) \right] =_a P_b^{(\alpha)} \int_a^b g(\alpha) d\alpha \left[\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau)) \dot{f}(\tau) d\tau \right] \tag{7.1} $$

where $g(\alpha)$ is a weight function and $0 < a < b < 1$.

Following the method of Caputo [3], [42] is readily seen that for the Fubini-Tonelli theorem, we may change the order of integration in $d\alpha$ and dt provided

$$ \int_a^b g(\alpha) d\alpha \left[\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau)) f^{(m+1)}(\tau) d\tau \right] $$

is summable with respect to τ in the interval $[a,b]$ with $0 < a < b < 1$, which is readily verified.

The solution if found using the LT of (7.1) which is

$$ LT_aP_b^{(\alpha)} f(t) = \int_0^\infty \int_a^b g(\alpha) d\alpha \left[D_t^{(\alpha)} f(t) \right] \exp(-pt) dt = $$
$$ = \int_0^\infty \int_a^b g(\alpha) d\alpha \left[\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau)) \dot{f}(\tau) d\tau \right] \exp(-pt) dt \tag{7.2} $$

or

$$ LT_aP_b^{(\alpha)} f(t) = \int_a^b \left\{ \int_0^\infty \left[\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau)) \dot{f}(\tau) d\tau \right] \exp(-pt) dt \right\} g(\alpha) d\alpha $$

and finally obtain

$$ LT_aP_b^{(\alpha)} f(t) = \int_a^b \frac{p}{p+\alpha} F(p) d\alpha = F(p) \int_a^b \frac{pg(\alpha)}{\log(p+\alpha)} d\alpha \tag{7.3} $$

which represents the filtering properties of the operator and is simpler than that obtained using the Caputo derivative. As an example we may consider the simple case $g(\alpha) = 1$ which gives

$$ LT_aP_b^{(\alpha)} f(t) = pF(p) \int_a^b \frac{g(\alpha)}{p+\alpha} d\alpha = pF(p) \log \frac{p+b}{p+a} $$

hence, the response is

$$ p \log \frac{p+b}{p+a} $$

whose filtering properties are readily computed.

Other cases of practical interest, as in the note of Caputo [42], may be considered such as when $g(\alpha)$ is a linear function of α. Also the distributed order SFDF could be readily formulated, however its expression is somewhat complicated and we believe that it may be of scarce practical use.

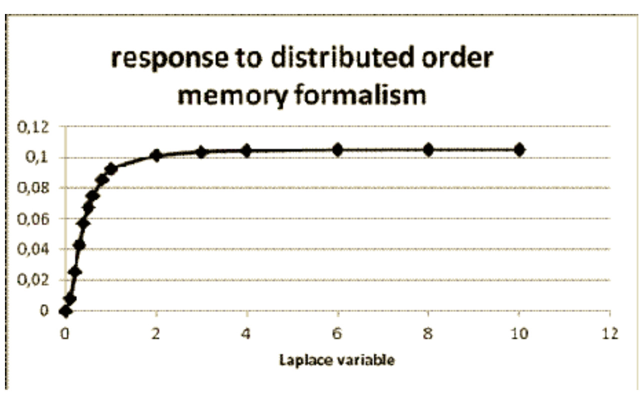

Fig 11. LT domain response function of the distributed order fractional derivative with b = 0.8, a = 0.3.

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8 Appendix

1 - We rewrite the definition (2.3) in the new form

$$\tilde{\mathscr{D}}_t^{(\nu)} f(t) = V(\nu) \int_a^t \dot{f}(\tau) \exp -\nu(t-\tau) d\tau \tag{8.1}$$

obtained from equation (2.2) or (2.3) with $\nu = \frac{1}{\sigma} > 0$, where $V(\nu) = .\nu N(1/\nu)$.
Then, we have

Theorem 3. *If the function $f \in W^{1,1}(a,b)$, then the integral in (8.1) exists for $t \in [a,b]$ and $\tilde{\mathscr{D}}_t^{(\nu)} f(t) \in L^1[a,b]$.*

Proof. Let us write

$$\begin{aligned}\tilde{\mathscr{D}}_t^{(\nu)} f(t) &= V(\nu) \int_a^t \dot{f}(\tau) \exp -\nu(t-\tau) d\tau = \\ &= V(\nu) \int_{-\infty}^{\infty} p_\nu(t-s) q(s) ds\end{aligned} \tag{8.2}$$

where $p_\nu(y) = \exp(-\nu y)$, when $0 < y < b-a$, with $p(y) = 0$ when $y < 0$ or $y > b-a$, $q(y) = \dot{f}(y)$ when $a \le y \le b$. Finally, $q(y) = 0$ when $y < a$ or $y > b$. Hence under the hypotheses of the theorem, the functions p_ν, $q \in L^1(a,b)$. Then, by the classical results on Lebesgue integrals (see [41]), the integral (8.1) exists almost everywhere in $t \in [a,b]$ and $\tilde{\mathscr{D}}_t^{(\nu)} f(t) \in L^1[a,b]$.

2 - It is of some interest to see the fractional derivatives of the elementary and transcendental functions according to the new definition (2.2). We begin with $\sin \omega t$, whose fractional derivative is given by

$$\mathscr{D}_t^{(\alpha)}(\sin \omega t) = E(\alpha) \int_0^t \omega \exp(-\frac{\alpha}{1-\alpha}(t-s)) \cos \omega s \, ds$$

where $E(\alpha) = \frac{M(\alpha)}{1-\alpha}$. Then

$$\begin{aligned}\mathscr{D}_t^{(\alpha)}(\sin \omega t) &= E(\alpha)\omega \exp\left[-\frac{\alpha}{1-\alpha}t\right] \int_0^t \exp(\frac{\alpha}{1-\alpha}s) \cos \omega s \, ds = \\ &= \frac{E(\alpha)\omega}{(\frac{\alpha}{1-\alpha})^2 + \omega^2}(\frac{\alpha}{1-\alpha} \cos \omega t + \omega \sin \omega t - \frac{\alpha}{1-\alpha} \exp(-\frac{\alpha}{1-\alpha}t)) = \\ &= \frac{E(\alpha)\omega}{((\frac{\alpha}{1-\alpha})^2 + \omega^2)}(((\frac{\alpha}{1-\alpha})^2 + \omega^2)^{0.5} \sin(\omega t + a) - \frac{\alpha}{1-\alpha} \exp\left[-\frac{\alpha}{1-\alpha}t\right]) = \\ &= E(\alpha) \cos a (\sin(\omega t + a) - \sin a \exp\left[-\frac{\alpha}{1-\alpha}t\right])\end{aligned} \tag{8.3}$$

where a is such that

$$\tan a = \frac{\alpha\omega}{(1-\alpha)}, \ \sin a = \frac{\alpha/1-\alpha}{((\alpha/1-\alpha))^2 + \omega^2)^{0.5}}, \ \cos a = \frac{\omega}{((\alpha/1-\alpha))^2 + \omega^2)^{0.5}}$$

We note that the new derivative of $\sin \omega t$ implies only a change of the phase a and the amplitude variation $\frac{\omega E(\alpha)}{((\frac{\alpha}{1-\alpha})^2 + \omega^2)^{0.5}}$

Now we see $\mathscr{D}_t^{(\alpha)}(\cos \omega t)$. With the same procedure we find

$$\begin{aligned}\mathscr{D}_t^{(\alpha)}(\cos \omega t) &= E(\alpha) \exp(-\frac{\alpha}{1-\alpha})t \int_0^t \exp(\frac{\alpha}{1-\alpha}s) \sin \omega s ds = \\ &= E(\alpha) \cos a \left[\cos(\omega t + b) - \cos a \exp(-\frac{\alpha}{1-\alpha}t)\right]\end{aligned}$$

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where again we note the same phase change and amplitude variation noted for the case of $\sin \omega t$. Moreover, we observe that b is related to a by

$$\tan a = \frac{1}{\tan b}$$

Hence, we consider $\mathscr{D}_t^{(\alpha)}(\exp \omega t)$, then we find

$$\mathscr{D}_t^{(\alpha)}(\exp \omega t) = \frac{E(\alpha)\omega}{\frac{\alpha}{1-\alpha}+\omega}\left\{\exp(\omega t) - \exp\left[-\frac{\alpha}{1-\alpha}t\right]\right\} =$$

$$\frac{E(\alpha)\omega}{\frac{\alpha}{1-\alpha}+\omega}\exp(\omega t)\left\{1 - \exp-(\omega + \frac{\alpha}{1-\alpha}t)\right\}$$

Finally

$$\mathscr{D}_t^{(\alpha)} t = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))ds =$$

$$\frac{M(\alpha)}{\alpha}(1 - \exp - \frac{\alpha}{1-\alpha}t). \;\; 0 < \alpha \leq 1$$

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