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The k-Metric Dimension of a Graph

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Abstract: As a generalization of the concept of a metric basis, this article introduces the notion of k-metric basis in graphs. Given a connected graph $G=(V,E)$, a set $S\subseteq V$ is said to be a k-metric generator for G if the elements of any pair of different vertices of G are distinguished by at least k elements of S, *i.e.*, for any two different vertices $u,v\in V$, there exist at least k vertices $w_1,w_2,\dots,w_k\in S$ such that $d_G(u,w_i)\neq d_G(v,w_i)$ for every $i\in\{1,\dots,k\}$. A k-metric generator of minimum cardinality is called a k-metric basis and its cardinality the k-metric dimension of G. A connected graph G is *k-metric dimensional* if k is the largest integer such that there exists a k-metric basis for G. We give a necessary and sufficient condition for a graph to be k-metric dimensional and we obtain several results on the r-metric dimension, $r\in\{1,\dots,k\}$.

Keywords: k-metric generator; k-metric dimension; k-metric dimensional graph; metric dimension; resolving set; locating set; metric basis

1 Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater in [19, 20], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [9], where metric generators were called resolving sets.

Nevertheless, the concept of a metric generator, in its primary version, has a weakness related with the possible uniqueness of the vertex identifying a pair of different vertices of the graph. Consider, for instance, some robots which are navigating, moving from node to node of a network. On a graph, however, there is neither the concept of direction nor that of visibility. We assume that robots have communication with a set of landmarks S (a subset of nodes) which provide them the distance to the landmarks in order to facilitate the navigation. In this sense, one aim is that each robot is uniquely determined by the landmarks. Suppose that in a specific moment there are two robots x,y whose positions are only distinguished by one landmark $s\in S$. If the communication between x and s is unexpectedly blocked, then the robot x will get lost in the sense that it can assume that it has the position of y. So, for a more realistic settings it could be desirable to consider a set of landmarks where each pair of nodes is distinguished by at least two landmarks.

A natural solution regarding that weakness is the location of one landmark in every node of the graph. But, such a solution, would have a very high cost. Thus, the choice of a correct set of landmarks is convenient for a satisfiable performance of the navigation system. That is, in order to achieve a reasonable efficiency, it would be convenient to have a set of as few landmarks as possible, always having the guarantee that every object of the network will be properly distinguished.

From now on we consider a simple and connected graph $G=(V,E)$. It is said that a vertex $v\in V$ distinguishes two different vertices $x,y\in V$, if $d_G(v,x)\neq d_G(v,y)$, where $d_G(a,b)$ represents the length of a shortest $a-b$ path. A set $S\subseteq V$ is a *metric generator* for G if any pair of different vertices of G is distinguished by some element of S. Such a name for S raises from the concept of *generator* of metric spaces, that is, a set S of points in the space with the property that every point of the space is uniquely determined by its "distances" from the elements of S. For our specific case, in a simple and connected graph $G=(V,E)$, we consider the metric $d_G: V\times V\rightarrow \mathbb{N}\cup\{0\}$, where $d_G(x,y)$ is defined as

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mentioned above and $\mathbb{N}$ is the set of positive integers. With this metric, (V,d_G) is clearly a metric space. A metric generator of minimum cardinality is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

Other useful terminology to define the concept of a metric generator in graphs is given at next. Given an ordered set $S=\{s_1,s_2,\ldots,s_d\}\subset V(G)$, we refer to the d-vector (ordered d-tuple) $r(u|S) = (d_G(u,s_1), d_G(u,s_2),\ldots,d_G(u,s_d))$ as the *metric representation* of u with respect to S. In this sense, S is a metric generator for G if and only if for every pair of different vertices u,v of G, it follows $r(u|S)\neq r(v|S)$.

In order to avoid the weakness of metric basis described above, from now on we consider an extension of the concept of metric generators in the following way. Given a simple and connected graph $G=(V,E)$, a set $S\subseteq V$ is said to be a *k-metric generator* for G if and only if any pair of different vertices of G is distinguished by at least k elements of S, *i.e.*, for any pair of different vertices $u,v\in V$, there exist at least k vertices $w_1,w_2,\ldots,w_k\in S$ such that

$$d_G(u,w_i)\neq d_G(v,w_i),\ \text{for every } i\in\{1,\ldots,k\}. \quad (1)$$

A k-metric generator of the minimum cardinality in G will be called a *k-metric basis* and its cardinality the *k-metric dimension* of G, which will be denoted by $\dim_k(G)$.

As an example we take the cycle graph C_4 with vertex set $V=\{x_1,x_2,x_3,x_4\}$ and edge set $E=\{x_ix_j : j-i=1 \pmod 2\}$. We claim that $\dim_2(C_4)=4$. That is, if we take the pair of vertices x_1,x_3, then they are distinguished only by themselves. So, x_1,x_3 must belong to every 2-metric generator for C_4. Analogously, x_2,x_4 also must belong to every 2-metric generator for C_4. Other example is the graph G in Figure 1, for which $\dim_2(G)=4$. To see this, note that v_3 does not distinguish any pair of different vertices of $V(G)-\{v_3\}$ and for each pair v_i,v_3, $1\leq i\leq 5, i\neq 3$, there exist two elements of $V(G)-\{v_3\}$ that distinguish them. Hence, v_3 does not belong to any 2-metric basis for G. To conclude that $V(G)-\{v_3\}$ must be a 2-metric basis for G we proceed as in the case of C_4.

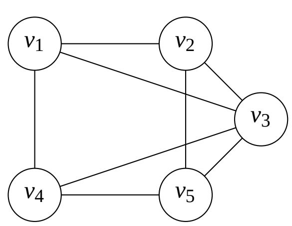

Fig. 1: A graph G where $V(G)-\{v_3\}$ is a 2-metric basis for G.

Note that every k-metric generator S satisfies that $|S|\geq k$ and, if $k>1$, then S is also a $(k-1)$-metric generator. Moreover, 1-metric generators are the standard metric generators (resolving sets or locating sets as defined in [9] or [19], respectively). Notice that if $k=1$, then the problem of checking if a set S is a metric generator reduces to check condition (1) only for those vertices $u,v\in V-S$, as every vertex in S is distinguished at least by itself. Also, if $k=2$, then condition (1) must be checked only for those pairs having at most one vertex in S, since two vertices of S are distinguished at least by themselves. Nevertheless, if $k\geq 3$, then condition (1) must be checked for every pair of different vertices of the graph.

The literature about metric dimension in graphs shows several of its usefulness, for instance, applications to the navigation of robots in networks are discussed in [13] and applications to chemistry in [11, 12], among others. This invariant was studied further in a number of other papers including [1, 3–5, 7, 8, 10, 16, 17, 21–23]. Several variations of metric generators including resolving dominating sets [2], independent resolving sets [6], local metric sets [16], and strong resolving sets [14, 15, 18], etc. have been introduced and studied. It is therefore our goal to introduce this extension of metric generators in graphs as a possible future tool for other possibly more general variations of the applications described above.

We introduce now some other more necessary terminology for the article and the rest of necessary concepts will be introduced the first time they are mentioned in the work. We will use the notation K_n, $K_{r,s}$, C_n, N_n and P_n for complete graphs, complete bipartite graphs, cycle graphs, empty graphs and path graphs, respectively. If two vertices u,v are adjacent in $G=(V,E)$, then we write $u\sim v$ or we say that $uv\in E(G)$. Given $x\in V(G)$ we define $N_G(x)$ to be the *open neighbourhood* of x in G. That is, $N_G(x)=\{y\in V(G) : x\sim y\}$. The *closed neighbourhood*, denoted by $N_G[x]$, equals $N_G(x)\cup\{x\}$. If there is no ambiguity, we will simply write $N(x)$ or $N[x]$. We also refer to the degree of v as $\delta(v)=|N(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a non-empty set $S\subseteq V(G)$, and a vertex $v\in V(G)$, $N_S(v)$ denotes the set of neighbors that v has in S, *i.e.*, $N_S(v)=S\cap N(v)$.

2 k-metric dimensional graphs

It is clear that it is not possible to find a k-metric generator in a connected graph G for every integer k. That is, given a connected graph G, there exists an integer t such that G does not contain any k-metric generator for every $k>t$. According to that fact, a connected graph G is said to be a *k-metric dimensional graph*, if k is the largest integer such that there exists a k-metric basis for G. Notice that, if G is a k-metric dimensional graph, then for every positive integer $k'\leq k$, G has at least a k'-metric basis. Since for every pair of different vertices x,y of a graph G we have that they are distinguished at least by themselves, it follows that the whole vertex set $V(G)$ is a 2-metric generator for

G and, as a consequence it follows that every graph G is k-metric dimensional for some $k \geq 2$. On the other hand, for any connected graph G of order $n > 2$ there exists at least one vertex $v \in V(G)$ such that $\delta(v) \geq 2$. Since v does not distinguish any pair of different neighbours $x,y \in N_G(v)$, there is no n-metric dimensional graph of order $n > 2$.

Remark 1 *Let G be a k-metric dimensional graph of order n. If $n \geq 3$, then $2 \leq k \leq n-1$. Moreover, G is n-metric dimensional if and only if $G \cong K_2$.*

Next we give a characterization of k-metric dimensional graphs. To do so, we need some additional terminology. Given two different vertices $x,y \in V(G)$, we say that the set of *distinctive vertices* of x,y is

$$\mathscr{D}_G(x,y) = \{z \in V(G) : d_G(x,z) \neq d_G(y,z)\}$$

and the set of *non-trivial distinctive vertices* of x,y is

$$\mathscr{D}^*_G(x,y) = \mathscr{D}_G(x,y) - \{x,y\}.$$

Theorem 1. *A connected graph G is k-metric dimensional if and only if $k = \min_{x,y\in V(G),x\neq y} |\mathscr{D}_G(x,y)|$.*

Proof. (Necessity) If G is a k-metric dimensional graph, then for any k-metric basis B and any pair of different vertices $x,y \in V(G)$, we have $|B \cap \mathscr{D}_G(x,y)| \geq k$. Thus, $k \leq \min_{x,y\in V(G),x\neq y} |\mathscr{D}_G(x,y)|$. Now, we suppose that $k < \min_{x,y\in V(G),x\neq y} |\mathscr{D}_G(x,y)|$. In such a case, for every $x',y' \in V(G)$ such that $|B \cap \mathscr{D}_G(x',y')| = k$, there exists a distinctive vertex $z_{x'y'}$ of x',y' with $z_{x'y'} \in \mathscr{D}_G(x',y') - B$. Hence, the set

$$B \cup \left(\bigcup_{x',y'\in V(G):|B\cap\mathscr{D}_G(x',y')|=k} \{z_{x'y'}\} \right)$$

is a $(k+1)$-metric generator for G, which is a contradiction. Therefore, $k = \min_{x,y\in V(G),x\neq y} |\mathscr{D}_G(x,y)|$.

(Sufficiency) Let $a,b \in V(G)$ such that $\min_{x,y\in V(G),x\neq y} |\mathscr{D}_G(x,y)| = |\mathscr{D}_G(a,b)| = k$. Since the set

$$\bigcup_{x,y\in V(G)} \mathscr{D}_G(x,y)$$

is a k-metric generator for G and the pair a,b is not distinguished by $k' > k$ vertices of G, we conclude that G is a k-metric dimensional graph.

2.1 On some families of k-metric dimensional graphs for some specific values of k

The characterization proved in Theorem 1 gives a result on general graphs. Thus, next we particularize this for some specific classes of graphs or we bound its possible value in terms of other parameters of the graph. To this end, we need the following concepts. Two vertices x,y are called *false twins* if $N(x) = N(y)$ and x,y are called *true twins* if $N[x] = N[y]$. Two vertices x,y are *twins* if they are false twins or true twins. A vertex x is said to be a *twin* if there exists a vertex $y \in V(G) - \{x\}$ such that x and y are twins in G. Notice that two vertices x,y are twins if and only if $\mathscr{D}^*_G(x,y) = \emptyset$.

Corollary 1. *A connected graph G of order $n \geq 2$ is 2-metric dimensional if and only if G has twin vertices.*

It is clear that P_2 and P_3 are 2-metric dimensional. Now, a specific characterization for 2-dimensional trees is obtained from Theorem 1 (or from Corollary 1). A *leaf* in a tree is a vertex of degree one, while a *support vertex* is a vertex adjacent to a leaf.

Corollary 2. *A tree T of order $n \geq 4$ is 2-metric dimensional if and only if T contains a support vertex which is adjacent to at least two leaves.*

An example of a 2-metric dimensional tree is the star graph $K_{1,n-1}$, whose 2-metric dimension is $\dim_2(K_{1,n-1}) = n-1$ (see Corollary 8). On the other side, an example of a tree T which is not 2-metric dimensional is drawn in Figure 2. Notice that $S = \{v_1,v_3,v_5,v_6,v_7\}$ is a 3-metric basis of T. Moreover, T is 3-metric dimensional since $|\mathscr{D}_T(v_1,v_3)| = 3$.

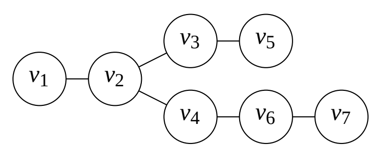

Fig. 2: $S = \{v_1,v_3,v_5,v_6,v_7\}$ is a 3-metric basis of T.

A *cut vertex* in a graph is a vertex whose removal increases the number of components of the graph and an *extreme vertex* is a vertex v such that the subgraph induced by $N[v]$ is isomorphic to a complete graph. Also, a *block* is a maximal biconnected subgraph[1] of the graph. Now, let $\mathfrak{F}$ be the family of sequences of connected graphs $G_1,G_2,\ldots,G_t$, $t \geq 2$, such that G_1 is a complete graph K_{n_1}, $n_1 \geq 2$, and G_i, $i \geq 2$, is obtained recursively from G_{i-1} by adding a complete graph K_{n_i}, $n_i \geq 2$, and identifying one vertex of G_{i-1} with one vertex of K_{n_i}.

From this point we will say that a connected graph G is a *generalized tree*[2] if and only if there exists a sequence $\{G_1,G_2,\ldots,G_t\} \in \mathfrak{F}$ such that $G_t = G$ for some $t \geq 2$. Notice that in these generalized trees every vertex is

[1] A biconnected graph is a connected graph having no articulation vertices.

[2] In some works these graphs are called block graphs.

either, a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that, if every K_{n_i} is isomorphic to K_2, then G_t is a tree, justifying the terminology used. With these concepts we give the following consequence of Theorem 1, which is a generalization of Corollary 2.

Corollary 3. *A generalized tree G is 2-metric dimensional if and only if G contains at least two extreme vertices being adjacent to a common cut vertex.*

The *Cartesian product graph* $G\Box H$, of two graphs $G=(V_1,E_1)$ and $H=(V_2,E_2)$, is the graph whose vertex set is $V(G\Box H)=V_1\times V_2$ and any two distinct vertices $(x_1,x_2),(y_1,y_2)\in V_1\times V_2$ are adjacent in $G\Box H$ if and only if either:

(a) $x_1=y_1$ and $x_2\sim y_2$, or
(b) $x_1\sim y_1$ and $x_2=y_2$.

Proposition 1. *Let G and H be two connected graphs of order $n\geq 2$ and $n'\geq 3$, respectively. If $G\Box H$ is k-metric dimensional, then $k\geq 3$.*

Proof. Notice that for any vertex $(a,b)\in V(G\Box H)$, $N_{G\Box H}((a,b))=(N_G(a)\times\{b\})\cup(\{a\}\times N_H(b))$. Now, for any two distinct vertices $(a,b),(c,d)\in V(G\Box H)$ at least $a\neq c$ or $b\neq d$ and since H is a connected graph of order greater than two, we have that at least $N_H(b)\neq\{d\}$ or $N_H(d)\neq\{b\}$. Thus, we obtain that $N_{G\Box H}((a,b))\neq N_{G\Box H}((c,d))$. Therefore, $G\Box H$ does not contain any twins and, by Remark 1 and Corollary 1, if $G\Box H$ is k-metric dimensional, then $k\geq 3$.

Proposition 2. *Let C_n be a cycle graph of order n. If n is odd, then C_n is $(n-1)$-metric dimensional and if n is even, then C_n is $(n-2)$-metric dimensional.*

Proof. We consider two cases:

(1) n is odd. For any pair of different vertices $u,v\in V(C_n)$ there exists only one vertex $w\in V(C_n)$ such that w does not distinguish u and v. Therefore, by Theorem 1, C_n is $(n-1)$-metric dimensional.
(2) n is even. In this case, C_n is 2-antipodal[3]. For any pair of vertices $u,v\in V(C_n)$, such that $d(u,v)=2l$, we can take a vertex x such that $d(u,x)=d(v,x)=l$. So, $\mathscr{D}_G(u,v)=V(C_n)-\{x,y\}$, where y is antipodal to x. On the other hand, if $d(u,v)$ is odd, then $\mathscr{D}_G(u,v)=V(C_n)$. Therefore, by Theorem 1, the graph C_n is $(n-2)$-metric dimensional.

Now, according to Remark 1 we have that every graph of order n, different from K_2, is k-metric dimensional for some $k\leq n-1$. Next we characterize those graphs being $(n-1)$-metric dimensional.

[3] The diameter of $G=(V,E)$ is defined as $D(G)=\max_{u,v\in V(G)}\{d_G(u,v)\}$. We say that u and v are antipodal vertices or mutually antipodal if $d_G(u,v)=D(G)$. We recall that $G=(V,E)$ is 2-antipodal if for each vertex $x\in V$ there exists exactly one vertex $y\in V$ such that $d_G(x,y)=D(G)$.

Theorem 2. *A graph G of order $n\geq 3$ is $(n-1)$-metric dimensional if and only if G is a path or G is an odd cycle.*

Proof. Since $n\geq 3$, by Remark 1, G is k-metric dimensional for some $k\in\{2,\ldots,n-1\}$. Now, for any pair of different vertices $u,v\in V(P_n)$ there exists at most one vertex $w\in V(P_n)$ such that w does not distinguish u and v. Then P_n is $(n-1)$-metric dimensional. By Proposition 2, we have that if G is an odd cycle, then G is $(n-1)$-metric dimensional.

On the contrary, let G be a $(n-1)$-metric dimensional graph. Hence, for every pair of different vertices $x,y\in V(G)$ there exists at most one vertex which does not distinguish x,y. Suppose $\Delta(G)>2$ and let $v\in V(G)$ such that $\{u_1,u_2,u_3\}\subset N(v)$. Figure 3 shows all the possibilities for the links between these four vertices. Figures 3 (a), 3 (b) and 3 (d) show that v,u_1 do not distinguish u_2,u_3. Figure 3 (c) shows that u_1,u_2 do not distinguish v,u_3. Thus, from the cases above we deduce that there is a pair of different vertices which is not distinguished by at least two other different vertices. Thus G is not a $(n-1)$-metric dimensional graph, which is a contradiction. As a consequence, $\Delta(G)\leq 2$ and we have that G is either a path or a cycle graph. Finally, by Proposition 2, we have that if G is a cycle, then G has odd order.

2.2 Bounding the value k for which a graph is k-metric dimensional

In order to continue presenting our results, we need to introduce some definitions. A vertex of degree at least three in a graph G will be called a *major vertex* of G. Any end-vertex (a vertex of degree one) u of G is said to be a *terminal vertex* of a major vertex v of G if $d_G(u,v)<d_G(u,w)$ for every other major vertex w of G. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\mathscr{M}(G)$ be the set of exterior major vertices of G having terminal degree greater than one.

Given $w\in\mathscr{M}(G)$ and a terminal vertex u_j of w, we denote by $P(u_j,w)$ the shortest path that starts at u_j and ends at w. Let $l(u_j,w)$ be the length of $P(u_j,w)$. Now, given $w\in\mathscr{M}(G)$ and two terminal vertices u_j,u_r of w we denote by $P(u_j,w,u_r)$ the shortest path from u_j to u_r containing w, and by $\varsigma(u_j,u_r)$ the length of $P(u_j,w,u_r)$. Notice that, by definition of exterior major vertex, $P(u_j,w,u_r)$ is obtained by concatenating the paths $P(u_j,w)$ and $P(u_r,w)$, where w is the only vertex of degree greater than two lying on these paths.

Finally, given $w\in\mathscr{M}(G)$ and the set of terminal vertices $U=\{u_1,u_2,\ldots,u_k\}$ of w, for $j\neq r$ we define $\varsigma(w)=\min_{u_j,u_r\in U}\{\varsigma(u_j,u_r)\}$ and $l(w)=\min_{u_j\in U}\{l(u_j,w)\}$.

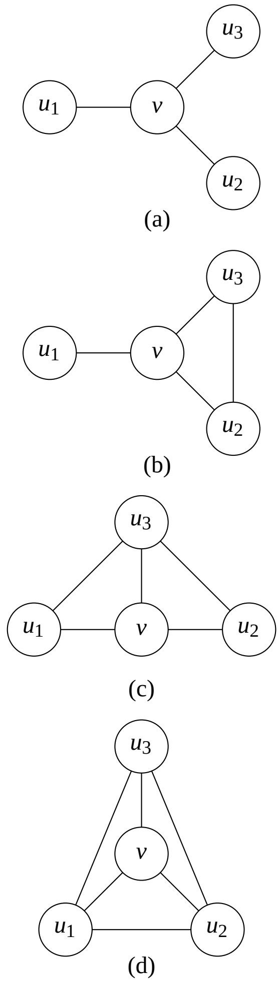

Fig. 3: Possible cases for a vertex v with three adjacent vertices u_1, u_2, u_3.

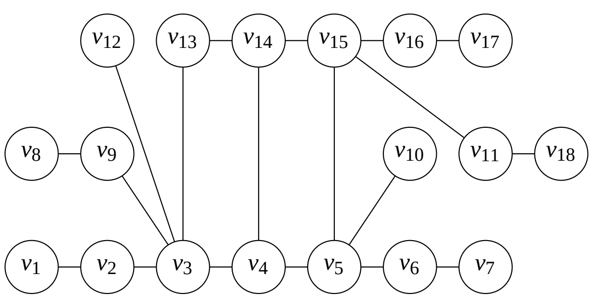

Fig. 4: A graph G where $\varsigma(G) = 3$.

From the local parameters above we define the following global parameter

$$\varsigma(G) = \min_{w \in \mathscr{M}(G)} \{\varsigma(w)\}.$$

An example which helps to understand the notation above is given in Figure 4. In such a case we have $\mathscr{M}(G) = \{v_3, v_5, v_{15}\}$ and, for instance, $\{v_1, v_8, v_{12}\}$ are terminal vertices of v_3. So, v_3 has terminal degree three ($\mathrm{ter}(v_3) = 3$) and it follows that

$$\begin{aligned} l(v_3) &= \min\{l(v_{12}, v_3), l(v_8, v_3), l(v_1, v_3)\} \\ &= \min\{1, 2, 2\} = 1, \end{aligned}$$

and

$$\begin{aligned} \varsigma(v_3) &= \min\{\varsigma(v_{12}, v_1), \varsigma(v_{12}, v_8), \varsigma(v_8, v_1)\} \\ &= \min\{3, 3, 4\} = 3. \end{aligned}$$

Similarly, it is possible to observe that $\mathrm{ter}(v_5) = 2$, $l(v_5) = 1$, $\varsigma(v_5) = 3$, $\mathrm{ter}(v_{15}) = 2$, $l(v_{15}) = 2$ and $\varsigma(v_{15}) = 4$. Therefore, $\varsigma(G) = 3$.

According to this notation we present the following result.

Theorem 3. *Let G be a connected graph such that $\mathscr{M}(G) \neq \emptyset$. If G is k-metric dimensional, then $k \leq \varsigma(G)$.*

Proof. We claim that there exists at least one pair of different vertices $x, y \in V(G)$ such that $|\mathscr{D}_G(x,y)| = \varsigma(G)$. To see this, let $w \in \mathscr{M}(G)$ and let u_1, u_2 be two terminal vertices of w such that $\varsigma(G) = \varsigma(w) = \varsigma(u_1, u_2)$. Let u_1' and u_2' be the vertices adjacent to w in the shortest paths $P(u_1, w)$ and $P(u_2, w)$, respectively. Notice that it could happen $u_1' = u_1$ or $u_2' = u_2$. Since every vertex $v \notin V(P(u_1, w, u_2)) - \{w\}$ satisfies that $d_G(u_1', v) = d_G(u_2', v)$, and the only distinctive vertices of u_1', u_2' are those ones belonging to $P(u_1', u_1)$ and $P(u_2', u_2)$, we have that $|\mathscr{D}_G(u_1', u_2')| = \varsigma(G)$. Therefore, by Theorem 1, if G is k-metric dimensional, then $k \leq \varsigma(G)$.

The upper bound of Theorem 3 is tight. For instance, it is achieved for every tree different from a path as it is proved further in Section 4, where the k-metric dimension of trees is studied.

A *clique* in a graph G is a set of vertices S such that the subgraph induced by S, denoted by $\langle S \rangle$, is isomorphic to a complete graph. The maximum cardinality of a clique in a graph G is the *clique number* and it is denoted by $\omega(G)$. We will say that S is an $\omega(G)$-clique if $|S| = \omega(G)$.

Theorem 4. *Let G be a graph of order n different from a complete graph. If G is k-metric dimensional, then $k \leq n - \omega(G) + 1$.*

Proof. Let S be an $\omega(G)$-clique. Since G is not complete, there exists a vertex $v \notin S$ such that $N_S(v) \subsetneq S$. Let $u \in S$ with $v \not\sim u$. If $N_S(v) = S - \{u\}$, then $d(u,x) = d(v,x) = 1$ for every $x \in S - \{u\}$. Thus, $|\mathscr{D}_G(u,v)| \leq n - \omega(G) + 1$. On the other hand, if $N_S(v) \neq S - \{u\}$, then there exists $u' \in S - \{u\}$ such that $u' \not\sim v$. Thus, $d(u,v) = d(u',v) = 2$ and for every $x \in S - \{u, u'\}$, $d(u,x) = d(u',x) = 1$. So, $|\mathscr{D}_G(u,u')| \leq n - \omega(G) + 1$. Therefore, Theorem 1 leads to $k \leq n - \omega(G) + 1$.

Examples where the previous bound is achieved are those connected graphs G of order n and clique number $\omega(G)=n-1$. In such a case, $n-\omega(G)+1=2$. Notice that in this case there exists at least two twin vertices. Hence, by Corollary 1 these graphs are 2-metric dimensional.

The *girth* of a graph G is the length of a shortest cycle in G.

Theorem 5. *Let G be a graph of minimum degree $\delta(G)\geq 2$, maximum degree $\Delta(G)\geq 3$ and girth $\mathrm{g}(G)\geq 4$. If G is k-metric dimensional, then*

$$k\leq n-1-(\Delta(G)-2)\sum_{i=0}^{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-2}(\delta(G)-1)^i.$$

Proof. Let $v\in V$ be a vertex of maximum degree in G. Since $\Delta(G)\geq 3$ and $\mathrm{g}(G)\geq 4$, there are at least three different vertices adjacent to v and $N(v)$ is an independent set[4]. Given $u_1,u_2\in N(v)$ and $i\in\{0,\ldots,\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-2\}$ we define the following sets.

$$\begin{aligned}
A_0 &= N(v)-\{u_1,u_2\}.\\
A_1 &= \bigcup_{x\in A_0} N(x)-\{v\}.\\
A_2 &= \bigcup_{x\in A_1} N(x)-A_0.\\
&\cdots\\
A_{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-2} &= \bigcup_{x\in A_{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-3}} N(x)-A_{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-4}.
\end{aligned}$$

Now, let $A=\{v\}\cup\left(\bigcup_{i=0}^{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-2}A_i\right)$. Since $\delta(G)\geq 2$, we have that $|A|\geq 1+(\Delta(G)-2)\sum_{i=0}^{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-2}(\delta(G)-1)^i$. Also, notice that for every vertex $x\in A$, $d(u_1,x)=d(u_2,x)$. Thus, u_1,u_2 can be only distinguished by themselves and at most $n-|A|-2$ other vertices. Therefore, $|\mathscr{D}_G(u_1,u_2)|\leq n-|A|$ and the result follows by Theorem 1.

The bound of Theorem 5 is sharp. For instance, it is attained for the graph in Figure 5. Since in this case $n=8$, $\delta(G)=2$, $\Delta(G)=3$ and $\mathrm{g}(G)=5$, we have that $k\leq n-1-(\Delta(G)-2)\sum_{i=0}^{\left\lfloor\frac{\mathrm{g}(G)}{2}\right\rfloor-2}(\delta(G)-1)^i=6$. Table 1 shows every pair of different vertices of this graph and their corresponding non-trivial distinctive vertices. Notice that by Theorem 1 the graph is 6-metric dimensional.

[4] An independent set or stable set is a set of vertices in a graph, no two of which are adjacent.

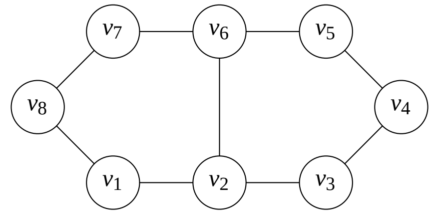

Fig. 5: A graph that satisfies the equality in the upper bound of Theorem 5.

Table 1: Pairs of vertices of the graph in Figure 5 and their non-trivial distinctive vertices.

x,y	$\mathscr{D}^*_G(x,y)$
v_1,v_3	$\{v_4,v_5,v_7,v_8\}$
v_1,v_5	$\{v_2,v_4,v_6,v_8\}$
v_1,v_6	$\{v_4,v_5,v_7,v_8\}$
v_1,v_7	$\{v_2,v_3,v_5,v_6\}$
v_1,v_8	$\{v_2,v_3,v_4,v_7\}$
v_2,v_5	$\{v_1,v_3,v_4,v_8\}$
v_2,v_6	$\{v_1,v_3,v_5,v_7\}$
v_2,v_7	$\{v_1,v_3,v_4,v_8\}$
v_3,v_4	$\{v_1,v_2,v_5,v_8\}$
v_3,v_5	$\{v_1,v_2,v_6,v_7\}$
v_3,v_6	$\{v_4,v_5,v_7,v_8\}$
v_3,v_7	$\{v_2,v_4,v_6,v_8\}$
v_4,v_5	$\{v_3,v_6,v_7,v_8\}$
v_4,v_8	$\{v_1,v_3,v_5,v_7\}$
v_5,v_7	$\{v_1,v_3,v_4,v_8\}$
v_7,v_8	$\{v_1,v_4,v_5,v_6\}$

x,y	$\mathscr{D}^*_G(x,y)$
v_1,v_2	$\{v_3,v_4,v_5,v_6,v_8\}$
v_1,v_4	$\{v_2,v_3,v_5,v_7,v_8\}$
v_2,v_3	$\{v_1,v_4,v_6,v_7,v_8\}$
v_2,v_4	$\{v_1,v_5,v_6,v_7,v_8\}$
v_2,v_8	$\{v_3,v_4,v_5,v_6,v_7\}$
v_3,v_8	$\{v_1,v_2,v_4,v_5,v_7\}$
v_4,v_6	$\{v_1,v_2,v_3,v_7,v_8\}$
v_4,v_7	$\{v_1,v_3,v_5,v_6,v_8\}$
v_5,v_6	$\{v_1,v_2,v_4,v_7,v_8\}$
v_5,v_8	$\{v_1,v_3,v_4,v_6,v_7\}$
v_6,v_7	$\{v_2,v_3,v_4,v_5,v_8\}$
v_6,v_8	$\{v_1,v_2,v_3,v_4,v_5\}$

3 The k-metric dimension of graphs

In this section we present some results that allow to compute the k-metric dimension of several families of graphs. We also give some tight bounds on the k-metric dimension of a graph.

Theorem 6(Monotony of the k-metric dimension). *Let G be a k-metric dimensional graph and let k_1,k_2 be two integers. If $1\leq k_1<k_2\leq k$, then $\dim_{k_1}(G)<\dim_{k_2}(G)$.*

Proof. Let B be a k-metric basis of G. Let $x\in B$. Since all pairs of different vertices in $V(G)$ are distinguished by at least k vertices of B, we have that $B-\{x\}$ is a $(k-1)$-metric generator for G and, as a consequence, $\dim_{k-1}(G)\leq|B-\{x\}|<|B|=\dim_k(G)$. Proceeding analogously, we obtain that $\dim_{k-1}(G)>\dim_{k-2}(G)$ and, by a finite repetition of the process we obtain the result.

Corollary 4. *Let G be a k-metric dimensional graph of order n.*

(i) *For every $r\in\{1,\ldots,k\}$, $\dim_r(G)\geq\dim(G)+(r-1)$.*
(ii) *For every $r\in\{1,\ldots,k-1\}$, $\dim_r(G)<n$.*
(iii) *If $G\ncong P_n$, then for any $r\in\{1,\ldots,k\}$, $\dim_r(G)\geq r+1$.*

Proposition 3. *Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = 2$ if and only if $G \cong P_n$.*

Proof. It was shown in [5] that $\dim(G) = 1$ if and only if $G \cong P_n$.

(Necessity) If $\dim_2(G) = 2$, then by Corollary 4 (i) we have that $\dim(G) = 1$, *i.e.*,

$$2 = \dim_2(G) \geq \dim(G) + 1 \geq 2.$$

Hence, G must be isomorphic to a path graph.

(Sufficiency) By Corollary 4 (i) we have $\dim_2(P_n) \geq \dim(P_n) + 1 = 2$ and, since the leaves of P_n distinguish every pair of different vertices of P_n, we conclude that $\dim_2(P_n) = 2$.

Let $\mathscr{D}_k(G)$ be the set obtained as the union of the sets of distinctive vertices $\mathscr{D}_G(x,y)$ whenever $|\mathscr{D}_G(x,y)| = k$, *i.e.*,

$$\mathscr{D}_k(G) = \bigcup_{|\mathscr{D}_G(x,y)|=k} \mathscr{D}_G(x,y).$$

Remark 2 *If G is a k-metric dimensional graph, then* $\dim_k(G) \geq |\mathscr{D}_k(G)|$.

Proof. Since every pair of different vertices x,y is distinguished only by the elements of $\mathscr{D}_G(x,y)$, if $|\mathscr{D}_G(x,y)| = k$, then for any k-metric basis B we have $\mathscr{D}_G(x,y) \subseteq B$ and, as a consequence, $\mathscr{D}_k(G) \subseteq B$. Therefore, the result follows.

The bound given in Remark 2 is tight. For instance, in Proposition 6 we will show that there exists a family of trees attaining this bound for every k. Other examples can be derived from the following result.

Proposition 4. *Let G be a k-metric dimensional graph of order n. Then $\dim_k(G) = n$ if and only if $V(G) = \mathscr{D}_k(G)$.*

Proof. Suppose that $V(G) = \mathscr{D}_k(G)$. Now, since every k-metric dimensional graph G satisfies that $\dim_k(G) \leq n$, by Remark 2 we obtain that $\dim_k(G) = n$.

On the contrary, let $\dim_k(G) = n$. Note that for every $a,b \in V(G)$, we have $|\mathscr{D}_G(a,b)| \geq k$. If there exists at least one vertex $x \in V(G)$ such that $x \notin \mathscr{D}_k(G)$, then for every $a,b \in V(G)$, we have $|\mathscr{D}_G(a,b) - \{x\}| \geq k$ and, as a consequence, $V(G) - \{x\}$ is a k-metric generator for G, which is a contradiction. Therefore, $V(G) = \mathscr{D}_k(G)$.

Corollary 5. *Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = n$ if and only if every vertex is a twin.*

We will show other examples of graphs that satisfy Proposition 4 for $k \geq 3$. To this end, we recall that the *join graph* $G + H$ of the graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G+H) = V_1 \cup V_2$ and edge set $E(G+H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. We give now some examples of graphs satisfying the assumptions of Proposition 4. Let $W_{1,n} = C_n + K_1$ be the *wheel graph* and $F_{1,n} = P_n + K_1$ be the *fan graph*. The vertex of K_1 is called the central vertex of the wheel or the fan, respectively. Since $V(F_{1,4}) = \mathscr{D}_3(F_{1,4})$ and $V(W_{1,5}) = \mathscr{D}_4(W_{1,5})$, by Proposition 4 we have that $\dim_3(F_{1,4}) = 5$ and $\dim_4(W_{1,5}) = 6$, respectively.

Given two non-trivial graphs G and H, it holds that any pair of twin vertices $x,y \in V(G)$ or $x,y \in V(H)$ are also twin vertices in $G+H$. As a direct consequence of Corollary 5, the next result holds.

Remark 3 *Let G and H be two nontrivial graphs of order n_1 and n_2, respectively. If all the vertices of G and H are twin vertices, then $G+H$ is 2-metric dimensional and*

$$\dim_2(G+H) = n_1 + n_2.$$

Note that in Remark 3, the graphs G and H could be non connected. Moreover, G and H could be nontrivial empty graphs. For instance, $N_r + N_s$, where N_r, N_s, $r,s > 1$, are empty graphs, is the complete bipartite graph $K_{r,s}$ which satisfies that $\dim_2(K_{r,s}) = r + s$.

3.1 Bounding the k-metric dimension of graphs

We begin this subsection with a necessary definition of the *twin equivalence relation* $\mathscr{R}$ on $V(G)$ as follows:

$$x\mathscr{R}y \longleftrightarrow N_G[x] = N_G[y] \text{ or } N_G(x) = N_G(y).$$

We have three possibilities for each twin equivalence class U:

(a) U is singleton, or
(b) $N_G(x) = N_G(y)$, for any $x,y \in U$ (and case (a) does not apply), or
(c) $N_G[x] = N_G[y]$, for any $x,y \in U$ (and case (a) does not apply).

We will refer to the type (c) classes as the *true twin equivalence classes i.e.*, U is a true twin equivalence class if and only if U is not singleton and $N_G[x] = N_G[y]$, for any $x,y \in U$.

Let us see three different examples where every vertex is a twin. An example of a graph where every equivalence class is a true twin equivalence class is $K_r + (K_s \cup K_t)$, $r,s,t \geq 2$. In this case, there are three equivalence classes composed by r,s and t true twin vertices, respectively. As an example where no class is composed by true twin vertices we take the complete bipartite graph $K_{r,s}$, $r,s \geq 2$. Finally, the graph $K_r + N_s$, $r,s \geq 2$, has two equivalence classes and one of them is composed by r true twin vertices. On the other hand, $K_1 + (K_r \cup N_s)$, $r,s \geq 2$, is an example where one class is singleton, one class is composed by true twin vertices and the other one is composed by false twin vertices.

In general, we can state the following result.

Remark 4 *Let G be a connected graph and let $U_1, U_2, \ldots, U_t$ be the non-singleton twin equivalence classes of G. Then*

$$\dim_2(G) \geq \sum_{i=1}^{t} |U_i|.$$

Proof. Since for two different vertices $x,y\in V(G)$ we have that $\mathscr{D}_2(x,y)=\{x,y\}$ if and only if there exists an equivalence class U_i such that $x,y\in U_i$, we deduce

$$\mathscr{D}_2(G)=\bigcup_{i=1}^{t}U_i.$$

Therefore, by Remark 2 we conclude the proof.

Notice that the result above leads to Corollary 5, so this bound is tight. Now we consider the connected graph G of order $r+s$ obtained from a null graph N_r of order $r\geq 2$ and a path P_s of order $s\geq 1$ by connecting every vertex of N_r to a given leaf of P_s. In this case, there are s singleton classes and one class, say U_1, of cardinality r. By the previous result we have $\dim_2(G)\geq |U_1|=r$ and, since U_1 is a 2-metric generator for G, we conclude that $\dim_2(G)=r$.

We recall that the *strong product graph* $G\boxtimes H$ of two graphs $G=(V_1,E_1)$ and $H=(V_2,E_2)$ is the graph with vertex set $V(G\boxtimes H)=V_1\times V_2$, where two distinct vertices $(x_1,x_2),(y_1,y_2)\in V_1\times V_2$ are adjacent in $G\boxtimes H$ if and only if one of the following holds.

– $x_1=y_1$ and $x_2\sim y_2$, or
– $x_1\sim y_1$ and $x_2=y_2$, or
– $x_1\sim y_1$ and $x_2\sim y_2$.

Theorem 7. *Let G and H be two nontrivial connected graphs of order n and n', respectively. Let $U_1,U_2,\ldots,U_t$ be the true twin equivalence classes of G. Then*

$$\dim_2(G\boxtimes H)\geq n'\sum_{i=1}^{t}|U_i|.$$

Moreover, if every vertex of G is a true twin, then

$$\dim_2(G\boxtimes H)=nn'.$$

Proof. For any two vertices $a,c\in U_i$ and $b\in V(H)$,

$$\begin{aligned}N_{G\boxtimes H}[(a,b)]&=N_G[a]\times N_H[b]\\&=N_G[c]\times N_H[b]\\&=N_{G\boxtimes H}[(c,b)].\end{aligned}$$

Thus, (a,b) and (c,b) are true twin vertices. Hence,

$$\mathscr{D}_2(G\boxtimes H)\supseteq\bigcup_{i=1}^{t}U_i\times V(H).$$

Therefore, by Remark 2 we conclude $\dim_2(G\boxtimes H)\geq n'\sum_{i=1}^{t}|U_i|$.

Finally, if every vertex of G is a true twin, then $\bigcup_{i=1}^{t}U_i=V(G)$ and, as a consequence, we obtain $\dim_2(G\boxtimes H)=nn'$.

We now present a lower bound for the k-metric dimension of a k'-metric dimensional graph G with $k'\geq k$. To this end, we require the use of the following function for any exterior major vertex $w\in V(G)$ having terminal degree greater than one, *i.e.*, $w\in\mathscr{M}(G)$. Notice that this function uses the concepts already defined in Section 2.2. Given an integer $r\leq k'$,

$$I_r(w)=\begin{cases}(\mathrm{ter}(w)-1)\,(r-l(w))+l(w), & \text{if } l(w)\leq\lfloor\frac{r}{2}\rfloor,\\[4pt](\mathrm{ter}(w)-1)\lceil\frac{r}{2}\rceil+\lfloor\frac{r}{2}\rfloor, & \text{otherwise.}\end{cases}$$

In Figure 4 we give an example of a graph G, which helps to clarify the notation above. Since every graph is at least 2-metric dimensional, we can consider the integer $r=2$ and we have the following.

– Since $l(v_3)=1\leq\lfloor\frac{r}{2}\rfloor$, it follows that $I_r(v_3)=(\mathrm{ter}(v_3)-1)\,(r-l(v_3))+l(v_3)=(3-1)(2-1)+1=3$.
– Since $l(v_5)=1\leq\lfloor\frac{r}{2}\rfloor$, it follows that $I_r(v_5)=(\mathrm{ter}(v_5)-1)\,(r-l(v_5))+l(v_5)=(2-1)(2-1)+1=2$.
– Since $l(v_{15})=2>\lfloor\frac{r}{2}\rfloor$, it follows that $I_r(v_{15})=(\mathrm{ter}(v_{15})-1)\lceil\frac{r}{2}\rceil+\lfloor\frac{r}{2}\rfloor=(2-1)\lceil\frac{2}{2}\rceil+\lfloor\frac{2}{2}\rfloor=2$.

Therefore, according to the result below, $\dim_2(G)\geq 3+2+2=7$.

Theorem 8. *If G is a k-metric dimensional graph such that $|\mathscr{M}(G)|\geq 1$, then for every $r\in\{1,\ldots,k\}$,*

$$\dim_r(G)\geq\sum_{w\in\mathscr{M}(G)}I_r(w).$$

Proof. Let S be an r-metric basis of G. Let $w\in\mathscr{M}(G)$ and let u_i,u_s be two different terminal vertices of w. Let u'_i,u'_s be the vertices adjacent to w in the paths $P(u_i,w)$ and $P(u_s,w)$, respectively. Notice that $\mathscr{D}_G(u'_i,u'_s)=V(P(u_i,w,u_s))-\{w\}$ and, as a consequence, it follows that $|S\cap(V(P(u_i,w,u_s))-\{w\})|\geq r$. Now, if $\mathrm{ter}(w)=2$, then we have

$$|S\cap(V(P(u_i,w,u_s))-\{w\})|\geq r=I_r(w).$$

Now, we assume $\mathrm{ter}(w)>2$. Let W be the set of terminal vertices of w, and let u'_j be the vertex adjacent to w in the path $P(u_j,w)$ for every $u_j\in W$. Let $U(w)=\bigcup_{u_j\in W}V(P(u_j,w))-\{w\}$ and let $x=\min_{u_j\in W}\{|S\cap V(P(u_j,w))|\}$. Since S is an r-metric generator of minimum cardinality (it is an r-metric basis of G), it is satisfied that $0\leq x\leq\min\{l(w),\lfloor\frac{r}{2}\rfloor\}$. Let u_α be a terminal vertex such that $|S\cap(V(P(u_\alpha,w))-\{w\})|=x$. Since for every terminal vertex $u_\beta\in W-\{u_\alpha\}$ we have that $|S\cap\mathscr{D}_G(u'_\beta,u'_\alpha)|\geq r$, it follows that $\left|S\cap\left(V\left(P(u_\beta,w)\right)-\{w\}\right)\right|\geq r-x$. Thus,

$$\begin{aligned}|S\cap U(w)| &= |S\cap(V(P(u_\alpha,w))-\{w\})|+\\ &+\sum_{\beta=1,\beta\neq\alpha}^{\mathrm{ter}(w)}\left|S\cap\left(V\left(P(u_\beta,w)\right)-\{w\}\right)\right|\\ &\geq(\mathrm{ter}(w)-1)(r-x)+x.\end{aligned}$$

Now, if $x=0$, then $|S\cap U(w)|\geq(\mathrm{ter}(w)-1)r>I_r(w)$. On the contrary, if $x>0$, then the function $f(x)=(\mathrm{ter}(w)-1)(r-x)+x$ is decreasing with respect to x. So, the minimum value of f is achieved in the highest possible value of x. Thus, $|S\cap U(w)|\geq I_r(w)$. Since $\bigcap_{w\in\mathcal{M}(G)}U(w)=\emptyset$, it follows that

$$\dim_r(G)\geq\sum_{w\in\mathcal{M}(G)}|S\cap U(w)|\geq\sum_{w\in\mathcal{M}(G)}I_r(w).$$

Now, in order to give some consequences of the bound above we will use some notation defined in Section 2.2 to introduce the following parameter.

$$\mu(G)=\sum_{v\in\mathcal{M}(G)}\mathrm{ter}(v).$$

Notice that, for $k=1$, Theorem 8 leads to the bound on the metric dimension of a graph, established by Chartrand *et al.* in [5]. In such a case, $I_1(w)=\mathrm{ter}(w)-1$ for all $w\in\mathcal{M}(G)$ and thus,

$$\dim(G)\geq\sum_{w\in\mathcal{M}(G)}(\mathrm{ter}(w)-1)=\mu(G)-|\mathcal{M}(G)|.$$

Next we give the particular cases of Theorem 8 for $r=2$ and $r=3$.

Corollary 6. *If G is a connected graph, then*

$$\dim_2(G)\geq\mu(G).$$

Proof. If $\mathcal{M}(G)=\emptyset$, then $\mu(G)=0$ and the result is direct. Suppose that $\mathcal{M}(G)\neq\emptyset$. Since $I_2(w)=\mathrm{ter}(w)$ for all $w\in\mathcal{M}(G)$, we deduce that

$$\dim_2(G)\geq\sum_{w\in\mathcal{M}(G)}\mathrm{ter}(w)=\mu(G).$$

Corollary 7. *If G is k-metric dimensional for some $k\geq 3$, then*

$$\dim_3(G)\geq 2\mu(G)-|\mathcal{M}(G)|.$$

Proof. If $\mathcal{M}(G)=\emptyset$, then the result is direct. Suppose that $\mathcal{M}(G)\neq\emptyset$. Since $I_3(w)=2\,\mathrm{ter}(w)-1$ for all $w\in\mathcal{M}(G)$, we obtain that

$$\dim_3(G)\geq\sum_{w\in\mathcal{M}(G)}(2\,\mathrm{ter}(w)-1)=2\mu(G)-|\mathcal{M}(G)|.$$

In the next section we give some results on trees which show that the bounds proved in Theorem 8 and Corollaries 6 and 7 are tight. Specifically those results are Theorem 10 and Corollaries 8 and 9, respectively.

4 The particular case of trees

To study the k-metric dimension of a tree it is of course necessary to know first the value k for which a given tree is k-metric dimensional. That is what we do next. In this sense, from now on we need the terminology and notation already described in Section 2.2 and also the following one. Given an exterior major vertex v in a tree T and the set of its terminal vertices $v_1,\ldots,v_\alpha$, the subgraph induced by the set $\bigcup_{i=1}^{\alpha}V(P(v,v_i))$ is called a *branch* of T at v (a v-branch for short).

Theorem 9. *If T is a k-metric dimensional tree different from a path, then $k=\varsigma(T)$.*

Proof. Since T is not a path, $\mathcal{M}(T)\neq\emptyset$. Let $w\in\mathcal{M}(T)$ and let u_1,u_2 be two terminal vertices of w such that $\varsigma(T)=\varsigma(w)=\varsigma(u_1,u_2)$. Notice that, for instance, the two neighbours of w belonging to the paths $P(w,u_1)$ and $P(w,u_2)$, say u_1' and u_2' satisfy $|\mathcal{D}_T(u_1',u_2')|=\varsigma(T)$.

It only remains to prove that for every $x,y\in V(T)$ it holds that $|\mathcal{D}_T(x,y)|\geq\varsigma(T)$. Let $w\in\mathcal{M}(T)$ and let $T_w=(V_w,E_w)$ be the w-branch. Also we consider the set of vertices $V'=V(T)-\bigcup_{w\in\mathcal{M}(T)}V_w$. Note that $|V_w|\geq\varsigma(T)+1$ for every $w\in\mathcal{M}(T)$. With this fact in mind, we consider three cases.

Case 1: $x\in V_w$ and $y\in V_{w'}$ for some $w,w'\in\mathcal{M}(T)$, $w\neq w'$. In this case x,y are distinguished by w or by w'. Now, if w distinguishes the pair x,y, then at most one element of V_w does not distinguish x,y (see Figure 6). So, x and y are distinguished by at least $|V_w|-1$ vertices of T or by at least $|V_{w'}|-1$ vertices of T.

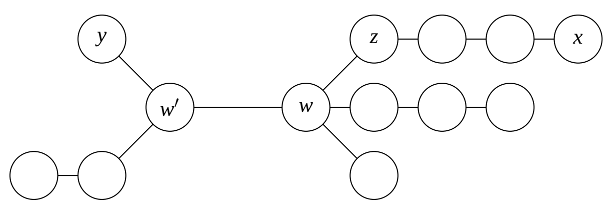

Fig. 6: In this example, w distinguishes the pair x,y, and z is the only vertex in V_w that does not distinguish x,y.

Case 2: $x\in V'$ or $y\in V'$. Thus, $V'\neq\emptyset$ and, as a consequence, $|\mathcal{M}(T)|\geq 2$. Hence, we have one of the following situations.

– There exist two vertices $w,w'\in\mathcal{M}(T)$, $w\neq w'$, such that the shortest path from x to w and the shortest path from y to w' have empty intersection, or
– for every vertex $w''\in\mathcal{M}(T)$, it follows that either y belongs to the shortest path from x to w'' or x belongs to the shortest path from y to w''.

In the first case, x,y are distinguished by vertices in V_w or by vertices in $V_{w'}$ and in the second one, x,y are distinguished by vertices in $V_{w''}$.

Case 3: $x,y \in V_w$ for some $w \in \mathscr{M}(T)$. If $x,y \in V(P(u_l,w))$ for some $l \in \{1,\ldots,\mathrm{ter}(w)\}$, then there exists at most one vertex of $V(P(u_l,w))$ which does not distinguish x,y. Since $\mathrm{ter}(w) \geq 2$, the vertex w has a terminal vertex u_q with $q \neq l$. So, x,y are distinguished by at least $|V(P(u_l,w,u_q))| - 1$ vertices, and since $|V(P(u_l,w,u_q))| \geq \varsigma(T)+1$, we are done. If $x \in V(P(u_l,w)$ and $y \in V(P(u_q,w)$ for some $l,q \in \{1,\ldots,\mathrm{ter}(w)\}$, $l \neq q$, then there exists at most one vertex of $V(P(u_l,w,u_q))$ which does not distinguish x,y. Since $|V(P(u_l,w,u_q))| \geq \varsigma(T)+1$, the result follows.

Therefore, $\varsigma(T) = \min\limits_{x,y\in V(T)} |\mathscr{D}_T(x,y)|$ and by Theorem 1 the result follows.

Since any path is a particular case of a tree and its behavior with respect to the k-metric dimension is different, here we analyze them in first instance. In Proposition 3 we noticed that the 2-metric dimension of a path $P_n (n \geqslant 2)$ is two. Here we give a formula for the k-metric dimension of any path graph for $k \geq 3$.

Proposition 5. *Let $k \geq 3$ be an integer. For any path graph P_n of order $n \geq k+1$,*

$$\dim_k(P_n) = k+1.$$

Proof. Let v_1 and v_n be the leaves of P_n and let S be a k-metric basis of P_n. Since $|S| \geq k \geq 3$, there exists at least one vertex $w \in S \cap (V(P_n) - \{v_1,v_n\})$. For any vertex $w \in V(P_n) - \{v_1,v_n\}$ there exist at least two vertices $u,v \in V(P_n)$ such that w does not distinguish u and v. Hence, $|S| = \dim_k(P_n) \geq k+1$.

Now, notice that for any pair of different vertices $u,v \in V(P_n)$ there exists at most one vertex $w \in V(P_n) - \{v_1,v_n\}$ such that w does not distinguish u and v. Thus, we have that for every $S \subseteq V(P_n)$ such that $|S| = k+1$ and every pair of different vertices $x,y \in V(P_n)$, there exists at least k vertices of S such that they distinguish x,y. So S is a k-metric generator for P_n. Therefore, $\dim_k(P_n) \leq |S| = k+1$ and, consequently, the result follows.

Once studied the path graphs, we are now able to give a formula for the r-metric dimension of any k-metric dimensional tree different from a path which, among other usefulness, shows that Theorem 8 is tight.

Theorem 10. *If T is a tree which is not a path, then for any $r \in \{1,\ldots,\varsigma(T)\}$,*

$$\dim_r(T) = \sum_{w\in\mathscr{M}(T)} I_r(w).$$

Proof. Since T is not a path, T contains at least one vertex belonging to $\mathscr{M}(T)$. Let $w \in \mathscr{M}(T)$ and let $T_w = (V_w,E_w)$ be the w-branch. Also we consider the set $V' = V(T) - \bigcup_{w\in\mathscr{M}(T)} V_w$. For every $w \in \mathscr{M}(T)$, we suppose u_1 is a terminal vertex of w such that $l(u_1,w) = l(w)$. Let $U(w) = \{u_1,u_2,\ldots,u_s\}$ be the set of terminal vertices of w. Now, for every $u_j \in U(w)$, let the path $P(u_j,w) = u_ju_j^1u_j^2\ldots u_j^{l(u_j,w)-1}w$ and we consider the set $S(u_j,w) \subset V(P(u_j,w)) - \{w\}$ given by:

$$S(u_1,w) = \begin{cases} \left\{u_1,u_1^1,\ldots,u_1^{l(w)-1}\right\}, & \text{if } l(w) \leq \lfloor \frac{r}{2} \rfloor \\ \left\{u_1,u_1^1,\ldots,u_1^{\lfloor \frac{r}{2} \rfloor -1}\right\}, & \text{if } l(w) > \lfloor \frac{r}{2} \rfloor. \end{cases}$$

and for $j \neq 1$,

$$S(u_j,w) = \begin{cases} \left\{u_j,u_j^1,\ldots,u_j^{r-l(w)-1}\right\}, & \text{if } l(w) \leq \lfloor \frac{r}{2} \rfloor, \\ \left\{u_j,u_j^1,\ldots,u_j^{\lceil \frac{r}{2} \rceil -1}\right\}, & \text{if } l(w) > \lfloor \frac{r}{2} \rfloor. \end{cases}$$

According to this we have,

$$|S(u_j,w)| = \begin{cases} l(w), & \text{if } l(w) \leq \lfloor \frac{r}{2} \rfloor \text{ and } u_j = u_1, \\ r-l(w), & \text{if } l(w) \leq \lfloor \frac{r}{2} \rfloor \text{ and } u_j \neq u_1, \\ \lfloor \frac{r}{2} \rfloor, & \text{if } l(w) > \lfloor \frac{r}{2} \rfloor \text{ and } u_j = u_1, \\ \lceil \frac{r}{2} \rceil, & \text{if } l(w) > \lfloor \frac{r}{2} \rfloor \text{ and } u_j \neq u_1. \end{cases}$$

Let $S(w) = \bigcup\limits_{u_j\in U(w)} S(u_j,w)$ and $S = \bigcup\limits_{w\in\mathscr{M}(T)} S(w)$. Since for every $w \in \mathscr{M}(T)$ it follows that $\bigcap\limits_{u_j\in U(w)} S(u_j,w) = \emptyset$ and $\bigcap\limits_{w\in\mathscr{M}(T)} S(w) = \emptyset$, we obtain that $|S| = \sum\limits_{w\in\mathscr{M}(T)} I_r(w)$.

Also notice that for every $w \in \mathscr{M}(T)$, such that $\mathrm{ter}(w) = 2$ we have $|S(w)| = r$ and, if $\mathrm{ter}(w) > 2$, then we have $|S(w)| \geq r+1$. We claim that S is an r-metric generator for T. Let u,v be two distinct vertices of T. We consider the following cases.

Case 1: $u,v \in V_w$ for some $w \in \mathscr{M}(T)$. We have the following subcases.

Subcase 1.1: $u,v \in V(P(u_j,w))$ for some $j \in \{1,\ldots,\mathrm{ter}(w)\}$. Hence, there exists at most one vertex of $S(w) \cap V(P(u_j,w))$ which does not distinguish u,v. If $\mathrm{ter}(w) = 2$, then there exists at least one more exterior major vertex $w' \in \mathscr{M}(T) - \{w\}$. So, the elements of $S(w')$ distinguish u,v. Since $|S(w')| \geq r$, we deduce that at least r elements of S distinguish u,v. On the other hand, if $\mathrm{ter}(w) > 2$, then since $|S(w)| \geq r+1$, we obtain that at least r elements of $S(w)$ distinguish u,v.

Subcase 1.2: $u \in V(P(u_j,w))$ and $v \in V(P(u_l,w))$ for some $j,l \in \{1,\ldots,\mathrm{ter}(w)\}$, $j \neq l$. According to the construction of the set $S(w)$, there exists at most one vertex of $(S(w) \cap (V(P(u_j,w,u_l)))$ which does not distinguish u,v.

Now, if $\mathrm{ter}(w) = 2$, then there exists $w' \in \mathscr{M}(T) - \{w\}$. If $d(u,w) = d(v,w)$, then the r

elements of $S(w)$ distinguish u,v and, if $d(u,w) \neq d(v,w)$, then the elements of $S(w')$ distinguish u,v.

On the other hand, if $\mathrm{ter}(w) > 2$, then since $|S(w)| \geq r+1$, we deduce that at least r elements of $S(w)$ distinguish u,v.

Case 2: $u \in V_w, v \in V_{w'}$, for some $w,w' \in \mathcal{M}(T)$ with $w \neq w'$. In this case, either the vertices in $S(w)$ or the vertices in $S(w')$ distinguish u,v. Since $|S(w)| \geq r$ and $|S(w')| \geq r$ we have that u,v are distinguished by at least r elements of S.

Case 3: $u \in V'$ or $v \in V'$. Without loss of generality we assume $u \in V'$. Since $V' \neq \emptyset$, we have that there exist at least two different vertices in $\mathcal{M}(T)$. Hence, we have either one of the following situations.

– There exist two vertices $w,w' \in \mathcal{M}(T)$, $w \neq w'$, such that the shortest path from u to w and the shortest path from v to w' have empty intersection, or
– for every vertex $w'' \in \mathcal{M}(T)$, it follows that either v belongs to every shortest path from u to w'' or u belongs to every shortest path from v to w''.

Notice that in both situations, since $|S(w)| \geq r$, for every $w \in \mathcal{M}(T)$), we have that u,v are distinguished by at least r elements of S. In the first case, u and v are distinguished by the elements of $S(w)$ or by the elements of $S(w')$ and, in the second one, u and v are distinguished by the elements of $S(w'')$.

Therefore, S is an r-metric generator for T and, by Theorem 8, the proof is complete.

In the case $r = 1$, the formula of Theorem 10 leads to

$$\dim(T) = \mu(T) - |\mathcal{M}(T)|,$$

which is a result obtained in [5]. Other interesting particular cases are the following ones for $r = 2$ and $r = 3$, respectively. That is, by Theorem 10 we have the next results.

Corollary 8. *If T is a tree different from a path, then*

$$\dim_2(T) = \mu(T).$$

Corollary 9. *If T is a tree different from a path with $\varsigma(T) \geq 3$, then*

$$\dim_3(T) = 2\mu(T) - |\mathcal{M}(T)|.$$

As mentioned before, the two corollaries above show that the bounds given in Corollaries 6 and 7 are achieved. We finish our exposition with a formula for the k-metric dimension of a k-metric dimensional tree with some specific structure, also showing that the inequality $\dim_k(T) \geq |\mathcal{D}_k(T)|$, given in Remark 2, can be reached.

Proposition 6. *Let T be a tree different from a path and let $k \geq 2$ be an integer. If* $\mathrm{ter}(w) = 2$ *and* $\varsigma(w) = k$ *for every $w \in \mathcal{M}(T)$, then* $\dim_k(T) = |\mathcal{D}_k(T)|$.

Proof. Since every vertex $w \in \mathcal{M}(T)$ satisfies that $\mathrm{ter}(w) = 2$ and $\varsigma(w) = k$, we have that $\varsigma(T) = k$. Thus, by Theorem 9, T is k-metric dimensional tree. Since $I_k(w) = k$ for every $w \in \mathcal{M}(T)$, by Theorem 10 we have that $\dim_k(T) = k|\mathcal{M}(T)|$. Let u_r, u_s be the terminal vertices of w. As we have shown in the proof of Theorem 9, for every pair $x,y \in V(T)$ such that $x \notin V(P(u_r,w,u_s)) - \{w\}$ or $y \notin V(P(u_r,w,u_s)) - \{w\}$, it follows that x,y are distinguished by at least $k+1$ vertices of T and so $|\mathcal{D}_T^*(x,y)| > k-2$. Hence, if $|\mathcal{D}_T^*(x,y)| = k-2$, then $x,y \in V(P(u_r,w,u_s)) - \{w\}$ for some $w \in \mathcal{M}(T)$. If $d(x,w) \neq d(y,w)$, then x,y are distinguished by more than k vertices (those vertices not in $V(P(u_r,w,u_s)) - \{w\}$). Thus, if $|\mathcal{D}_T^*(x,y)| = k-2$, then $d(x,w) = d(y,w)$ and, as a consequence, $\mathcal{D}_T^*(x,y) = V(P(u_r,w,u_s)) - \{x,y,w\}$. Considering that $|V(P(u_r,w,u_s)) - \{w\}| = k$ and at the same time that $\bigcap_{w \in \mathcal{M}(T)} V(P(u_r,w,u_s)) = \emptyset$, we deduce $|\mathcal{D}_k(T)| = k|\mathcal{M}(T)|$. Therefore, $\dim_k(T) = |\mathcal{D}_k(T)|$.

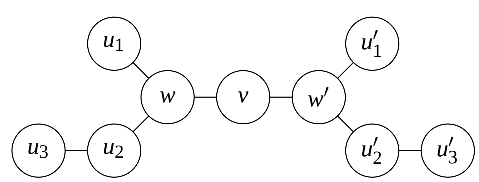

Fig. 7: A 3-metric dimensional tree T for which $\dim_3(T) = |\mathcal{D}_3(T)| = 6$.

Figure 7 shows an example of a 3-metric dimensional tree. In this case $\mathcal{M}(T) = \{w,w'\}$, $\mathrm{ter}(w) = \mathrm{ter}(w') = 2$ and $\varsigma(w) = \varsigma(w') = 3$. Then Proposition 6 leads to $\dim_3(T) = |\mathcal{D}_3(T)| = |\{u_1,u_2,u_3,u_1',u_2',u_3'\}| = 6$.

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