Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/050108

Hermite-Hadamard Inequality for Geometrically Quasiconvex Functions on a Rectangular Box

Ali Barani and Fatemeh Malmir*

Department of Mathematics, Lorestan University, P. O. Box 465, Khoramabad, Iran

Received: 27 Jul. 2015, Revised: 20 Oct. 2015, Accepted: 23 Oct. 2015
Published online: 1 Jan. 2016

Abstract: In this paper, we introduce the concept of geometrically quasiconvex functions on a rectangular box in $\mathbb{R}^3$. Then, some Hermite-Hadamard type inequalities for functions whose third derivatives in absolute value are geometrically convex are given.

Keywords: Hermite-Hadamard inequality, rectangular box, geometrically quasiconvex functions

1 Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if for every $x, y \in I$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y).$$

Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, we have the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This remarkable result is well known in the literature as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Since then some refinements of the Hermite-Hadamard inequality for convex functions have been extensively investigated by number of authors, see for example [1-10,14-16]). Let $I \subseteq \mathbb{R}_+ := (0,\infty)$ be an inerval and $f : I \to \mathbb{R}_+$ be a continuous function. Then, f is said to be geometrically convex on I, if for every $x, y \in I$ and $\lambda \in [0,1]$,

$$f(x^\lambda y^{1-\lambda}) \leq f(x)^\lambda f(y)^{1-\lambda}.$$

In [6], S.S. Dragomir defined convex functions on the co-ordinates (or co-ordinated convex functions) on the set $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mappings,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are convex. This means that for every $(x,y),(z,w) \in \Delta$ and $t, s \in [0,1]$,

$$\begin{aligned} & f(tx+(1-t)z, sy+(1-s)w) \\ \leq\ & tsf(x,y) + s(1-t)f(z,y) \\ & + t(1-s)f(x,w) + (1-t)(1-s)f(z,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. Since then several important generalizations introduced on this category, see [11, 18-20] and references therein. Recall that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be quasiconvex if for every $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

In [13], M.E. Özdemir et al. introduced the notion of co-ordinated quasiconvex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1.2. A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be quasiconvex on the co-ordinates on Δ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mapping,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

* Corresponding author e-mail: barani.a@lu.ac.ir

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are quasiconvex. This means that for every $(x,y),(z,w) \in \Delta$ and $s,t \in [0,1]$,

$$\begin{aligned} &f(tx+(1-t)z, sy+(1-s)w) \\ \leq\ &\max\{f(x,y), f(x,w), f(z,y), f(z,w)\}. \end{aligned}$$

Since then several important generalizations on this category proved by M.E. Özdemir et al. in [9,12,13]. On the other hand F. Qi and B.A. Xi in [19] introduced the notion of geometrically quasiconvex functions and established some integral inequalities of Hermite-Hadamard type.

Definition 1.3. A function $f : I \subseteq \mathbb{R}_0 := [0,\infty) \to \mathbb{R}_0$, is said to be geometrically quasiconvex on I if for every $x,y \in I$ and $\lambda \in [0,1]$,

$$f(x^\lambda y^{1-\lambda}) \leq \max\{f(x), f(y)\}.$$

Note that if f decreasing and geometrically quasiconvex then, it is quasiconvex. If f increasing and quasiconvex then, it is geometrically quasiconvex. We recall some results introduced in [19].

Lemma 1.1. Let $f : I \subseteq \mathbb{R}_+ := (0,\infty) \to \mathbb{R}$, be a differentiable function on I° and $a,b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ then,

(i)

$$\begin{aligned} &\frac{(\ln b) f(b) - (\ln a) f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \\ = &\int_0^1 a^{1-t}b^t \ln(a^{1-t}b^t) f'(a^{1-t}b^t)dt. \end{aligned} \tag{1}$$

(ii)

$$M(a,b) := \int_0^1 |\ln(a^{1-t}b^t)|dt, \tag{2}$$

$$N(a,b) := \int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)|dt. \tag{3}$$

In [14], M. E. Özdemir defined the concept of geometrically convex functions on the co-ordinates as follows:

Definition 1.4. Let $\Delta_+ := [a,b] \times [c,d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. A function $f : \Delta_+ \to \mathbb{R}$ is said to be geometrically convex on the co-ordinates if for every $y \in [c,d]$ and $x \in [a,b]$ the partial mappings,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are geometrically convex function. This means that for every $(x,y),(z,w) \in \Delta_+$ and $t,s \in [0,1]$,

$$\begin{aligned} &f(x^t z^{1-t}, y^s w^{1-s}) \\ \leq\ &tsf(x,y) + s(1-t)f(z,y) \\ &+ t(1-s)f(x,w) + (1-t)(1-s)f(z,w). \end{aligned}$$

In [15], M.E. Özdemir defined convex functions on a rectangular box as follows:
Let us consider the rectangular box $\Omega := [a,b] \times [c,d] \times [e,f]$ in $\mathbb{R}^3$. The mapping $g : \Omega \to \mathbb{R}$ is said to be convex on Ω if for every $(x,y,z),(u,v,w) \in \Omega$ and $\lambda \in [0,1]$,

$$\begin{aligned} &g(\lambda x + (1-\lambda)u, \lambda y + (1-\lambda)v, \lambda z + (1-\lambda)w) \\ \leq\ &\lambda g(x,y,z) + (1-\lambda)g(u,v,w). \end{aligned}$$

A function $g : \Omega \to \mathbb{R}$ is said to convex on the co-ordinates on Ω if for every $(x,y) \in [a,b] \times [c,d]$, $(x,z) \in [a,b] \times [e,f]$ and $(y,z) \in [c,d] \times [e,f]$, the partial mapping,

$$\begin{aligned} &g_z : [a,b] \times [c,d] \to \mathbb{R}, \quad g_z(u,v) = g(u,v,z), \quad z \in [e,f], \\ &g_y : [a,b] \times [e,f] \to \mathbb{R}, \quad g_y(u,w) = g(u,y,w), \quad y \in [c,d], \\ &g_x : [c,d] \times [e,f] \to \mathbb{R}, \quad g_x(v,w) = g(x,v,w), \quad x \in [a,b], \end{aligned}$$

are convex. In [4], the notion of co-ordinated quasiconvex functions on a rectangular box in $\mathbb{R}^3$ which generalize the notion of co-ordinated convex functions is given as follows:

Definition 1.5. A function $g : \Omega \to \mathbb{R}$ is said to quasiconvex on the co-ordinates on Ω if for every $(x,y) \in [a,b] \times [c,d]$, $(x,z) \in [a,b] \times [e,f]$ and $(y,z) \in [c,d] \times [e,f]$, the partial mapping,

$$\begin{aligned} &g_z : [a,b] \times [c,d] \to \mathbb{R}, \quad g_z(u,v) = g(u,v,z), \quad z \in [e,f], \\ &g_y : [a,b] \times [e,f] \to \mathbb{R}, \quad g_y(u,w) = g(u,y,w), \quad y \in [c,d], \\ &g_x : [c,d] \times [e,f] \to \mathbb{R}, \quad g_x(v,w) = g(x,v,w), \quad x \in [a,b], \end{aligned}$$

are quasiconvex. This means that for every $(x,y,z),(u,v,w) \in \Omega$ and $t,s,r \in [0,1]$

$$\begin{aligned} &g(tx+(1-t)u, sy+(1-s)v, rz+(1-r)w) \\ \leq\ &\max\{g(x,y,z), g(x,y,w), g(x,v,z), g(u,y,z), \\ &g(u,v,w), g(u,v,z), g(u,y,w), g(x,v,w)\}. \end{aligned}$$

Motivated by above results in this paper we introduce the notion of "geometrically quasiconvex functions on a rectangular box" in $\mathbb{R}^3$. Then, some results in connection to Hermite-Hadamard inequality are given.

2 Main results

In this section we introduce the notion; "geometrically quasiconvex functions on a rectangular box" for a functions defined on a rectangular in $\mathbb{R}^3_+$, which is a generalization of the notion "geometrically convex functions on a rectangular box". Then, we establish some Hermite-Hadamard type inequalities for this class of functions. In this section, Ω_+ is a rectangular box in $\mathbb{R}^3$ defined by

$$\Omega_+ := [a,b] \times [c,d] \times [e,f],$$

where $a,b,c,d \in \mathbb{R}_+$. In [5] the notion of co-ordinated geometrically quasiconvex functions on the plan in $\mathbb{R}^2$ is given as follows:

Definition 2.1. A function $f : \Delta_+ \to \mathbb{R}$ is said geometrically quasiconvex function on $\Delta_+ \subseteq \mathbb{R}^2_+$ if for every $(x,y),(z,w) \in \Delta_+$ and $\lambda \in [0,1]$,

$$f(x^\lambda z^{1-\lambda}, y^\lambda w^{1-\lambda}) \leq \max\left\{f(x,y), f(z,w)\right\}.$$

Definition 2.2. Let $\Delta_+ := [a,b] \times [c,d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. A function $f : \Delta_+ \to \mathbb{R}$ is said to be geometrically quasiconvex on the co-ordinates on $\Delta_+ \subseteq \mathbb{R}^2_+$ if for every $y \in [c,d]$ and $x \in [a,b]$ the partial mappings

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are geometrically quasiconvex. This means that for every $(x,y),(z,w) \in \Delta_+$ and $s,t \in [0,1]$,

$$f(x^t z^{1-t}, y^s w^{1-s}) \leq \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}.$$

Definition 2.3 A function $g : \Omega_+ \to \mathbb{R}$ will be called geometrically quasiconvex on the co-ordinates on $\Omega_+ \subseteq \mathbb{R}^3$ if for every $(x,y) \in [a,b] \times [c,d]$, $(x,z) \in [a,b] \times [e,f]$ and $(y,z) \in [c,d] \times [e,f]$, the partial mapping,

$$\begin{aligned}
&g_z : [a,b] \times [c,d] \to \mathbb{R}, \quad g_z(u,v) = g(u,v,z), \quad z \in [e,f],\\
&g_y : [a,b] \times [e,f] \to \mathbb{R}, \quad g_y(u,w) = g(u,y,w), \quad y \in [c,d],\\
&g_x : [c,d] \times [e,f] \to \mathbb{R}, \quad g_x(v,w) = g(x,v,w), \quad x \in [a,b],
\end{aligned}$$

are geometrically quasiconvex. This means that for every $(x,y,z),(u,v,w) \in \Omega_+$ and $t,s,r \in [0,1]$

$$\begin{aligned}
&f(x^t u^{1-t}, y^s v^{1-s}, z^r w^{1-r})\\
\leq\ &\max\{g(x,y,z), g(x,y,w), g(x,v,z), g(u,y,z)\\
&g(u,v,w), g(u,v,z), g(u,y,w), g(x,v,w)\}.
\end{aligned}$$

Definition 2.4. A function $g : \Omega_+ \to \mathbb{R}$ is said geometrically quasiconvex function on $\Omega_+ \subseteq \mathbb{R}^3_+$ if for every $(x,y,z),(u,v,w) \in \Omega_+$ and $\lambda \in [0,1]$,

$$g(x^\lambda u^{1-\lambda}, y^\lambda v^{1-\lambda}, z^\lambda w^{1-\lambda}) \leq \max\left\{g(x,y,z), g(u,v,w)\right\}.$$

The following lemma holds.

Lemma 2.1. Every geometrically quasiconvex mapping $g : \Omega_+ \to \mathbb{R}$ is geometrically quasiconvex on the co-ordinates.

Proof. Suppose that $g : \Omega_+ \to \mathbb{R}$ is geometrically quasiconvex on Ω_+. Then for every $(x,y) \in [a,b] \times [c,d]$, $(x,z) \in [a,b] \times [e,f]$ and $(y,z) \in [c,d] \times [e,f]$, the partial mapping,

$$\begin{aligned}
&g_z : [a,b] \times [c,d] \to \mathbb{R}, \quad g_z(u,v) = g(u,v,z), \quad z \in [e,f],\\
&g_y : [a,b] \times [e,f] \to \mathbb{R}, \quad g_y(u,w) = g(u,y,w), \quad y \in [c,d],\\
&g_x : [c,d] \times [e,f] \to \mathbb{R}, \quad g_x(v,w) = g(x,v,w), \quad x \in [a,b],
\end{aligned}$$

are geometrically quasiconvex on Ω_+. For $\lambda \in [0,1]$ and $v_1, v_2 \in [c,d]$, $w_1, w_2 \in [e,f]$, one has

$$\begin{aligned}
g_x(v_1^\lambda v_2^{1-\lambda}, w_1^\lambda w_2^{1-\lambda}) &= g(x, v_1^\lambda v_2^{1-\lambda}, w_1^\lambda w_2^{1-\lambda})\\
&= g(x^\lambda x^{1-\lambda}, v_1^\lambda v_2^{1-\lambda}, w_1^\lambda w_2^{1-\lambda})\\
&\leq \max\{g(x,v_1,w_1), g(x,v_2,w_2)\}\\
&= \max\{g_x(v_1,w_1), g_x(v_2,w_2)\},
\end{aligned}$$

which completes the proof of geometrically quasiconvexity of g_x on $[c,d] \times [e,f]$. Therefore g_y and g_z is also geometrically quasiconvex on $[a,b] \times [e,f]$ and $[a,b] \times [c,d]$ for all $y \in [c,d]$ and $z \in [e,f]$ goes likewise and we shall omit the details. $\square$.

Now we introduce the following new lemma which we need to reach our goal.

Lemma 2.2. Suppose that $g : \Omega_+ \to \mathbb{R}$ be a partial differentiable mapping on $\mathrm{int}(\Omega_+)$. If $\frac{\partial^3 g}{\partial t \partial s \partial r} \in L(\Omega_+)$ then,

$$\begin{aligned}
&\frac{1}{(\ln b - \ln a)(\ln d - \ln c)(\ln f - \ln e)}\\
&\times \left(C + D - \iiint_\Omega \frac{g(x,y,z)}{xyz} dxdydz + E - F\right)\\
= &\int_0^1 \int_0^1 \int_0^1 a^{1-t} b^t c^{1-s} d^s e^{1-r} f^r\\
&\times \ln(a^{1-t} b^t) \ln(c^{1-s} d^s) \ln(e^{1-r} f^r)\\
&\times \frac{\partial^3 g}{\partial t \partial s \partial r}(a^{1-t} b^t, c^{1-s} d^s, e^{1-r} f^r) dtdsdr,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
C := (\ln f)\Big(&(\ln d)\big[(\ln b) g(b,d,f) - (\ln a) g(a,d,f)\big]\\
&- (\ln c)\big[(\ln b) g(b,c,f) - (\ln a) g(a,c,f)\big]\Big),\\
D := (\ln e)\Big(&(\ln d)\big[(\ln a) g(a,d,e) - (\ln b) g(b,d,e)\big]\\
&- (\ln c)\big[(\ln a) g(b,c,e) - (\ln b) g(a,c,e)\big]\Big),
\end{aligned}$$

$$\begin{aligned}
E := \quad &(\ln f) \iint_{\Delta_{+1}} \frac{g(x,y,f)}{xy}\\
&-(\ln e) \iint_{\Delta_{+1}} \frac{g(x,y,e)}{xy} dxdy\\
&+(\ln d) \iint_{\Delta_{+2}} \frac{g(x,d,z)}{xz} dxdz\\
&-(\ln c) \iint_{\Delta_{+2}} \frac{g(x,c,z)}{xz} dxdz\\
&+(\ln b) \iint_{\Delta_{+3}} \frac{g(b,y,z)}{yz} dydz\\
&-(\ln a) \iint_{\Delta_{+3}} \frac{g(a,y,z)}{yz} dydz,
\end{aligned}$$

where $\Delta_{+_1} = [a,b]\times[c,d]$, $\Delta_{+_2} = [a,b]\times[e,f]$ and $\Delta_{+_3} = [c,d]\times[e,f]$ are subsets of $\mathbb{R}^2_+$ with $a<b$, $c<d$ and $e<f$, and

$$\begin{aligned}
F := &(\ln b)\left\{(\ln d)\int_e^f \frac{g(b,d,z)}{z}dz\right.\\
&\left.-(\ln c)\int_e^f \frac{g(b,c,z)}{z}dz\right\}\\
&-(\ln a)\left\{(\ln d)\int_e^f \frac{g(a,d,z)}{z}dz\right.\\
&\left.-(\ln c)\int_e^f \frac{g(a,c,z)}{z}dz\right\}\\
&+(\ln f)\left\{(\ln b)\int_c^d \frac{g(b,y,f)}{y}dy\right.\\
&\left.-(\ln a)\int_c^d \frac{g(a,y,f)}{y}dy\right\}\\
&-(\ln e)\left\{(\ln b)\int_c^d \frac{g(b,y,e)}{y}dy\right.\\
&\left.-(\ln a)\int_c^d \frac{g(a,y,e)}{y}dy\right\}\\
&+(\ln d)\left\{(\ln f)\int_a^b \frac{g(x,d,,f)}{x}dx\right.\\
&\left.-(\ln e)\int_a^b \frac{g(x,d,e)}{x}dx\right\}\\
&-(\ln c)\left\{(\ln f)\int_a^b \frac{g(x,c,,f)}{x}dx\right.\\
&\left.-(\ln e)\int_a^b \frac{g(x,c,e)}{x}dx\right\}.
\end{aligned}$$

Proof. We denote the right hand side of (4) by I. Letting $x=a^{1-t}b^t$, $y=c^{1-s}d^s$ and $z=e^{1-r}f^r$ for $t,s,r\in[0,1]$. By integration by parts on Ω_+, we have

$$\begin{aligned}
&(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)\times I\\
=\ &(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)\\
&\times\int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\
&\times\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\\
&\times\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)dtdsdr\\
=\ &\int_e^f\int_c^d\int_a^b(\ln x)(\ln y)(\ln z)\frac{\partial^3 g}{\partial x\partial y\partial z}(x,y,z)dxdydz\\
=\ &\int_e^f\int_c^d(\ln z)(\ln y)\left\{(\ln x)\frac{\partial^2 g}{\partial y\partial z}(x,y,z)\Big|_a^b\right.\\
&\left.-\int_a^b\frac{1}{x}\frac{\partial^2 g}{\partial y\partial z}(x,y,z)dx\right\}dydz\\
=\ &\int_e^f\int_c^d(\ln z)(\ln y)\left\{(\ln b)\frac{\partial^2 g}{\partial y\partial z}(b,y,z)\right.\\
&\left.-(\ln a)\frac{\partial^2 g}{\partial y\partial z}(a,y,z)-\int_a^b\frac{1}{x}\frac{\partial^2 g}{\partial y\partial z}(x,y,z)dx\right\}dydz\\
=\ &(\ln b)\int_e^f\int_c^d(\ln z)(\ln y)\frac{\partial^2 g}{\partial y\partial z}(b,y,z)dydz\\
&-(\ln a)\int_e^f\int_c^d(\ln z)(\ln y)\frac{\partial^2 g}{\partial y\partial z}(a,y,z)dydz\\
&-\int_e^f\int_c^d\int_a^b\frac{(\ln z)(\ln y)}{x}\frac{\partial^2 g}{\partial y\partial z}(x,y,z)dxdydz.
\end{aligned}\tag{5}$$

Similarly integration by parts in the right side of (5) deduce that

$$\begin{aligned}
&(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)\times I\\
=\ &(\ln b)\int_e^f(\ln z)\left\{(\ln y)\frac{\partial g}{\partial z}(b,y,z)\Big|_c^d\right.\\
&\left.-\int_c^d\frac{1}{y}\frac{\partial g}{\partial z}(b,y,z)dy\right\}dz\\
&-(\ln a)\int_e^f(\ln z)\left\{(\ln y)\frac{\partial g}{\partial z}(a,y,z)\Big|_c^d\right.\\
&\left.-\int_c^d\frac{1}{y}\frac{\partial g}{\partial z}(a,y,z)dy\right\}dz\\
&-\int_e^f\int_a^b(\ln z)\frac{1}{x}\left\{(\ln y)\frac{\partial g}{\partial z}(x,y,z)\Big|_c^d\right.\\
&\left.-\int_c^d\frac{1}{y}\frac{\partial g}{\partial z}(x,y,z)dy\right\}dxdz
\end{aligned}\tag{6}$$

$$
\begin{aligned}
&= (\ln b)\int_e^f (\ln z)\Bigg\{[(\ln d)\frac{\partial g}{\partial z}(b,d,z)-(\ln c)\frac{\partial g}{\partial z}(b,c,z)]\\
&\quad -\int_c^d \frac{1}{y}\frac{\partial g}{\partial z}(b,y,z)dy\Bigg\}dz\\
&\quad -(\ln a)\int_e^f (\ln z)\Bigg\{[(\ln d)\frac{\partial g}{\partial z}(a,d,z)\\
&\quad -(\ln c)\frac{\partial g}{\partial z}(a,c,z)]-\int_c^d \frac{1}{y}\frac{\partial g}{\partial z}(a,y,z)dy\Bigg\}dz\\
&\quad -(\ln d)\int_e^f\int_a^b \frac{(\ln z)}{x}\frac{\partial g}{\partial z}(x,d,z)dxdz\\
&\quad +(\ln c)\int_e^f\int_a^b \frac{(\ln z)}{x}\frac{\partial g}{\partial z}(x,c,z)dxdz\\
&\quad +\int_e^f\int_c^d\int_a^b \frac{(\ln z)}{xy}\frac{\partial g}{\partial z}(x,y,z)dxdydz.
\end{aligned}
$$

Again integration by parts in the right side of (6) deduce that

$$
\begin{aligned}
&(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)\times I\\
&= (\ln b)(\ln d)\Bigg\{(\ln z)g(b,d,z)\Big|_e^f-\int_e^f\frac{g(b,d,z)}{z}dz\Bigg\}\\
&\quad -(\ln b)(\ln c)\Bigg\{(\ln z)g(b,c,z)\Big|_e^f-\int_e^f\frac{g(b,c,z)}{z}dz\Bigg\}\\
&\quad -\int_c^d\frac{(\ln b)}{y}\Bigg\{(\ln z)g(b,y,z)\Big|_e^f-\int_e^f\frac{g(b,y,z)}{z}dz\Bigg\}dy\\
&\quad -(\ln a)(\ln d)\Bigg\{(\ln z)g(a,d,z)\Big|_e^f-\int_e^f\frac{g(a,d,z)}{z}dz\Bigg\}\\
&\quad +(\ln a)(\ln c)\Bigg\{(\ln z)g(a,c,z)\Big|_e^f-\int_e^f\frac{g(a,c,z)}{z}dz\Bigg\}\\
&\quad +\int_c^d\frac{(\ln a)}{y}\Bigg\{(\ln z)g(a,y,z)\Big|_e^f-\int_e^f\frac{g(a,y,z)}{z}dz\Bigg\}dy\\
&\quad -\int_a^b\frac{(\ln d)}{x}\Bigg\{(\ln z)g(x,d,z)\Big|_e^f-\int_e^f\frac{g(x,d,z)}{z}dz\Bigg\}dx\\
&\quad +\int_a^b\frac{(\ln c)}{x}\Bigg\{(\ln z)g(x,c,z)\Big|_e^f-\int_e^f\frac{g(x,c,z)}{z}dz\Bigg\}dx\\
&\quad +\int_c^d\int_a^b\frac{1}{xy}\Bigg\{(\ln z)g(x,y,z)\Big|_e^f-\int_e^f\frac{g(x,y,z)}{z}dz\Bigg\}dxdy\\
&= (\ln b)(\ln d)\Bigg\{\big[(\ln f)g(b,d,f)-(\ln e)g(b,d,e)\big]\\
&\quad -\int_e^f\frac{g(b,d,z)}{z}dz\Bigg\}\\
&\quad -(\ln b)(\ln c)\Bigg\{\big[(\ln f)g(b,c,f)-(\ln e)g(b,c,e)\big]\\
&\quad -\int_e^f\frac{g(b,c,z)}{z}dz\Bigg\}\\
&\quad -\int_c^d\frac{(\ln b)}{y}\Bigg\{(\ln f)g(b,y,f)-(\ln e)g(b,y,e)\\
&\quad -\int_e^f\frac{g(b,y,z)}{z}dz\Bigg\}dy\\
&\quad -(\ln a)(\ln d)\Bigg\{\big[(\ln f)g(a,d,f)-(\ln e)g(a,d,e)\big]\\
&\quad -\int_e^f\frac{g(a,d,z)}{z}dz\Bigg\}\\
&\quad +(\ln a)(\ln c)\Bigg\{\big[(\ln f)g(a,c,f)-(\ln e)g(a,c,e)\big]\\
&\quad -\int_e^f\frac{g(a,c,z)}{z}dz\Bigg\}\\
&\quad +\int_c^d\frac{(\ln a)}{y}\Bigg\{(\ln f)g(a,y,f)-(\ln e)g(a,y,e)\\
&\quad -\int_e^f\frac{g(a,y,z)}{z}dz\Bigg\}dy\\
&\quad -\int_a^b\frac{(\ln d)}{x}\Bigg\{(\ln f)g(x,d,f)-(\ln e)g(x,d,e)\\
&\quad -\int_e^f\frac{g(x,d,z)}{z}dz\Bigg\}dx\\
&\quad +\int_a^b\frac{(\ln c)}{x}\Bigg\{(\ln f)g(x,c,f)-(\ln e)g(x,c,e)\\
&\quad -\int_e^f\frac{g(x,c,z)}{z}dz\Bigg\}dx\\
&\quad +\int_c^d\int_a^b\frac{1}{xy}\Bigg\{(\ln f)g(x,y,f)-(\ln e)g(x,y,e)\\
&\quad -\int_e^f\frac{g(x,y,z)}{z}dz\Bigg\}dxdy.
\end{aligned}
\tag{7}
$$

Dividing both sides of (7) by

$$(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)$$

implies that the equation (4) holds and proof is completed. □.

Theorem 2.1. Let $g:\Omega_+\to\mathbb{R}$ be a partial differentiable mapping on $\mathrm{int}(\Omega_+)$ and $\frac{\partial^3 g}{\partial t\partial s\partial r}\in L(\Omega_+)$. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|$ is a geometrically quasiconvex function on the co-ordinates on

Ω_+ then the following inequality holds:

$$\begin{aligned}&\Bigg|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\\&-\frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\Bigg|\\&\le N(a,b)\,N(c,d)\,N(e,f)\\&\quad\times\max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|,\\&\quad\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|,\\&\quad\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|,\\&\quad\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\},\end{aligned} \tag{8}$$

where, C,D, and $N(a,b)$ are defined, respectively, in Lemma 2.2 and Lemma 1.1, and

$$\tilde{E}=\frac{E}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)},$$
$$\tilde{F}=\frac{F}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)},$$

where E,F are defined, in Lemma 2.2.

Proof. From Lemma 2.2, it follows that

$$\begin{aligned}&\Bigg|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\\&-\frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\Bigg|\\\le&\int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|\\&\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|dtdsdr.\end{aligned}$$

Since $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|$ is geometrically quasiconvex on the co-ordinates on Ω_+, we have

$$\begin{aligned}&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|\\\le&\max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\},\end{aligned}$$

where $t,s,r\in[0,1]$. From this inequality and Lemma 1.1, it follows that

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|\\&\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|dtdsdr\\\le&\max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\}\\&\times\int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|\\=&\;N(a,b)\,N(c,d)\,N(e,f)\\&\times\max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\},\end{aligned}$$

which is the required inequality (8), since

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\\=&\left(\int_0^1 a^{(1-t)}b^t|\ln(a^{1-t}b^t)|dt\right)\\&\times\left(\int_0^1 c^{(1-s)}d^s|\ln(c^{1-s}d^s)|ds\right)\\&\times\left(\int_0^1 e^{1-r}f^r|\ln(e^{1-r}f^r)|dt\right)\\=&\;N(a,b)\,N(c,d)\,N(e,f).\end{aligned}$$

The proof of theorem is completed. □.

The following corollary is an immediate consequence of theorem 2.1.

Corollary 2.1. Suppose the conditions of the Theorem 2.1 are satisfied. Additionally, if

(1) $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|$ is increasing on the co-ordinates on Ω_+, then

$$\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} - \frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right| \leq N(a,b)\,N(c,d)\,N(e,f)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}g(b,d,f)\right|. \tag{9}$$

(2) $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|$ is decreasing on the co-ordinates on Ω_+, then

$$\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} - \frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right| \leq N(a,b)\,N(c,d)\,N(e,f)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}g(a,c,e)\right|. \tag{10}$$

$\square$.

Theorem 2.2. Let $g:\Omega_+\to\mathbb{R}$ be a partial differentiable mapping on $\mathrm{int}(\Omega_+)$ and $\frac{\partial^3 g}{\partial t\partial s\partial r}\in L(\Omega_+)$. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Ω_+ and $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} - \frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right| \\
&\leq \left[N(a^p,b^p)\,N(c^p,d^p)\,N(e^p,f^p)\right]^{1/p} \\
&\times\Bigg[\max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q, \\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q, \\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q, \\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg\}\Bigg]^{1/q},
\end{aligned}\tag{11}$$

where, C, D, $\tilde{E},\tilde{F}$ and $N(a,b)$ are defined respectively, in Lemma 2.1, Theorem 2.1 and Lemma 1.1.

Proof. Suppose that $p>1$. From Lemma 2.2 and well-known Hölder inequality for triple integrals, we obtain

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} - \frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right| \\
&\leq \int_0^1\int_0^1\int_0^1 a^{1-t}b^t c^{1-s}d^s e^{1-r}f^r \\
&\quad\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right| \\
&\quad\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|dtdsdr \\
&\leq \Bigg(\int_0^1\int_0^1\int_0^1 a^{p(1-t)}b^{pt}c^{p(1-s)}d^{ps}e^{p(1-r)}f^{pr} \\
&\quad\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{p(1-r)}f^{pr})\right|^p dtdsdr\Bigg)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\int_0^1\int_0^1\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|^q dtdsdr\right)^{\frac{1}{q}}.
\end{aligned}\tag{12}$$

Since $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is geometrically quasiconvex on the co-ordinates on Ω_+, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|^q dtdsdr \\
&\leq \max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q, \\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q, \\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q, \\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg\}.
\end{aligned}\tag{13}$$

We also notice that

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 a^{p(1-t)}b^{pt}c^{p(1-s)}d^{ps}e^{p(1-r)}f^{pr} \\
&\quad\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{p(1-r)}f^{pr})\right|^p dtdsdr \\
&=\left(\int_0^1 a^{p(1-t)}b^{pt}|\ln(a^{1-t}b^t)|^p dt\right) \\
&\quad\times\left(\int_0^1 c^{p(1-s)}d^{ps}|\ln(c^{1-s}d^s)|^p ds\right) \\
&\quad\times\left(\int_0^1 e^{p(1-r)}f^{pr}\ln(e^{p(1-r)}f^{pr})dr\right) \\
&= N(a^p,b^p)\,N(c^p,d^p)\,N(e^p,f^p).
\end{aligned}\tag{14}$$

Combination of (12), (13) and (14), gives the desired inequality (11). Hence the proof of the theorem is completed. $\square$.

The following corollary is an immediate consequence of theorem 2.2.

Corollary 2.2. Suppose the conditions of the Theorem 2.2 are satisfied. Additionally, if

(1) $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is increasing on the co-ordinates on Ω_+, then

$$\begin{aligned}&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\right.\\&\left.-\frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right|\\&\leq \left[N(a^p,b^p)\,N(c^p,d^p)\,N(e^p,f^p)\right]^{1/p}\\&\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}g(b,d,f)\right|.\end{aligned}\tag{15}$$

(2) $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is decreasing on the co-ordinates on Ω_+, then

$$\begin{aligned}&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\right.\\&\left.-\frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right|\\&\leq \left[N(a^p,b^p)\,N(c^p,d^p)\,N(e^p,f^p)\right]^{1/p}\\&\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}g(a,c,e)\right|.\end{aligned}\tag{16}$$

□.

Theorem 2.3. Let $g:\Omega_+\to\mathbb{R}$ be a partial differentiable mapping on $\mathrm{int}(\Omega_+)$ and $\frac{\partial^3 g}{\partial t\partial s\partial r}\in L(\Omega_+)$. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Ω_+ for $q>1$, then the following inequality holds:

$$\begin{aligned}&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\right.\\&\left.-\frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right|\\&\leq [M(a,b)\,M(c,d)\,M(e,f)]^{1/q}\\&\times\left[\left(\frac{q-1}{q}\right)^3 N\left(a^{q/(q-1)},b^{q/(q-1)}\right)\right.\\&\left.\times N\left(c^{q/(q-1)},d^{q/(q-1)}\right)N\left(e^{q/(q-1)},f^{q/(q-1)}\right)\right]^{1-1/q}\\&\times\left[\max\left\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q,\right.\right.\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q,\\&\left.\left.\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\right\}\right]^{1/q},\end{aligned}\tag{17}$$

where, C, D, $\tilde{E}$,$\tilde{F}$ and $M(a,b),N(a,b)$ are defined, respectively, in Lemma 2.2, Theorem 2.1 and Lemma 1.1.

Proof. By Lemma 2.2, Hölder's inequality, and the geometric quasiconvexity of $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ on Ω_+, we have

$$\begin{aligned}&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\right.\\&\left.-\frac{\iiint_\Omega \frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right|\\&\leq \int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|\\&\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|dtdsdr\\&\leq\left[\int_0^1\int_0^1\int_0^1 a^{q(1-t)(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}\right.\\&\times d^{qs/(q-1)}e^{q(1-r)/(q-1)}f^{qr/(q-1)}\\&\left.\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\right]^{1-1/q}\\&\times\left[\int_0^1\int_0^1\int_0^1\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|\right.\\&\left.\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|^q dtdsdr\right]^{1/q}\\&\leq\left[\int_0^1\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}\right.\\&\times d^{qs/(q-1)}e^{q(1-r)/(q-1)}f^{qr/(q-1)}\\&\left.\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\right]^{1-1/q}\\&\times\left[\max\left\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q,\right.\right.\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q,\\&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q,\\&\left.\left.\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\right\}\right]^{1/q}.\end{aligned}$$

Note that by Lemma 1.1 we deduce that,

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}\\
&\times d^{qs/(q-1)}e^{q(1-r)/(q-1)}f^{qr/(q-1)}\\
&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\\
=&\left(\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}\left|\ln(a^{1-t}b^t)\right|dt\right)\\
&\times\left(\int_0^1 c^{q(1-s)/(q-1)}d^{qs/(q-1)}\left|\ln(c^{1-s}d^s)\right|ds\right)\\
&\times\left(\int_0^1 e^{q(1-r)/(q-1)}f^{qr/(q-1)}\left|\ln(e^{1-r}f^r)\right|dr\right)\\
=&\frac{(q-1)^3}{q^3}N\left(a^{q/(q-1)},b^{q/(q-1)}\right)\\
&\times N\left(c^{q/(q-1)},d^{q/(q-1)}\right)N\left(e^{q/(q-1)},f^{q/(q-1)}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtds\\
=&M(a,b)\,M(c,d)\,M(e,f).
\end{aligned}$$

The proof of Theorem 2.3 is completed. □.

Theorem 2.4. Let $g:\Omega_+\to\mathbb{R}$ be a partial differentiable mapping on Ω and $\frac{\partial^3 g}{\partial t\partial s\partial r}\in L(\Omega_+)$. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Ω_+ and $q>\ell>0$, then

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\right.\\
&\left.-\frac{\iiint_\Omega\frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right|\\
\le&\left(\frac{q-1}{q-\ell}\right)^{3(1-1/q)}\left(\frac{1}{\ell}\right)^{3/q}\\
&\times\left[N\left(a^\ell,b^\ell\right)N\left(c^\ell,d^\ell\right)N\left(e^\ell,f^\ell\right)\right]^{1/q}\\
&\times\left[N\left(a^{(q-\ell)/(q-1)},b^{(q-\ell)/(q-1)}\right)\right.\\
&\times N\left(c^{(q-\ell)/(q-1)},d^{(q-\ell)/(q-1)}\right)\\
&\left.\times N\left(e^{(q-\ell)/(q-1)},f^{(q-\ell)/(q-1)}\right)\right]^{(1-1/q)}\\
&\times\left[\max\left\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q,\right.\right.\\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q,\\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q,\\
&\left.\left.\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\right\}\right]^{1/q},
\end{aligned}\tag{18}$$

where, C, D, $\tilde{E},\tilde{F}$ and $N(a,b)$ are defined, respectively, in Lemma 2.2, Theorem 2.1 and Lemma 1.1.

Proof. From Lemma 2.2, Hölder's inequality, and the geometric quasiconvexity of $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ on the co-ordinates on Ω_+ we get,

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}\right.\\
&\left.-\frac{\iiint_\Omega\frac{g(x,y,z)}{xyz}dxdydz}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)}+\tilde{E}-\tilde{F}\right|\\
\le&\int_0^1\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^se^{1-r}f^r\\
&\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|\\
&\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|dtdsdr\\
\le&\left[\int_0^1\int_0^1\int_0^1 a^{(q-\ell)(1-t)/(q-1)}b^{(q-\ell)t/(q-1)}c^{(q-\ell)(1-s)/(q-1)}\right.\\
&\times d^{(q-\ell)s/(q-1)}e^{(q-\ell)(1-r)/(q-1)}f^{(q-\ell)r/(q-1)}\\
&\left.\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\right]^{1-1/q}\\
&\times\left[\int_0^1\int_0^1\int_0^1\left|\ln(a^{\ell(1-t)}b^{\ell t})\ln(c^{\ell(1-s)}d^{\ell s})\ln(e^{\ell(1-r)}f^{\ell r})\right|\right.\\
&\left.\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)\right|^q dtdsdr\right]^{1/q}\\
\le&\left[\int_0^1\int_0^1\int_0^1 a^{(q-\ell)(1-t)/(q-1)}b^{(q-\ell)t/(q-1)}c^{(q-\ell)(1-s)/(q-1)}\right.\\
&\times d^{(q-\ell)s/(q-1)}e^{(q-\ell)(1-r)/(q-1)}f^{(q-\ell)r/(q-1)}\\
&\left.\times\left|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\right]^{1-1/q}\\
&\times\left[\int_0^1\int_0^1\int_0^1\left|a^{\ell(1-t)}b^{\ell t}c^{\ell(1-s)}d^{\ell s}e^{\ell(1-r)}f^{\ell r}\right.\right.\\
&\left.\left.\times\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\ln(e^{1-r}f^r)\right|dtdsdr\right]^{1/q}\\
&\times\left[\max\left\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q,\right.\right.\\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q,\\
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q,\\
&\left.\left.\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q,\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\right\}\right]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{q-1}{q-\ell}\right)^{3(1-1/q)} \Big[N\big(a^{(q-\ell)/(q-1)}, b^{(q-\ell)/(q-1)}\big) \\
&\quad \times N\big(c^{(q-\ell)/(q-1)}, d^{(q-\ell)/(q-1)}\big) \\
&\quad \times N\big(e^{(q-\ell)/(q-1)}, f^{(q-\ell)/(q-1)}\big)\Big]^{1-1/q} \\
&\quad \times \left(\frac{1}{\ell}\right)^{3/q} \Big[N\big(a^{\ell}, b^{\ell}\big)\; N\big(c^{\ell}, d^{\ell}\big)\; N\big(e^{\ell}, f^{\ell}\big)\Big]^{1/q} \\
&\quad \times \Bigg[\max\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q, \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q, \\
&\quad \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q, \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q, \\
&\quad \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q, \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q, \\
&\quad \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q, \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg\}\Bigg]^{1/q},
\end{aligned}
$$

The proof of theorem is completed. □.

Theorem 2.5. Let $f:\Omega_+\to\mathbb{R}$ be a geometrically quasiconvex function on the co-ordinates on Ω_+. If $g\in L(\Omega_+)$, then

$$
\begin{aligned}
&f\big((ab)^{1/2},(cd)^{1/2},(ef)^{1/2}\big) \\
&\le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} \\
&\quad \times \int_e^f\int_c^d\int_a^b \frac{f(x,y,z)}{xyz}dxdydz \\
&\le \max\{g(a,c,e),g(a,c,f),g(b,c,e),g(b,c,f), \\
&\qquad g(a,d,e),g(a,d,f),g(b,d,e),g(b,d,f)\}.
\end{aligned} \tag{19}
$$

Proof. By geometric quasiconvexity of f on co-ordinates on Ω_+, we have

$$
\begin{aligned}
&f\big((ab)^{1/2},(cd)^{1/2},(ef)^{1/2}\big) \\
&\le \max\{f(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r), \\
&\qquad f(a^tb^{1-t},c^sd^{1-s},e^{1-r}f^r)\} \\
&\le \max\{g(a,c,e),g(a,c,f),g(b,c,e),g(b,c,f), \\
&\qquad g(a,d,e),g(a,d,f),g(b,d,e),g(b,d,f)\}.
\end{aligned} \tag{20}
$$

Since

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1 f(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)dtdsdr \\
&= \int_0^1\int_0^1\int_0^1 f(a^tb^{1-t},c^sd^{1-s},e^{1-r}f^r)dtdsdr \\
&= \frac{1}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} \\
&\quad \times \int_e^f\int_c^d\int_a^b \frac{f(x,y,z)}{xyz}dxdydz,
\end{aligned}
$$

by integrating in (20) we get

$$
\begin{aligned}
&f\big((ab)^{1/2},(cd)^{1/2},(ef)^{1/2}\big) \\
&\le \max\Big\{\int_0^1\int_0^1\int_0^1 f(a^tb^{1-t},c^sd^{1-s},e^{1-r}f^r)dtdsdr, \\
&\qquad \int_0^1\int_0^1\int_0^1 f(a^tb^{1-t},c^sd^{1-s},e^{1-r}f^r)dtdsdr, \\
&\qquad \int_0^1\int_0^1\int_0^1 f(a^tb^{1-t},c^sd^{1-s},e^{1-r}f^r)dtdsdr\Big\} \\
&= \frac{1}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} \\
&\quad \times \int_e^f\int_c^d\int_a^b \frac{f(x,y,z)}{xyz}dxdydz \\
&\le \max\{g(a,c,e),g(a,c,f),g(b,c,e),g(b,c,f), \\
&\qquad g(a,d,e),g(a,d,f),g(b,d,e),g(b,d,f)\},
\end{aligned}
$$

and proof is completed. □.

Theorem 2.6. Suppose that $g,h:\Omega_+\to\mathbb{R}$ are geometrically quasiconvex functions on the co-ordinates on Ω_+. If $gh\in L(\Omega_+)$. Then,

$$
\begin{aligned}
&\frac{1}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} \\
&\times \int_e^f\int_c^d\int_a^b \frac{g(x,y,z)}{xyz}dxdydz \\
&\le \max\{g(x,y,z)\;h(u,v,w)\mid x,u\in[a,b],\; y,v\in[c,d], \\
&\qquad\qquad w,z\in[e,f]\}.
\end{aligned}
$$

Proof. Let $x=a^{1-t}b^t,\ y=a^{1-s}b^s,\ z=e^{1-r}f^r,\ t,s,r\in[0,1]$ and using the geometric quasiconvexity of g,h on the co-ordinates on Ω_+ yields

$$
\begin{aligned}
&\frac{1}{(\ln b-\ln a)(\ln d-\ln c)(\ln f-\ln e)} \\
&\times \int_e^f\int_c^d\int_a^b \frac{g(x,y,z)}{xyz}dxdydz \\
&= \int_0^1\int_0^1\int_0^1 f(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r) \\
&\quad \times g(a^{1-t}b^t,c^{1-s}d^s,e^{1-r}f^r)dtdsdr \\
&\le \max\{g(a,c,e),g(a,c,f),g(b,c,e),g(b,c,f), \\
&\qquad g(a,d,e),g(a,d,f),g(b,d,e),g(b,d,f)\} \\
&\quad \times \max\{g(a,c,e),g(a,c,f),g(b,c,e),g(b,c,f), \\
&\qquad g(a,d,e),g(a,d,f),g(b,d,e),g(b,d,f)\},
\end{aligned}
$$

and proof is completed. □.

3 Conclusion

This paper is concerned with the celebrate Hermite-Hadamard inequality for functions of three variables. We obtained some results in connection to Hermite-Hadamard inequality byusing the notion of geometrically quasiconvex functions on a rectangular box. In our opinion, additional research should not only focus on how to weaken the partial differentiability condition in Lemma 2.2, but also how to generalize the convexity.

Acknowledgement

The authors are indebted to referees for carefully reading the paper and for making useful suggestions.

References

[1] A. Barani, S. Barani, Hermite-Hadamard inequalities for functions when a power of the absolute value of the first derivative is P-convex, Bull. Aust. Math. Soc. **86** (2012), 126-134.

[2] A. Barani, A. G. Ghazanfari and S. S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, J. Inequal. Appl 2012, 2012:247.

[3] A. Barani and F. Malmir, New some Hadamard's type inequalities for convex functions on a rectangular box, submitted.

[4] A. Barani and F. Malmir, on some new inequalities for quasiconvex functions on a rectangular box, submitted.

[5] A. Barani and F. Malmir, Hermite-Hadamard inequality for geometrically quasiconvex functions on co-ordinates, RGMIA Research Report Collection, **18** (2015), Article 37, 18 pp.

[6] S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plan, Taiwan. J. Math **5** (2001) 775-778.

[7] S. S. Dragomir and C.E.M. Pearce, Selected Topics on Hermit-Hadamard Type Inequalites and Applications, RGMIA (2000), Monographs, [ONLINE: http://ajmaa.org//RGMIA/monographs/hermit hadamard.htmI].

[8] U. S. Kirmaci, Inequalites for differentiable mappings and applications to special means of real numbers to midpoint formula, App. Math. Comput **147** (2004) 137-146.

[9] M. A. Latif, S. Hussain and S.S. Dragomir, On Some new inequality for co-ordinated quasi-convex functions, http://ajmaa.org/RGMIA/papers/ v14a55.pdf, 2011.

[10] M. S. Moslehian, Matrix Hermite-Hadamard type inequalities, Houston. J. Math **39** (2013) 177-189.

[11] M. E. Özdemir, A. O. Akdemir, Ç. Yıldız, On the co-ordinated convex functions, Appl. Math. Info. Sci **8** (2014) 1085-1091.

[12] M. E. Özdemir,Ç. Yıldız and A.O. Akdemir, On some new Hadamard-type inequalities for co-ordinated quasi-convex functions, Hacettepe. J. Math. Sta **41** (2012) 697-707.

[13] M. E. Özdemir, A.O. Akdemir, Ç. Yıldız, On co-ordinated quasi-convex functions, Czech. Math. J **62** (2012) 889-900.

[14] M. E. Özdemir, On the co-ordinated geometrically convex functions, Abstracts of MMA2013 and AMOE2013, May 27-30, 2013, Tartu, Estonia.

[15] M. E. Özdemir, A.O. Akdemir, On some Hadamard-type inequalities for convex functions on a rectangular box, volume 2011, year 2011 article ID jnaa-00101, 10 pages doi:10.5899/2011/jnaa-00101.

[16] M. A. Noor, K. I. Noor, M. U. Awan and J. Lib, On Hermite-Hadamard inequalities for h-preinvex functions, Filomat **28** (2014) 1463-1474.

[17] M. A. Noor, K. I. Noor, M. U. Awan, Hermite-Hadamard inequalities for relative semi-convex functions and applications, Filomat **28** (2014) 221-230

[18] J. Pečarić, F. Proschan and Y. L. Tong, Convex Function, Partial Orderings and Statistical Applications, Academic Press (1992), Inc.

[19] F. Qi and B.Y. Xi, Some Hermite-Hadamard type inequalities for geometrically quasi-convex functions, Proceedings of the Indian Academy of Science **124** (2014) 33-342.

[20] D. Y. Wang, K. L. Tseng, G. S. Yang, Some Hadamard's inequality for co-ordinated convex functions in a rectangle from the plane, Taiwan J. Math **11** (2007) 63-73.

[21] B-Y. Xi, J. Hua , F. Qi, Hermit-Hadamard type inequalities for extended s-convex functions on the co-ordinates in a rectangle. J. Appl. Anal **20** (2014) 29-39.

Ali Barani received the PhD degree in Mathematics for Isfahan University, Iran. His research interests are in the areas of inequalities and generalized convexity on Riemannian manifolds and linear spaces. He has published research articles in reputed international journals of mathematical sciences. He is referee of mathematical journals.

Fatemeh Malmir received the master degree in Mathematics for Lorestan university, Iran. His research interests are in the areas of generalized convexity. Now she is a PhD student of Mathematics.